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ON THE COMPUTATION OF JORDAN CANONICAL FORM

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Abstract: In this paper, it is shown that to compute the Jordan Canonical Form of a matrix in the case of numerical computation is an unsolvable problem.

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1. Introduction

Computation of Jordan Canonical Form is a subject proposed by Golub and Wilkinson in 1976 [2] (from now on we will name this subject as G-W problem). In the beginning people used to try to find a stable algorithm for computing the Jordan Canonical Form (JCF) of a matrix. But after Golub pointed out diagonalizable matrix is dense in set of matrices, people have to reconsider the exit of solving this subject, because above fact pointed by Golub put forward question efficiency about numerical computation for computing the JCF of a matrix.

In Fact, if matrix A has exact non-linear divisors, since diagonalizable matrix dense, the neighbourhood of A contains diagonalizable matrix, therefore the Jordan Block Structure (JBS) is not continuous function of A, thus following

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relation no longer holds

$$Lim\ JBS(A+\Delta A) \to JBS(A), as||\Delta A|| \to 0$$

as well as no longer holds

$$Lim\ JCF(A+\Delta A) \to JCF(A), as||\Delta A|| \to 0$$

and even if numerical result fl[JCF(A)] such that

$$||JCF(A) - fl[JCF(A)]|| < \varepsilon$$

Where $\varepsilon > 0$ is any given small number. we can not eliminate such possibility that fl[JCF(A)] is a diagonal matrix yet, because exist diagonalizable matrices (Say A') Such that

$$||JCF(A') - JCF(A)|| < \varepsilon$$

Especially such matter had been certificated by practices experience that fl [JCF(A)] usually is a diagonal matrix in practical work. As Wilkinson said: In our experience "matrix having exact non-linear divisors are almost non-existent in practical work" and "rounding errors will usually lead to a matrix which no longer has non-linear elementary divisors."

In this paper, we will find the theory background of above Wilkinson's experience first, then prove the algorithm for computing the JCF of a matrix in the case of numerical computation is non-existent.

2. Mathematical Background

Now we have shown that:

Theorem 1. (see [8]) *The probability that a random real matrix to be a diagonalizable matrix equals to 1.*

Corollary 1. (see [8]) *The probability that a random real matrix to be a matrix having non-linear elementary divisors equals to zero.*

Theorem 2. (see [9]) *The probability that a random rational matrix to be a diagonalizable matrix equals to 1.*

Corollary 2. (see [9]) *The probability that a random rational matrix to be a matrix having non-linear elementary divisors equals to zero.*

When the base field $F = Q$(rational field), R(real field)), above theorem 1, 2 can be written as

Theorem 3. *The probability that a random matrix $A \in F^{n\times n}(F = Q, R)$ to be a diagonalizable matrix equals to 1.*

Corollary 1, 2 can be uniformly written as

Theorem 4. *The probability that a random matrix $A \in F^{n\times n}(F = Q, R)$ bo be a matrix having non-linear elementary divisors equals to zero.*

Moreover according to the fundamental principle of numerical computation [3], the numerical result of computing JBS of matrix $A \in F^{n\times n}(F = Q, R)$ is

$$fl[JBS(A)] = JBS(A + \Delta A), \|\Delta A\| < \delta, \delta > 0, \tag{1}$$

where ΔA is the perturbation as a result from rounding errors in computing process, $\|\Delta A\| < \delta$, and $\delta > 0$ is the unique restriction on ΔA, and $\forall \Delta A \in \Omega = \{\Delta A| \ \|\Delta A\| < \delta, \delta > 0\}$, JBS$(A + \Delta A)$ can act as

$$fl[JBS(A)] \tag{2}$$

, thus ΔA is random in point set Ω, or $A + \Delta A = x$ in the neighborhood $\Omega' = \{x| \ \|x - A\| < \delta, \delta > 0\}$ is random. According to theorem 3 and theorem 4, we have as following conclusions:

$$\text{For } \forall x \in \Omega', \text{x} = \text{a diagonalizable matrix.} \tag{3}$$

holds with probability 1

$$\text{For } \forall x \in \Omega', \text{x} = \text{a matrix having exact non-linear divisors} \tag{4}$$

holds with probability zero.

From equation (1) and (3), we have

Theorem 5. $A \in F^{n\times n}(F = Q, R)$,

$$fl[JBS(A)] = JBS(x)$$

where x is a diagonalizable matrix with probability 1.

Theorem 5 just is the original mathematical form of above Wilkinson's experience "rounding errors will usually lead to a matrix which no longer has non-linear elementary divisors."

From equations (1) and (4) we have

Theorem 6. $A \in F^{n\times n}(F = Q, R)$,

$$fl[JBS(A)]) = JBS(x)$$

where x is a matrix having non-linear elementary divisors with probability zero.

Theorem 6 just is the original mathematical form of above Wilkinson's experience "matrix having exact non-linear elementary divisors are almost non-existent in practical work."

Obviously, theorem 5 and theorem 6 is equivalent each other, and in the future we will uniformly name these two theorem as Wilkinson Theorem.

3. The Answer of the G-W Problem

According to above Wilkinson theorem, we conclude that any numerical result ascertaining the JBS of a matrix $A \in F^{n\times n}, (F = Q, R)$ is inefficient.

In fact, the numerical result fl[JBS(A)] $= JBS(A'), A'$ usually is a diagonalizable matrix, it is false for the original matrix having non-linear elementary divisors, or it is true for the original matrix being diagonalizable, but before performing to compute the JBS of A, we know nothing about A whether or not is diagonalizable matrix. Thus fl[JBS(A)] may be true or false. Obviously such character of numerical result fl[JBS(A)] is useless at all for determining the JBS of A. In other words, the numerical algorithm ascertaining the JBS of a matrix $A \in F^{n\times n}, (F = Q, R)$ is non-existent.

Further for any numerical field $F, A \in F^{n\times n}$, the numerical algorithm ascertaining the JBS of A is non-existent too. Because if not, there exist a numerical algorithm ascertaining the JBS of $A, A \in F^{n\times n}$, then this numerical algorithm will be effect for $A \in Q^{n\times n}$(rational matrix), because rational field is the subfield of any numerical field,thus this is contrary to above result.

Finally we conclude that the numerical algorithm to compute the JCF of a matrix $A \in F^{n\times n}$,(F is any numerical field) is non-existent, because the JBS is a part of the JCF and we have proved that the numerical algorithm to compute the JBS of a matrix $A \in F^{n\times n}$ (F is any numerical field) is non-existent.

The above discussion provides the proof that the computation of Jordan canonical form in the case of numerical computation is unsolvable.

4. The Significance of G-W Problem

As people discover $x^2 - 2 = 0$ is unsolvable in rational field lead to expand from rational field to real field and discover $x^2 + 2 = 0$ is unsolvable in real field lead to expand from real field to complex field, the G-W problem is the first unsolvable computing problem in the case of numerical computation since numerical computation arise. This fact indicate that only numerical computation is insufficient for computational mathematics, Computational mathematics need to add new computational means except numerical computation.

Fortunately, the symbolic computation had aroused in end of last century, see [4]. Symbolic computation posses such function that it can exactly complete pure rational operation for formula and fraction without interference come from rounding errors. This function of symbolic computation is just the lack of numerical computation and we have shown that the JBS of a matrix A can be determined by symbolic computation, see [5], [6], [7]. But symbolic computation can not take the place of numerical computation in G-W problem, because when symbolic computation face the eigenvalue (the another part of JCF), this is unsolvable too. In fact, the eigenvalues is the roots of the characteristic polynomial and it is well known that when degree more than five it is impossible to give out the root of polynomial by formula. Now ask to compute the eigenvalue by symbolic computation namely ask give out the expression of eigenvalue by exact rational operations. Obviously this is impossible. In this time to compute the approximate value of eigenvalue by numerical computation is a regular subject in numerical analysis. We conclude from output that both symbolic computation and numerical computation are needed computation means for computational mathematics, they are independent and complement each other.

In fact, if we determine the JBS of matrix $A \in F^{\mathsf{n}\times\mathsf{n}}$ by symbolic computation [5] and computing the eigenvalue of A by numerical computation, after that embed the eigenvalues in to the corresponding Jordan black. Finally, we can obtain an approximate of the JCF of A(notice that here the approximate JCF of A imply its JBS is exactly identical with the JBS of the original matrix, only the corresponding eigenvalues is approximate). In a word, to compute the JCF of A will no longer be unsolvable, if the computational mathematics have based on algebraic operation (numerical computation $\smile$ symbolic computation). In another word, if computational mathematics allow to employ the algebraic operation (numerical operation $\smile$ symbolic computation), it will expand itself service scope.

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