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WEIGHTED COMPOSITION OPERATORS ON WEIGHTED BERGMAN SPACES OF THE UNIT BALL

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Abstract: Suppose φ is an analytic self map of $\mathbb{B}_n$ and ψ is analytic on $\mathbb{B}_n$. Then a weighted composition operator induced by φ with weight ψ is given by $(W_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z))$ for z in $\mathbb{B}_n$ and f analytic on $\mathbb{B}_n$. Given $W_{\psi,\varphi} : A^p_\alpha(\mathbb{B}_n) \to A^q_\beta(\mathbb{B}_n)$ we characterize boundedness and compactness of $W_{\psi,\varphi}$, where $0 < p, q < \infty$ and $-1 < \alpha, \beta < \infty$. We also characterize the Schatten p-class weighted composition operators $S_p\left(A^2_\alpha(\mathbb{B}_n)\right)$ for $0 < p < \infty$ and $-1 < \alpha < \infty$.

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1. Introduction

Let $\mathbb{B}_n$ denote the open unit ball in the complex n-dimensional Euclidean space $\mathbb{C}^n$. Let $H(\mathbb{B}_n)$ denote the space of all analytic functions in $\mathbb{B}_n$. For $\alpha > -1$,

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the weighted Lebesgue measure dv_α is defined by:

$$dv_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha dv(z),$$

where dv is a volume measure on $\mathbb{B}_n$ and

$$c_\alpha = \frac{\Gamma(n + \alpha + 1)}{n!\Gamma(\alpha + 1)}$$

is a normalizing constant so that dv_α is a probability measure on the unit ball.

For $-1 < \alpha < \infty$ and $0 < p < \infty$, the weighted Bergman space $A^p_\alpha(\mathbb{B}_n)$ consists of analytic functions in $L^p(\mathbb{B}_n, dv_\alpha)$. In other words $A^p_\alpha(\mathbb{B}_n)$ consists of those analytic functions f on $\mathbb{B}_n$ that satisfy

$$\|f\|^p_{A^p_\alpha} = \int_{\mathbb{B}_n} |f(z)|^p \, dv_\alpha(z) < \infty.$$

Here the assumption $\alpha > -1$ is needed, because the space $L^p(\mathbb{B}_n, dv_\alpha)$ does not contain any analytic function other than 0 when $\alpha \leq -1$. It is clear that $A^p_\alpha(\mathbb{B}_n)$ is a linear subspace of $L^p(\mathbb{B}_n, dv_\alpha)$. When $p = 2$ and $n = 1$ we get the weighted Bergman space $A^2_\alpha(\mathbb{D})$ of the unit disk. If φ is an analytic self-map of $\mathbb{B}_n$ and ψ belongs to $H(\mathbb{B}_n)$, the weighted composition operator $W_{\psi,\varphi}$ is defined on $H(\mathbb{B}_n)$ by $(W_{\psi,\varphi}f)(z) = \psi(z)f(\varphi(z))$, for all f in $H(\mathbb{B}_n)$ and z in $\mathbb{B}_n$. We call φ the inducing map and ψ the weight function of $W_{\psi,\varphi}$.

In this paper, we address the following: For $W_{\psi,\varphi} : A^p_\alpha(\mathbb{B}_n) \to A^q_\beta(\mathbb{B}_n)$, where $0 < p, q < \infty$ and $-1 < \alpha, \beta < \infty$, when is $W_{\psi,\varphi}$ bounded and when is it compact? Cuckovic and Zhao (see [3], [4]) studied the boundedness and compactness of the weighted composition operators acting on $A^2_\alpha(\mathbb{D})$. In Section 2 and 3 of this paper, we generalize the results of [3] and [4] to the unit ball. In terms of the Carleson-type measure, Ueki [14] characterized the boundedness and compactness of $W_{\psi,\varphi}$ from $A^p_\alpha(\mathbb{B}_n)$ into $A^q_\beta(\mathbb{B}_n)$ when $0 < p \leq q < \infty$. By using techniques similar to [4], Luo and Ueki [9] characterized the boundedness and compactness of $W_{\psi,\varphi}$ from $A^p_\alpha(\mathbb{B}_n)$ into $A^q_\beta(\mathbb{B}_n)$ when $0 < q < p < \infty$. Our results are more general than those in [9] and [14].

In Section 4, we characterize the Schatten $p-$class of weighted composition operators on $A^2_\alpha(\mathbb{B}_n)$. Luecking and Zhu [8] found a criteria of the Schatten $p-$class membership on Hardy and Bergman spaces of the unit disk, by using the Nevanlinna counting function of the inducing map. It is known that (weighted) composition operators are related to Toeplitz operators on weighted Bergman spaces and Hardy spaces as well. This relation is used to characterize Schatten $p-$class of composition operators on weighted Bergman spaces of the unit disk (see [8], [17], [18]) as well as weighted composition operators on Bergman spaces [3]. We will use this connection to Toeplitz operators, induced by positive measures and defined on the same space on which $W_{\psi,\varphi}$ acts, to describe the Schatten $p-$class of weighted composition operators on weighted Bergman spaces of the unit ball.

2. Boundedness and Compactness when $0 < p \leq q < \infty$

The results of this section concern boundedness and compactness of weighted composition operators mapping $A^p_\alpha(\mathbb{B}_n)$ into $A^q_\beta(\mathbb{B}_n)$ for $0 < p \leq q < \infty$. Our results will be expressed in terms of the following integral transform that generalizes the Berezin transform:

$$I_{\varphi,s,\alpha,\beta}(u)(z) = \int_{\mathbb{B}_n} \frac{|u(w)|^q (1-|z|^2)^s}{|1-\langle\varphi(w),z\rangle|^{s+(n+1+\alpha)q/p}} dv_\beta(w),$$

where s is a positive real number and α, β are real numbers such that $-1 < \alpha, \beta < \infty$. In order to prove our results, we need the following preliminaries and some lemmas.

For any $w \in \mathbb{B}_n$, let φ_w denote an automorphism of $\mathbb{B}_n$. It is well-known that the pseudo-hyperbolic metric on $\mathbb{B}_n$ is defined as $\rho(z,w) = |\varphi_w(z)|$. Let $\beta(z,w) = \frac{1}{2}\log\frac{1+|\varphi_w(z)|}{1-|\varphi_w(z)|}$ be the distance between z and w in the Bergman metric of $\mathbb{B}_n$, so that $\beta(z,w) = \tanh^{-1}|\varphi_w(z)|$. This shows $\rho(z,w) = \tanh\beta(z,w)$. It is clear that ρ is bounded, while β is not. So for $R > 0$ and $a \in \mathbb{B}_n$, let $D(a,R)$ denote the Bergman metric ball at a,

$$D(a,R) = \{z \in \mathbb{B}_n : \beta(z,a) < R\}.$$

Most of the results we obtain about weighted composition operators will be given in terms of a certain measure, which we define next. Suppose φ is an analytic self-map of the unit ball $\mathbb{B}_n$. Then for $0 < p < \infty$ and $-1 < \alpha < \infty$ we define for any $\psi \in A^p_\alpha(\mathbb{B}_n)$ a finite positive Borel measure $\mu_{\varphi,\psi,p,\alpha}$ on $\mathbb{B}_n$ by:

$$\mu_{\varphi,\psi,p,\alpha}(E) = \int_{\varphi^{-\mathbf{1}}(E)} |\psi|^p \, dv_\alpha,$$

where E is a Borel subset of $\mathbb{B}_n$.

Thus by using (see [5], Theorem III.10.4) we get the following change of variable formula,

$$\int_{\mathbb{B}_n} g d\mu_{\varphi,\psi,p,\alpha} = \int_{\mathbb{B}_n} |\psi|^p \, g(\varphi) dv_\alpha,$$

where g is an arbitrary measurable positive function on $\mathbb{B}_n$. So, for any $f \in A^p_\alpha(\mathbb{B}_n)$, by taking $g = |f|^p$ we get:

$$\int_{\mathbb{B}_n} |f(z)|^p \, d\mu_{\varphi,\psi,p,\alpha}(z) = \int_{\mathbb{B}_n} |\psi(z)|^p \, |f(\varphi(z))|^p \, dv_\alpha(z). \tag{1}$$

An important tool in the study of weighted composition operators on analytic functions spaces is the notion of Carleson measure. We will consider Carleson measure on weighted Bergman spaces.

Definition 1. Let μ be a positive Borel measure on $\mathbb{B}_n$ and let X be a Banach space of analytic functions on $\mathbb{B}_n$. Then for $q > 0$, μ is an $(X, q)-$Carleson measure if there is a constant $c > 0$ such that for any $f \in X$,

$$\int_{\mathbb{B}_n} |f(z)|^q \, d\mu(z) \leq c\|f\|^q_X.$$

Let $\mathbb{S}_n = \partial\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| = 1\}$. For any $\zeta \in \mathbb{S}_n$ and $r > 0$ the nonisotropic metric balls are defined as:

$$Q_r(\zeta) = \{z \in \mathbb{B}_n : |1 - \langle z, \zeta\rangle| < r\}.$$

These are the higher dimensional analogue of Carleson squares in the unit disk. As a special case of (see [15], Theorem 50) we have the following Lemma 2.

Lemma 2. *Suppose* $0 < p \leq q < \infty$, $\alpha > -1$ *and* μ *is a positive Borel measure on* $\mathbb{B}_n$*. Then the following conditions are equivalent:*

1. μ *is* $(A^p_\alpha, q)-$*Carleson measure, that is, there is a constant* $c_1 > 0$ *such that*
$$\int_{\mathbb{B}_n} |f(z)|^q \, d\mu(z) \leq c_1\|f\|^q_{A^p_\alpha}.$$
2. *There is a constant* $c_2 > 0$ *such that*
$$\mu(Q_r(\zeta)) \leq c_2 \; r^{(n+1+\alpha)q/p},$$
for all $r > 0$ *and* $\zeta \in \mathbb{S}_n$*.*

Lemma 3. (see [15], Theorem 45) *Suppose* $(n + 1 + \alpha) > 0$ *and* μ *is a positive Borel measure on* $\mathbb{B}_n$*. Then the following conditions are equivalent:*

1. *There exists a constant* $c_1 > 0$ *such that*
$$\mu(Q_r(\zeta)) \leq c_1 \; r^{(n+1+\alpha)},$$
for all $r > 0$ *and for* $\zeta \in \mathbb{S}_n$*.*

2. *For each (or some) $s > 0$ there exists a constant $c_2 > 0$ such that, for all $z \in \mathbb{B}_n$,*

$$\int_{\mathbb{B}_n} \frac{(1-|z|^2)^s}{|1-\langle w,z\rangle|^{s+(n+1+\alpha)}} d\mu(w) \leq c_2.$$

In the next theorem we characterize the boundedness of weighted composition operators.

Theorem 4. *Let $0 < p \leq q < \infty$, and $-1 < \alpha, \beta < \infty$. Suppose that φ is an analytic self-map of $\mathbb{B}_n$ and $\psi \in A^q_\beta(\mathbb{B}_n)$. Then the weighted composition operator $W_{\psi,\varphi}$ is bounded from $A^p_\alpha(\mathbb{B}_n)$ into $A^q_\beta(\mathbb{B}_n)$ if and only if for any real $s > 0$*

$$\sup_{a\in\mathbb{B}_n} I_{\varphi,s,\alpha,\beta}(\psi)(a) < \infty.$$

Proof. By definition $W_{\psi,\varphi}$ is bounded from A^p_α into A^q_β if and only if for any $f \in A^p_\alpha$, there exist a constant $c > 0$ such that $\|W_{\psi,\varphi}(f)\|^q_{A^q_\beta} \leq c\|f\|^q_{A^p_\alpha}$, that is,

$$\int_{\mathbb{B}_n} |\psi(z)|^q \, |f(\varphi(z))|^q \, dv_\beta \leq c\|f\|^q_{A^p_\alpha}.$$

By the change of variables formula 1 we get

$$\int_{\mathbb{B}_n} |f(z)|^q \, d\mu_{\varphi,\psi,q,\beta}(z) \leq c\|f\|^q_{A^p_\alpha}.$$

By Definition 1 this means $\mu_{\varphi,\psi,q,\beta}$ is $(A^p_\alpha, q)-$Carleson measure. By Lemma 2, this is equivalent to

$$\mu_{\varphi,\psi,q,\beta}(Q_r(\zeta)) \leq c\; r^{(n+1+\alpha)q/p},$$

for all $r > 0$, $\zeta \in \mathbb{S}_n$ and some constant $c > 0$. By Lemma 3, this is equivalent to

$$\int_{\mathbb{B}_n} \frac{(1-|a|^2)^s}{|1-\langle w,a\rangle|^{s+(n+1+\alpha)q/p}} d\mu_{\varphi,\psi,q,\beta}(w) \leq c,$$

for all $a \in \mathbb{B}_n$ and for each $s > 0$. It follows by using the change of variables formula 1,

$$\sup_{a\in\mathbb{B}_n} \int_{\mathbb{B}_n} \frac{|\psi(w)|^q \, (1-|a|^2)^s}{|1-\langle \varphi(w),a\rangle|^{s+(n+1+\alpha)q/p}} dv_\beta(w) < \infty,$$

which gives the desired result, $\sup_{a\in\mathbb{B}_n} I_{\varphi,s,\alpha,\beta}(\psi)(a) < \infty$. □

As a consequence of Theorem 4, Lemma 3, and Lemma 2 we have the following corollary.

Corollary 5. *Let $0 < p \le q < \infty$, and $-1 < \alpha, \beta < \infty$. Suppose that φ is an analytic self-map of $\mathbb{B}_n$ and $\psi \in A^q_\beta(\mathbb{B}_n)$. Then the following are equivalent:*

1. *$W_{\psi,\varphi} : A^p_\alpha(\mathbb{B}_n) \to A^q_\beta(\mathbb{B}_n)$ is bounded.*
2. *$\mu_{\varphi,\psi,q,\beta}$ is $(A^p_\alpha, q)-$Carleson measure.*
3. *There exists a constant $c > 0$ so that for each $s > 0$,*

$$\int_{\mathbb{B}_n} \frac{|\psi(w)|^q}{|1 - \langle \varphi(w), z\rangle|^{s+(n+1+\alpha)q/p}} dv_\beta(w) < \frac{c}{(1-|z|^2)^s},$$

for every z in $\mathbb{B}_n$.

Remark 6. If we take $s = (n+1+\alpha)q/p$ we get Theorem 3.1 in [14].

When a result concerning Carleson measure is established, it is then relatively straight forward to formulate and prove the corresponding "little-oh" version. In order to do that, we introduce vanishing Carleson measure.

Definition 7. Let μ be a positive Borel measure on $\mathbb{B}_n$, and let X be a Banach space of analytic functions on $\mathbb{B}_n$. Then for $q > 0$, μ is called vanishing $(X, q)-$Carleson measure if

$$\lim_{k\to\infty} \int_{\mathbb{B}_n} |f_k|^q d\mu = 0,$$

whenever $\{f_k\}$ is a bounded sequence in X that converges to 0 uniformly on compact subsets of $\mathbb{B}_n$.

As a special case of (see [15], p.71) we have the following lemma 8.

Lemma 8. *Suppose $0 < p \le q < \infty$, $\alpha > -1$ and μ is a positive Borel measure on $\mathbb{B}_n$. Then the following are equivalent:*

1. *μ is vanishing $(A^p_\alpha, q)-$Carleson measure.*
2. *The limit*

$$\lim_{r\to 0^+} \frac{\mu(Q_r(\zeta))}{r^{(n+1+\alpha)q/p}} = 0,$$

holds uniformly for $\zeta \in \mathbb{S}_n$.

The essential norm of an operator, denoted by $\|.\|_e$, is its distance in the operator norm from the ideal of compact operators. So for any weighted composition operator $W_{\psi,\varphi} : A^p_\alpha \to A^q_\beta$, we define

$$\|W_{\psi,\varphi}\|_e = \inf_{K\in\mathbb{K}} \|W_{\psi,\varphi} - K\|,$$

where $\mathbb{K} = \mathbb{K}(A^p_\alpha, A^q_\beta)$ is the set of compact operators acting from $A^p_\alpha(\mathbb{B}_n)$ into $A^q_\beta(\mathbb{B}_n)$. Next theorem gives the lower bound of the essential norm of weighted composition operator.

Theorem 9. *Let ψ be an analytic function on $\mathbb{B}_n$ and φ be an analytic self-map of $\mathbb{B}_n$. Let $1 < p \le q < \infty$ and $-1 < \alpha, \beta < \infty$. Let $W_{\psi,\varphi}$ be bounded from $A^p_\alpha(\mathbb{B}_n)$ into $A^q_\beta(\mathbb{B}_n)$. Then*

$$||W_{\psi,\varphi}||^q_e \ge c \limsup_{|a|\to 1} I_{\varphi,s,\alpha,\beta}(\psi)(a)$$

for any real $s > 0$.

Proof. For any $a \in \mathbb{B}_n$, consider the analytic function

$$f_a(z) = \frac{(1-|a|^2)^{s/q}}{(1-\langle z,a\rangle)^{s/q+(n+1+\alpha)/p}}.$$

It is clear that $\{f_a\}$ converges uniformly to 0 on compact subsets of $\mathbb{B}_n$, as $|a| \to 1$. We claim that $\sup_{a\in\mathbb{B}_n} \|f_a\|_{A^p_\alpha} \le c$. By using [12], Proposition 1.4.10, there is a constant $c > 0$ such that

$$\begin{aligned}\int_{\mathbb{B}_n} \frac{(1-|a|^2)^{sp/q}}{(1-\langle z,a\rangle)^{sp/q+(n+1+\alpha)}} dv_\alpha(z) &= \int_{\mathbb{B}_n} \frac{(1-|a|^2)^{sp/q}(1-|z|^2)^\alpha}{(1-\langle z,a\rangle)^{sp/q+(n+1+\alpha)}} dv(z)\\ &= (1-|a|^2)^{sp/q} \int_{\mathbb{B}_n} \frac{(1-|z|^2)^\alpha}{(1-\langle z,a\rangle)^{sp/q+(n+1+\alpha)}} dv(z)\\ &\le c(1-|a|^2)^{sp/q}(1-|a|^2)^{-sp/q}\\ &= c,\end{aligned}$$

which proves the claim.

Now, fix a compact operator K acting from A^p_α into A^q_β. Then $\|Kf_a\|_{A^q_\beta} \to 0$ as $|a| \to 1$. Therefore for a positive constant c_1 we have,

$$c_1\|W_{\psi,\varphi} - K\| \ge \limsup_{|a|\to 1} \|(W_{\psi,\varphi} - K)f_a\|_{A^q_\beta}$$

$$\geq \limsup_{|a|\to 1}\left(\|(W_{\psi,\varphi})f_a\|_{A^q_\beta} - \|Kf_a\|_{A^q_\beta}\right)$$
$$= \limsup_{|a|\to 1}\|(W_{\psi,\varphi})f_a\|_{A^q_\beta}.$$

Hence, by taking infimum over all compact operators K, we get

$$\begin{aligned}\|W_{\psi,\varphi}\|_e^q &\geq c_2 \limsup_{|a|\to 1}\|(W_{\psi,\varphi})f_a\|_{A^q_\beta}\\ &= c_2 \limsup_{|a|\to 1}\int_{\mathbb{B}_n}\frac{|\psi(z)|^q(1-|a|^2)^s}{(1-\langle\varphi(z),a\rangle)^{s+(n+1+\alpha)q/p}}\\ &= c_2 \sup_{|a|\to 1} I_{\varphi,s,\alpha,\beta}(\psi)(a),\end{aligned}$$

where c_2 is a constant that depends only on c_1. □

Our second main result in this section characterizes the compactness of the weighted composition operator. Before doing this we need the following lemma whose proof can be obtained by adapting the proof [2], Proposition 3.11.

Lemma 10. *For $0 < p, q < \infty$ and $-1 < \alpha, \beta < \infty$. Let φ be an analytic self-map of $\mathbb{B}_n$ and $\psi \in A^q_\beta(\mathbb{B}_n)$ such that $W_{\psi,\varphi}$ is bounded from $A^p_\alpha(\mathbb{B}_n)$ into $A^q_\beta(\mathbb{B}_n)$. Then $W_{\psi,\varphi}$ is compact if and only if whenever $\{f_n\}$ is bounded sequence in $A^p_\alpha(\mathbb{B}_n)$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{B}_n$, then $\|W_{\psi,\varphi}(f_n)\|_{A^q_\beta} \to 0$.*

Theorem 11. *Let ψ be an analytic function on $\mathbb{B}_n$ and φ be an analytic self-map of $\mathbb{B}_n$. Let $1 < p \leq q < \infty$ and $-1 < \alpha, \beta < \infty$. Let $W_{\psi,\varphi}$ be bounded from $A^p_\alpha(\mathbb{B}_n)$ into $A^q_\beta(\mathbb{B}_n)$. Then $W_{\psi,\varphi}$ is compact if and only if*

$$\limsup_{|a|\to 1} I_{\varphi,s,\alpha,\beta}(\psi)(a) = 0.$$

Proof. The necessary condition follows from Theorem 9. We consider the sufficient condition, to do this we are going to use Definition 7 and Lemma 8. We claim that $\mu_{\varphi,\psi,q,\beta}$ is vanishing $(A^p_\alpha, q)-$ Carleson measure. If $\zeta \in \partial\mathbb{B}_n$ and $0 < r < 1$, then for all $a \in \mathbb{B}_n$ we get

$$\begin{aligned}&\int_{Q_r(\zeta)}\frac{|\psi(z)|^q \quad (1-|a|^2)^s}{(1-\langle\varphi(z),a\rangle)^{s+(n+1+\alpha)q/p}}dv_\beta(z)\\ &\leq \int_{\mathbb{B}_n}\frac{|\psi(z)|^q \quad (1-|a|^2)^s}{(1-\langle\varphi(z),a\rangle)^{s+(n+1+\alpha)q/p}}dv_\beta(z)\end{aligned}$$

$$= I_{\varphi,s,\alpha,\beta}(\psi)(a).$$

Then by using the hypothesis (that is, $\limsup_{|a|\to 1} I_{\varphi,s,\alpha,\beta}(\psi)(a) = 0$), we get

$$\begin{aligned}
&\limsup_{|a|\to 1} \int_{Q_r(\zeta)} \frac{|\psi(z)|^q \quad (1-|a|^2)^s}{(1-\langle \varphi(z), a\rangle)^{s+(n+1+\alpha)q/p}} dv_\beta(z) \\
&= \limsup_{|a|\to 1} \int_{Q_r(\zeta)} \frac{(1-|a|^2)^s}{(1-\langle z, a\rangle)^{s+(n+1+\alpha)q/p}} d\mu_{\varphi,\psi,q,\beta}(z) \\
&= 0.
\end{aligned}$$

Hence, for all $\epsilon > 0$, there exists r in $(0,1)$ such that for $|a| > r$ we have

$$\left| \int_{Q_r(\zeta)} \frac{(1-|a|^2)^s}{(1-\langle z, a\rangle)^{s+(n+1+\alpha)q/p}} d\mu_{\varphi,\psi,q,\beta}(z) \right| < \epsilon. \tag{2}$$

So, if we choose $a = (1-r)\zeta$, then

$$\begin{aligned}
1 - \langle a, z\rangle &= 1 - \langle \zeta - r\zeta, z\rangle \\
&= 1 - \langle \zeta, z\rangle + r\langle \zeta, z\rangle \\
&= (1-r)(1 - \langle \zeta, z\rangle) + r.
\end{aligned}$$

Thus, for all $z \in Q_r(\zeta)$ we have

$$\begin{aligned}
|1 - \langle a, z\rangle| &\le (1-r)\,|1 - \langle \zeta, z\rangle| + r \\
&< (1-r)r + r \\
&< 2r.
\end{aligned}$$

Note that,

$$\begin{aligned}
(1-|a|^2)^s = (1-(1-r)^2)^s &= (2r - r^2)^s \\
&= r^s(2-r)^s \\
&\ge r^s.
\end{aligned}$$

Thus for all $z \in Q_r(\zeta)$,

$$\begin{aligned}
\frac{(1-|a|^2)^s}{(1-\langle z, a\rangle)^{s+(n+1+\alpha)q/p}} &\ge \frac{r^s}{(2r)^{s+(n+1+\alpha)q/p}} \\
&= \frac{2^{-s-(n+1+\alpha)q/p}}{r^{(n+1+\alpha)q/p}}.
\end{aligned} \tag{3}$$

Hence, from Equations 2 and 3, we get for all $\zeta \in \partial\mathbb{B}_n$

$$\begin{aligned}\epsilon &> \int_{Q_r(\zeta)} \frac{2^{-s-(n+1+\alpha)q/p}}{r^{(n+1+\alpha)q/p}} d\mu_{\varphi,\psi,q,\beta}(z)\\ &\geq c\, \frac{\mu_{\varphi,\psi,q,\beta}(Q_r(\zeta))}{r^{(n+1+\alpha)q/p}},\end{aligned}$$

which gives, $\lim_{r\to 0^+} \frac{\mu_{\varphi,\psi,q,\beta}(Q_r(\zeta))}{r^{(n+1+\alpha)q/p}} = 0$. Therefore, by Lemma 8, we get $\mu_{\varphi,\psi,q,\beta}$ is a vanishing $(A^p_\alpha, q)-$ Carleson measure. This proves the claim.

Now, by Definition 7, for any bounded sequence $\{f_k\}$ in A^p_α that converges uniformly to 0 on compact subsets of $\mathbb{B}_n$ we have,

$$\lim_{k\to\infty} \int_{\mathbb{B}_n} |f_k(z)|^q d\mu_{\varphi,\psi,q,\beta}(z) = 0,$$

which is equivalent to

$$0 = \lim_{k\to\infty} \int_{\mathbb{B}_n} |\psi(z)|^q \quad |f_k(\varphi(z))|^q \, dv_\beta(z) = \lim_{k\to\infty} \|W_{\psi,\varphi}(f_k)\|^q_{A^q_\beta}.$$

Therefore, by Lemma 10 we see that $W_{\psi,\varphi}$ is compact. □

As a result of Definition 7, Lemma 8, and Theorem 11 we have the following corollary.

Corollary 12. *Let ψ be an analytic function on $\mathbb{B}_n$ and φ be an analytic self-map of $\mathbb{B}_n$. Let $1 < p \leq q < \infty$ and $-1 < \alpha, \beta < \infty$. Let $W_{\psi,\varphi}$ be bounded from $A^p_\alpha(\mathbb{B}_n)$ into $A^q_\beta(\mathbb{B}_n)$. Then the following are equivalent:*

1. *$W_{\psi,\varphi}$ is compact.*
2. *$\mu_{\varphi,\psi,q,\beta}$ is vanishing $(A^p_\alpha, q)-$Carleson measure.*
3. *$\limsup_{|a|\to 1} I_{\varphi,s,\alpha,\beta}(\psi)(a) = 0$.*

Remark 13. Again if we take $s = (n+1+\alpha)q/p$ we get Theorem 4.1 in [14].

3. Boundedness and Compactness when $0 < q < p < \infty$

In this section we characterize the boundedness and compactness of the weighted composition operator $W_{\psi,\varphi}$ mapping $A^p_\alpha(\mathbb{B}_n)$ into $A^q_\beta(\mathbb{B}_n)$ when $0 < q < p < \infty$. To deal with this case, we associate two functions to any positive Borel measure μ on $\mathbb{B}_n$. More specifically, for any real γ and s we define for $z \in \mathbb{B}_n$,

$$B_{s,\gamma}(\mu)(z) = \int_{\mathbb{B}_n} \frac{(1-|z|^2)^s}{|1-\langle z,w\rangle|^{(n+1+s+\gamma)}} d\mu(w),$$

and for any real γ and positive R we define for $z \in \mathbb{B}_n$,

$$\hat{\mu}_{R,\gamma}(z) = \frac{\mu(D(z,R))}{(1-|z|^2)^{n+1+\gamma}}.$$

It is clear that$B_{s,\gamma}(\mu)(z)$ and $\hat{\mu}_{R,\gamma}(z)$ are certain averages of μ near the point z (for example see, [15] and [19]). The function $B_{s,\gamma}(\mu)$ is called the Berezin transform of μ. If $s = n+1+\gamma$, we will use $B_\gamma(\mu)$ to represent $B_{s,\gamma}(\mu)$.

The results in this section will be expressed in terms of the weighted $\varphi-$Berezin transform of a measurable function h, which is defined as

$$B_{\varphi,s,\alpha,\beta}(h)(z) = \int_{\mathbb{B}_n} \frac{(1-|z|^2)^s \;\; h(w)}{|1-\langle \varphi(w),z\rangle|^{(n+1+s+\alpha)}} dv_\beta(w),$$

where s is a real number and $-1 < \alpha, \beta < \infty$. Note that if $\varphi(z) = z$, $B_{\varphi,s,\alpha,\beta}$ is the usual weighted Berezin transform $B_{s,\alpha,\beta}$.

To make the section self-contained we state the following four well-known lemmas. The first lemma is a special case of [15], Theorem 54.

Lemma 14. *Let $0 < q < p < \infty$ and $\alpha > -1$ be any real number, and let μ be a positive Borel measure on $\mathbb{B}_n$. Then the following conditions are equivalent:*

1. *There is a constant $c > 0$ such that for all $f \in A^p_\alpha(\mathbb{B}_n)$,*

$$\int_{\mathbb{B}_n} |f(w)|^q \, d\mu(w) \le c\|f\|^q_{A^p_\alpha}.$$

2. *For any bounded sequence $\{f_k\}$ in $A^p_\alpha(\mathbb{B}_n)$ with $f_k(z) \to 0$ for every $z \in \mathbb{B}_n$,*

$$\lim_{k\to\infty} \int_{\mathbb{B}_n} |f_k(w)|^q \, d\mu(w) = 0.$$

3. *For any fixed $r > 0$ the function $\hat{\mu}_{r,\alpha} \in L^{p/(p-q)}(\mathbb{B}_n, dv_\alpha)$.*

4. *For any fixed $s > 0$ the function $B_{s,\alpha}(\mu) \in L^{p/(p-q)}(\mathbb{B}_n, dv_\alpha)$.*

Lemma 15. (see [19], Lemma 2.24) *Suppose $r > 0$, $p > 0$, and $\alpha > -1$. Then there exists a constant $c > 0$ such that*

$$|f(z)|^p \leq \frac{c}{(1-|z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f(w)|^p \, dv_\alpha(w),$$

for all $f \in H(\mathbb{B}_n)$ and all $z \in \mathbb{B}_n$.

Lemma 16. ([20], Lemma 6) *For each $r > 0$ there exists a positive constant c (depending on r) such that*

$$c^{-1} \leq \frac{1-|z|^2}{1-|w|^2} \leq c,$$

and

$$c^{-1} \leq \frac{1-|z|^2}{|1-\langle z,w\rangle|} \leq c,$$

for all $w \in D(z,r)$.

Some special cases of the following lemma can be found in [6], [12], and [19].

Lemma 17. ([15], Proposition 8) *Suppose a and b are complex numbers. If S and T are integral operators defined by*

$$Sf(z) = (1-|z|^2)^a \int_{\mathbb{B}_n} \frac{(1-|w|^2)^b f(w)}{(1-\langle z,w\rangle)^{n+1+a+b}} dv(w)$$

and

$$Tf(z) = (1-|z|^2)^a \int_{\mathbb{B}_n} \frac{(1-|w|^2)^b f(w)}{|1-\langle z,w\rangle|^{n+1+a+b}} dv(w)$$

then for any $1 \leq p < \infty$ and α real, the following are equivalent:

1. *The operator S is bounded on $L^p(\mathbb{B}_n, dv_\alpha)$.*

2. *The operator T is bounded on $L^p(\mathbb{B}_n, dv_\alpha)$.*

3. *The complex numbers a and b satisfy $-p\Re(a) < \alpha + 1 < p(\Re(b) + 1)$.*

The following is the main result of this section, which characterizes the boundedness and compactness of the weighted composition operators.

Theorem 18. *Let ψ be an analytic function on $\mathbb{B}_n$ and φ be an analytic self-map of $\mathbb{B}_n$. Let $0 < q < p < \infty$ and $-1 < \alpha, \beta < \infty$. Then the following are equivalent:*

1. *$W_{\psi,\varphi}$ is bounded from $A^p_\alpha(\mathbb{B}_n)$ into $A^q_\beta(\mathbb{B}_n)$.*
2. *$W_{\psi,\varphi}$is compact from $A^p_\alpha(\mathbb{B}_n)$ into $A^q_\beta(\mathbb{B}_n)$.*
3. *For any fixed $r > 0$,*
$$\frac{\mu_{\varphi,\psi,q,\beta}(D(z,r))}{(1-|z|^2)^{n+1+\alpha}} \in L^{p/(p-q)}(\mathbb{B}_n, dv_\alpha).$$
4. *For any fixed $s > 0$, $B_{\varphi,s,\alpha,\beta}(|\psi|^q) \in L^{p/(p-q)}(\mathbb{B}_n, dv_\alpha)$.*

Proof. Suppose $W_{\psi,\varphi}$ is bounded from A^p_α into A^q_β. Then for any $f \in A^p_\alpha$ there exist a constant $c > 0$ such that $\|W_{\psi,\varphi}(f)\|^q_{A^q_\beta} \le c\, \|f\|^q_{A^p_\alpha}$, that is

$$\int_{\mathbb{B}_n} |\psi(z)|^q \, |f(\varphi(z))|^q \, dv_\beta(z) \le c\, \|f\|^q_{A^p_\alpha}$$

then

$$\int_{\mathbb{B}_n} |f(z)|^q \, d\mu_{\varphi,\psi,q,\beta}(z) \le c\, \|f\|^q_{A^p_\alpha}.$$

Thus, $\mu_{\varphi,\psi,q,\beta}$ is $(A^p_\alpha, q)-$Carleson measure. By Lemma 14, this is equivalent to $\hat{\mu}_{r,\alpha} \in L^{p/(p-q)}(\mathbb{B}_n, dv_\alpha)$, for any fixed $r > 0$. That is,

$$\frac{\mu_{\varphi,\psi,q,\beta}(D(z,r))}{(1-|z|^2)^{n+1+\alpha}} \in L^{p/(p-q)}(\mathbb{B}_n, dv_\alpha).$$

This proves that (1) and (3) are equivalent. Now we will prove (3) implies (2). Let $\{f_k\}$ be a bounded sequence in A^p_α with $f_k \to 0$ uniformly on every compact subset of $\mathbb{B}_n$. For $0 < t < 1$, let $A_t = \{z \in \mathbb{B}_n : 1 - |z|^2 < t\}$, note that by Lemma 16 A_t is non-empty subset of $\mathbb{B}_n$. By Lemma 15 we get,

$$\begin{aligned} \|W_{\psi,\varphi}(f_k)\|^q_{A^q_\beta} &= \int_{\mathbb{B}_n} |\psi(z)|^q \, |f_k(\varphi(z))|^q \, dv_\beta(z) \\ &= \int_{\mathbb{B}_n} |f_k(z)|^q \, d\mu_{\varphi,\psi,q,\beta}(z) \\ &\le \int_{\mathbb{B}_n} \frac{c}{(1-|z|^2)^{n+1+\alpha}} \int_{D(z,r)} |f_k(w)|^q \, dv_\alpha(w) d\mu_{\varphi,\psi,q,\beta}(z) \end{aligned}$$

$$= c \int_{\mathbb{B}_n} |f_k(w)|^q \int_{D(z,r)} \frac{1}{(1-|z|^2)^{n+1+\alpha}} d\mu_{\varphi,\psi,q,\beta}(z) dv_\alpha(w)$$
$$\leq c_1 \int_{\mathbb{B}_n} |f_k(w)|^q \int_{D(w,r)} \frac{1}{(1-|w|^2)^{n+1+\alpha}} d\mu_{\varphi,\psi,q,\beta}(z) dv_\alpha(w)$$
$$= c_1 \int_{\mathbb{B}_n} |f_k(w)|^q \frac{\mu_{\varphi,\psi,q,\beta}(D(w,r))}{(1-|w|^2)^{n+1+\alpha}} dv_\alpha(w). \tag{4}$$

Where the equality in the fourth line follows from Fubini's theorem, and the inequality in the fifth line can be verified by using the fact $1-|z|^2$ is comparable to $1-|w|^2$ whenever $w \in D(z,r)$ (see Lemma 16). Now, by Hölder's inequality we have

$$\int_{A_t} |f_k(w)|^q \frac{\mu_{\varphi,\psi,q,\beta}(D(w,r))}{(1-|w|^2)^{n+1+\alpha}} dv_\alpha(w)$$
$$\leq \left(\int_{A_t} |f_k(w)|^p \, dv_\alpha(w) \right)^{q/p} \left[\int_{A_t} \left(\frac{\mu_{\varphi,\psi,q,\beta}(D(w,r))}{(1-|w|^2)^{n+1+\alpha}} \right)^{p/(p-q)} dv_\alpha(w) \right]^{1-q/p}.$$

If condition (3) holds, then for any given $\epsilon > 0$ there is $t \in (0,1)$ such that

$$\int_{A_t} \left(\frac{\mu_{\varphi,\psi,q,\beta}(D(w,r))}{(1-|w|^2)^{n+1+\alpha}} \right)^{p/(p-q)} dv_\alpha(w) < \epsilon^{p/(p-q)}.$$

Thus for such t,

$$\int_{A_t} |f_k(w)|^q \frac{\mu_{\varphi,\psi,q,\beta}(D(w,r))}{(1-|w|^2)^{n+1+\alpha}} dv_\alpha(w)$$
$$\leq \epsilon \, \|f_k\|^q_{A^p_\alpha}$$
$$\leq c\epsilon. \tag{5}$$

Since $\mathbb{B}_n \setminus A_t$ is a compact subset of $\mathbb{B}_n$ and $f_k \to 0$ uniformly on every compact subset of $\mathbb{B}_n$, we have

$$\lim_{k\to\infty} \int_{\mathbb{B}_n \setminus A_t} |f_k(w)|^p \, dv_\alpha(w) = 0.$$

So by similar argument above, we get

$$\lim_{k\to\infty} \int_{\mathbb{B}_n \setminus A_t} |f_k(w)|^q \frac{\mu_{\varphi,\psi,q,\beta}(D(w,r))}{(1-|w|^2)^{n+1+\alpha}} dv_\alpha(w) = 0. \tag{6}$$

Hence by using Equations 4, 5, and 6 we get,

$$\limsup_{k\to\infty} \|W_{\psi,\varphi}(f_k)\|_{A^q_\beta} = 0,$$

that is, $W_{\psi,\varphi}$ is compact. This proves (3) implies (2), which gives (1) implies (2). Since (2) always implies (1), then (1) and (2) are equivalent. To complete the proof of the theorem, we need to proof conditions (3) and (4) are equivalent. Let $f(z) = \dfrac{\mu_{\varphi,\psi,q,\beta}(D(z,r))}{(1-|z|^2)^{n+1+\alpha}} \in L^{p/(p-q)}(\mathbb{B}_n, dv_\alpha)$. For $a \in \mathbb{B}_n$, define the function g as

$$g(z) = \frac{(1-|a|^2)^{s/q}}{(1-\langle z,a\rangle)^{(n+1+s+\alpha)/q}}.$$

By argument similar to the one used above, we get

$$\begin{aligned}
B_{\varphi,\alpha,s,\beta}(|\psi|^q)(a) &= \int_{\mathbb{B}_n} |\psi(z)|^q \ \frac{(1-|a|^2)^s}{|1-\langle\varphi(z),a\rangle|^{n+1+s+\alpha}} dv_\beta(z) \\
&= \int_{\mathbb{B}_n} |W_{\psi,\varphi}(g(z))|^q dv_\beta(z) \\
&\le c\int_{\mathbb{B}_n} |g(w)|^q \ \frac{\mu_{\varphi,\psi,q,\beta}(D(w,r))}{(1-|w|^2)^{n+1+\alpha}} dv_\alpha(w) \\
&= c\int_{\mathbb{B}_n} \frac{(1-|a|^2)^s}{|1-\langle w,a\rangle|^{n+1+s+\alpha}} f(w) dv_\alpha(w) \\
&= c(1-|a|^2)^s \int_{\mathbb{B}_n} \frac{(1-|w|^2)^\alpha}{|1-\langle w,a\rangle|^{n+1+s+\alpha}} f(w) dv(w) \\
&= cTf(a),
\end{aligned}$$

where in last equality we used Lemma 17 with $a = s$ and $b = \alpha$. Another application of Lemma 17 with $\frac{p}{p-q}$, we get $B_{\varphi,s,\alpha,\beta}(|\psi|^q)(a) \in L^{p/(p-q)}(\mathbb{B}_n, dv_\alpha)$. This proves (3) implies (4).

Finally, by applying Lemma 16 we have,

$$\begin{aligned}
\frac{\mu_{\varphi,\psi,q,\beta}(D(z,r))}{(1-|z|^2)^{n+1+\alpha}} &= \int_{D(z,r)} \frac{1}{(1-|z|^2)^{n+1+\alpha}} d\mu_{\varphi,\psi,q,\beta}(w) \\
&\le c \int_{D(z,r)} \frac{1}{|1-\langle z,w\rangle|^{n+1+\alpha}} d\mu_{\varphi,\psi,q,\beta}(w) \\
&\le c_1 \int_{D(z,r)} \frac{(1-|z|^2)^s}{|1-\langle z,w\rangle|^{n+1+s+\alpha}} d\mu_{\varphi,\psi,q,\beta}(w) \\
&\le c_1 \int_{\mathbb{B}_n} \frac{(1-|z|^2)^s}{|1-\langle z,w\rangle|^{n+1+s+\alpha}} d\mu_{\varphi,\psi,q,\beta}(w) \\
&= c_1 \int_{\mathbb{B}_n} |\psi(w)|^q \ \frac{(1-|z|^2)^s}{|1-\langle z,\varphi(w)\rangle|^{n+1+s+\alpha}} dv_\beta(w)
\end{aligned}$$

$$= c_1 B_{\varphi,s,\alpha,\beta}(|\psi|^q)(z).$$

Where second and third inequalities can be verified by using the fact that $(1-|z|^2)$ is comparable to $|1-\langle z,w\rangle|$ for all $w \in D(z,r)$ (see Lemma 16). Thus, if $B_{\varphi,s,\alpha,\beta}(|\psi|^q) \in L^{p/(p-q)}(\mathbb{B}_n, dv_\alpha)$, then $\dfrac{\mu_{\varphi,\psi,q,\beta}(D(z,r))}{(1-|z|^2)^{n+1+\alpha}} \in L^{p/(p-q)}(\mathbb{B}_n, dv_\alpha)$. □

Remark 19. Again if we take $s = n+1+\alpha$ we get Theorem 2.1 in [9].

4. Schatten $p-$Class of Weighted Composition Operators

A positive compact operator T on $A^2_\alpha(\mathbb{B}_n)$ is in the trace class if

$$tr(T) = \sum_{n=1}^{\infty} \langle Te_n, e_n\rangle < \infty,$$

for some orthonormal basis $\{e_n\}$ of $A^2_\alpha(\mathbb{B}_n)$. More generally, if $0 < p < \infty$ and T is a compact operator on $A^2_\alpha(\mathbb{B}_n)$, then we say that T belongs to the Schatten $p-$class S_p if $(T^*T)^{p/2}$ is in the trace class. Also, the S_p norm of T is given as

$$\|T\|_{S_p} = \left[tr(T^*T)^{p/2}\right]^{1/p}.$$

For more information refer to Schatten [13] and Ringrose [11].

The orthogonal projection P_α from the Hilbert space $L^2(\mathbb{B}_n, dv_\alpha)$ onto the closed subspace $A^2_\alpha(\mathbb{B}_n)$ is given by

$$(P_\alpha f)(z) = \int_{\mathbb{B}_n} K^\alpha_w(z) f(w) dv_\alpha(w), \qquad \text{where } \ f \in L^2(\mathbb{B}_n, dv_\alpha),$$

where

$$K^\alpha_w(z) = K^\alpha(z,w) = \frac{1}{(1-\langle z,w\rangle)^{n+1+\alpha}}, \qquad \text{where } \ z, w \in \mathbb{B}_n$$

is the reproducing kernel of $A^2_\alpha(\mathbb{B}_n)$. When $\alpha = 0$, the reproducing kernel $K(z,w) = K^0(z,w)$ is called Bergman kernel. We denote the normalized reproducing kernel of $A^2_\alpha(\mathbb{B}_n)$ by

$$k^\alpha_z(w) = \frac{(1-|z|^2)^{\frac{n+1+\alpha}{2}}}{(1-\langle w,z\rangle)^{n+1+\alpha}}, \qquad \text{where } \ z, w \in \mathbb{B}_n.$$

More generally, if μ is a finite complex Borel measure on $\mathbb{B}_n$, we can define the Toeplitz operator T_μ with symbol μ by

$$(T_\mu f)(z) = \int_{\mathbb{B}_n} K_w^\alpha(z) f(w) d\mu(w), \qquad \text{where } \ f \in A_\alpha^2(\mathbb{B}_n).$$

To every $\varphi \in L^\infty(\mathbb{B}_n)$ we associate a linear operator T_φ, Toeplitz operator with symbol φ, that is defined for each $f \in A_\alpha^2(\mathbb{B}_n)$ by

$$(T_\varphi f)(z) = \int_{\mathbb{B}_n} K_w^\alpha(z)\varphi(w) f(w) dv_\alpha(w).$$

It is clear that $T_\varphi(f) = P_\alpha(\varphi f)$. Since P_α is an orthogonal projection onto $L^2(\mathbb{B}_n, dv_\alpha)$, the operator T_φ is bounded on $A_\alpha^2(\mathbb{B}_n)$ with $\|T_\varphi\| \leq \|\varphi\|_\infty$. The reader may consult the references for more information about Toeplitz operator on the unit disk [10], the unit ball [12] and [6], and on bounded symmetric domains [20] and [16].

Lemma 20. *Suppose T is an operator in the trace class of $A_\alpha^2(\mathbb{B}_n)$, then* $tr(T) = \int_{\mathbb{B}_n} \langle T k_z^\alpha, k_z^\alpha \rangle d\lambda(z)$.

Proof. By the definition of trace class we get,

$$\begin{aligned} tr(T) &= \int_{\mathbb{B}_n} \langle T K_z^\alpha, K_z^\alpha \rangle dv_\alpha(z) \\ &= \int_{\mathbb{B}_n} \langle T k_z^\alpha, k_z^\alpha \rangle K^\alpha(z,z) dv_\alpha(z) \\ &= \int_{\mathbb{B}_n} \langle T k_z^\alpha, k_z^\alpha \rangle \frac{1}{(1-|z|^2)^{n+1+\alpha}} dv_\alpha(z) \\ &= \int_{\mathbb{B}_n} \langle T k_z^\alpha, k_z^\alpha \rangle \frac{(1-|z|^2)^\alpha}{(1-|z|^2)^{n+1+\alpha}} dv(z) \\ &= \int_{\mathbb{B}_n} \langle T k_z^\alpha, k_z^\alpha \rangle \frac{1}{(1-|z|^2)^{n+1}} dv(z) \\ &= \int_{\mathbb{B}_n} \langle T k_z^\alpha, k_z^\alpha \rangle d\lambda(z). \end{aligned}$$

□

We use the following result from the theory of Toeplitz operators on weighted Bergman spaces of the unit ball.

Lemma 21. ([20], Theorem 16) *Suppose μ is a positive Borel measure on $\mathbb{B}_n$, $0 < p < \infty$ and $0 < r < \infty$. Then the following conditions are equivalent:*

1. *T_μ belongs to the Schatten class $S_p(A^2_\alpha(\mathbb{B}_n))$.*
2. *$\hat{\mu_r} \in L^p(\mathbb{B}_n, d\lambda)$.*

Moreover if $p > \frac{n}{n+1+\alpha}$, then the above conditions are also equivalent to $B_\alpha(\mu) \in L^p(\mathbb{B}_n, d\lambda)$.

We have seen in section 3 that the weighted $\varphi-$Berezin transform can be used to characterize the boundedness and compactness of weighted composition operators. So it would be interesting to obtain a characterization of weighted composition operators to be in Schatten $p-$class in terms of that transform. Cuckovic and Zhao [3] found by using Luecking's technique [7] a criteria for the weighted composition operator to belong to the Schatten $p-$class $S_p(A^2_\alpha(\mathbb{D}))$. By using similar technique we have the following criterion for $W_{\psi,\varphi}$ to belong to the Schatten $p-$class $S_p(A^2_\alpha(\mathbb{B}_n))$. The idea of the proof is to translate the statement from weighted composition operators to Toeplitz operators where the result is known.

Theorem 22. *Let $p > \frac{n}{n+1+\alpha}$. Let ψ be an analytic function on $\mathbb{B}_n$ and let φ be an analytic self-map of $\mathbb{B}_n$. Let $W_{\psi,\varphi}$ be compact on $A^2_\alpha(\mathbb{B}_n)$. Then $W_{\psi,\varphi} \in S_p(A^2_\alpha(\mathbb{B}_n))$ if and only if $B_{\varphi,\alpha}(|\psi|^2) \in L^{p/2}(\mathbb{B}_n, d\lambda)$, where $d\lambda(z) = K^0(z,z)dv(z)$ is the Möbius invariant volume measure of $\mathbb{B}_n$.*

Proof. For any f, g in $A^2_\alpha(\mathbb{B}_n)$,

$$\begin{aligned}\langle (W_{\psi,\varphi})^*(W_{\psi,\varphi})f, g\rangle &= \langle W_{\psi,\varphi}f, W_{\psi,\varphi}g\rangle \\ &= \int_{\mathbb{B}_n} \psi(z)f(\varphi(z))\overline{\psi(z)g(\varphi(z))}dv_\alpha(z) \\ &= \int_{\mathbb{B}_n} |\psi(z)|^2 f(\varphi(z))\overline{g(\varphi(z))}dv_\alpha(z) \\ &= \int_{\mathbb{B}_n} f(z)\overline{g(z)}d\mu_{\varphi,\psi,2,\alpha}(z),\end{aligned}$$

where $\mu_{\varphi,\psi,2,\alpha}$ is a finite positive Borel measure defined in section 2. Now, consider the Toeplitz operator $\left(T_{\mu_{\varphi,\psi,\mathbf{2},\alpha}}\right) f(z) = \int_{\mathbb{B}_n} f(w)K^\alpha_w(z)d\mu_{\varphi,\psi,2,\alpha}(w)$. Then

$$\langle T_{\mu_{\varphi,\psi,\mathbf{2},\alpha}} f, g\rangle = \int_{\mathbb{B}_n} T_{\mu_{\varphi,\psi,\mathbf{2},\alpha}} f(z)\overline{g(z)}dv_\alpha(z)$$

$$
\begin{aligned}
&= \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} f(w) K_w^{\alpha}(z) d\mu_{\varphi,\psi,2,\alpha}(w) \right) \overline{g(z)} dv_{\alpha}(z) \\
&= \int_{\mathbb{B}_n} f(w) \int_{\mathbb{B}_n} \overline{g(z)} K_w^{\alpha}(z) dv_{\alpha}(z) d\mu_{\varphi,\psi,2,\alpha}(w) \\
&= \int_{\mathbb{B}_n} f(w) \overline{\int_{\mathbb{B}_n} g(z) K_z^{\alpha}(w) dv_{\alpha}(z)} d\mu_{\varphi,\psi,2,\alpha}(w)
\end{aligned}
$$

$$
\begin{aligned}
&= \int_{\mathbb{B}_n} f(w) \overline{\langle g, K_w^{\alpha} \rangle} d\mu_{\varphi,\psi,2,\alpha}(w) \\
&= \int_{\mathbb{B}_n} f(w) \overline{g(w)} d\mu_{\varphi,\psi,2,\alpha}(w).
\end{aligned}
$$

Therefore we can see $T_{\mu_{\varphi,\psi,\mathbf{2},\alpha}} = (W_{\psi,\varphi})^*(W_{\psi,\varphi})$. By definition, for $0 < p < \infty$ an operator $T \in S_p$ if and only if $(T^*T)^{p/2} \in S_1$, which is equivalent to saying that $T^*T \in S_{p/2}$. Thus, $W_{\psi,\varphi} \in S_p$ if and only if $T_{\mu_{\varphi,\psi,\mathbf{2},\alpha}} \in S_{p/2}$. By Lemma 21, for $\frac{n}{n+1+\alpha} < p < \infty$, $T_{\mu_{\varphi,\psi,\mathbf{2},\alpha}} \in S_{p/2}$ if and only if $B_{\alpha}(\mu_{\varphi,\psi,2,\alpha}) \in L^{p/2}(\mathbb{B}_n, d\lambda)$.

$$
\begin{aligned}
B_{\alpha}(\mu_{\varphi,\psi,2,\alpha})(z) &= \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z, w\rangle|^{2(n+1+\alpha)}} d\mu_{\varphi,\psi,2,\alpha}(w) \\
&= \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z, \varphi(w)\rangle|^{2(n+1+\alpha)}} |\psi(w)|^2 dv_{\alpha}(w) \\
&= B_{\varphi,\alpha}(|\psi|^2)(z).
\end{aligned}
$$

Thus $W_{\psi,\varphi} \in S_p$ if and only if $B_{\varphi,\alpha}(|\psi|^2) \in L^{p/2}(\mathbb{B}_n, d\lambda)$. □

By using the previous Theorem 22, we find a criteria for the weighted composition operator $W_{\psi,\varphi}$ to be a Hilbert-Schmidt operator in $A_{\alpha}^2(\mathbb{B}_n)$.

Corollary 23. *$W_{\psi,\varphi} \in S_2(A_{\alpha}^2(\mathbb{B}_n))$ if and only if*

$$
\frac{|\psi|^2}{(1-|\varphi|^2)^{n+1+\alpha}} \in L^1(\mathbb{B}_n, dv_{\alpha})
$$

that is,

$$
\int_{\mathbb{B}_n} \frac{|\psi|^2}{(1-|\varphi|^2)^{n+1+\alpha}} dv_{\alpha}(z) < \infty.
$$

Proof. We know from Theorem 22 that, $W_{\psi,\varphi} \in S_2$ if and only if $B_{\varphi,\alpha}(|\psi|^2) \in L^1(\mathbb{B}_n, d\lambda)$ which means that

$$
I = \int_{\mathbb{B}_n} B_{\varphi,\alpha}\left(|\psi|^2\right)(z) d\lambda(z)
$$

$$= \int_{\mathbb{B}_n} \left(\int_{\mathbb{B}_n} \frac{(1-|z|^2)^{n+1+\alpha}}{|1-\langle z, \varphi(w)\rangle|^{2(n+1+\alpha)}} |\psi(w)|^2 dv_\alpha(w) \right) d\lambda(z) < \infty.$$

By Fubini's theorem, we get

$$\begin{aligned} I &= \int_{\mathbb{B}_n} |\psi(w)|^2 \int_{\mathbb{B}_n} \frac{(1-|z|^2)^{\alpha}}{|1-\langle z, \varphi(w)\rangle|^{2(n+1+\alpha)}} dv(z) dv_\alpha(w) \\ &= \int_{\mathbb{B}_n} |\psi(w)|^2 \int_{\mathbb{B}_n} \frac{dv_\alpha(z)}{|1-\langle z, \varphi(w)\rangle|^{2(n+1+\alpha)}} dv_\alpha(w) \\ &= \int_{\mathbb{B}_n} |\psi(w)|^2 \ \|K_{\varphi(w)}(z)\|^2_{A^2_\alpha} dv_\alpha(w) \\ &= \int_{\mathbb{B}_n} \frac{|\psi(w)|^2}{(1-|\varphi(w)|^2)^{n+1+\alpha}} dv_\alpha(w) < \infty. \end{aligned}$$

Therefore we have proved that, $W_{\psi,\varphi} \in S_2$ if and only if

$$\int_{\mathbb{B}_n} \frac{|\psi(w)|^2}{(1-|\varphi(w)|^2)^{n+1+\alpha}} dv_\alpha(w)$$

is finite. □

Luecking and Zhu [8] proved that the following condition

$$\int_{\mathbb{D}} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2} \right)^{p(\alpha+2)/2} d\lambda(z) < \infty$$

is necessary when $p \geq 2$ and sufficient when $p \leq 2$ for C_φ to be in $S_p(A^2_\alpha(\mathbb{D}))$. By using similar argument we can see that the condition

$$\int_{\mathbb{B}_n} |\psi(z)|^p \left(\frac{1-|z|^2}{1-|\varphi(z)|^2} \right)^{p(n+1+\alpha)/2} d\lambda(z) < \infty$$

is necessary when $p \geq 2$ and sufficient when $0 < p \leq 2$ for $W_{\psi,\varphi}$ to be in $S_p(A^2_\alpha(\mathbb{B}_n))$.

Theorem 24. *Let $2 \leq p < \infty$ and let $W_{\psi,\varphi}$ be in $A^2_\alpha(\mathbb{B}_n)$. If $W_{\psi,\varphi} \in S_p(A^2_\alpha(\mathbb{B}_n))$, then*

$$|\psi|^2 \left(\frac{1-|z|^2}{1-|\varphi|^2} \right)^{n+1+\alpha} \in L^{p/2}(\mathbb{B}_n, d\lambda).$$

Proof. We know that $W_{\psi,\varphi} \in S_p$ if and only if $(W_{\psi,\varphi})^* \in S_p$ if and only if $(W_{\psi,\varphi})(W_{\psi,\varphi})^* \in S_1$ if and only if $tr\,((W_{\psi,\varphi})(W_{\psi,\varphi})^*)$ is finite. Take $T = W_{\psi,\varphi}$, then by Lemma 20 we get

$$\begin{aligned} tr(TT^*)^{p/2} &= \int_{\mathbb{B}_n} \langle (TT^*)^{p/2} k_z, k_z \rangle d\lambda(z) \\ &\geq \int_{\mathbb{B}_n} \langle (TT^*) k_z, k_z \rangle_{A^2_\alpha}^{p/2}\, d\lambda(z) \\ &= \int_{\mathbb{B}_n} \langle T^* k_z, T^* k_z \rangle_{A^2_\alpha}^{p/2}\, d\lambda(z) \\ &= \int_{\mathbb{B}_n} \|T^* k_z\|^p_{A^2_\alpha} d\lambda(z) \\ &= \int_{\mathbb{B}_n} \|(W_{\psi,\varphi})^* k_z\|^p_{A^2_\alpha} d\lambda(z). \end{aligned}$$

Where the inequality can be verified by using Hölder's inequality. Since

$$tr(TT^*) < \infty,$$

then $\|(W_{\psi,\varphi})^* k_z\|_{A^2_\alpha} \in L^p(\mathbb{B}_n, d\lambda)$, that is,

$$\|(W_{\psi,\varphi})^* k_z\|^2_{A^2_\alpha} \in L^{p/2}(\mathbb{B}_n, d\lambda),$$

which is equivalent to

$$|\psi|^2 \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{n+1+\alpha} \in L^{p/2}(\mathbb{B}_n, d\lambda).$$

□

By argument similar to the proof of Theorem 24, when $0 < p \leq 2$ the inequality will be reversed. We get

$$tr(TT^*)^{p/2} \leq \int_{\mathbb{B}_n} \|T^* k_z\|^p_{A^2_\alpha},$$

which gives the sufficient condition for $W_{\psi,\varphi}$ to be in $S_p(A^2_\alpha(\mathbb{B}_n))$.

Theorem 25. *Let $0 < p \leq 2$ and let $W_{\psi,\varphi}$ be in $A^2_\alpha(\mathbb{B}_n)$. If*

$$|\psi|^2 \left(\frac{1 - |z|^2}{1 - |\varphi|^2} \right)^{n+1+\alpha} \in L^{p/2}(\mathbb{B}_n, d\lambda),$$

then $W_{\psi,\varphi} \in S_p(A^2_\alpha(\mathbb{B}_n))$.

Remark 26. By using Theorem 24 and Theorem 25 we get another proof for Corollary 23.

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References

[1] Al-Rawashdeh, W., Weighted Composition Operators on Weighted Hardy and Bergman Spaces, Ph.D. dissertation, Central Michigan University, 2010.

[2] C.C. Cowen and B.D. MacCluer, *Composition Operators on Spaces of Analytic Functions*, CRC press, Boca Raton, 1995.

[3] Z. Cuckovic and R. Zhao, Weighted composition operators on the Bergman spaces, *J. London Math. Soc.* 70(2004), 499-511.

[4] Z. Cuckovic and R. Zhao, Weighted composition operators between different weighted Bergman spaces and different Hardy spcaes, *Illinois J. of Math.* 51(2007), 479-498.

[5] N. Dunford and T. Schwartz, *Linear Operators*, Part 1, Interscience, New York, 1967.

[6] F. Forelli and W. Rudin, Projections on spaces of holomorphic functions on balls, *Indiana Univ. Math. J.* 24(1974), 593-602.

[7] D. H. Luecking, Trace ideal criteria for Toeplitz operators, *J. Functional Analysis* 73(1987), 345-368.

[8] D.H. Luecking and K. Zhu, Compostion operators belonging to the Schatten ideals, *American J. of Math.* 114(1992), 1127-1145.

[9] L. Luo and Sei-ichiro Ueki, Weighted Composition operators between weighted Bergman spaces and Hardy spaces on the uint ball of $\mathbb{B}_n$, *J. Math. Anal. Appl.* 326(2007), 88-100.

[10] R. A. Martínez-Avendaño and P. Rosenthal, *An Introduction to Operators on the Hardy-Hilbert Spaces*, Springer, Now York, 2007.

[11] J. R. Ringrose, *Compact Non-self-adjoint Operators*, Van Nostrand-Reinhold, New York, 1971.

[12] W. Rudin, *Function theory in the unit ball of* $\mathbb{C}^n$, Springer-Verlag, New York, 1980.

[13] R. Schatten, *Norm Ideals of Completely Continuous Operators*, Springer-Verlag, Berlin, 1960.

[14] Sei-ichiro Ueki, Weighted Composition operators between weighted Bergman spaces in the unit ball of $\mathbb{B}_n$, *Nihonkai Math. J.* 16(2005), 31-48.

[15] R. Zhao and K. Zhu, Theory of Bergman spaces in the unit ball of $\mathbb{C}^n$, *Mémoire de la SMF 115* , Société Mathématique de France, France, 2008.

[16] K. Zhu, Positive Toeplitz operators on weighted Bergman spaces of bounded symmetric domains, *J. Operator Theory* 20(1988), 329-357.

[17] K. Zhu, *Operator Theory in Function Spaces.* Monograghs and Textbooks in Pure and Applied Mathematics, 139. Marcel Dekker, Inc.,New York, 1990.

[18] K. Zhu, Schatten class composition operators on weighted Bergman spaces of the disk, *J. Operator Theory* 46(2001) 173-181.

[19] K. Zhu, *Spaces of Holomorphic Functions in the Unit Ball*, Springer-Verlag, New York, 2005.

[20] K. Zhu, Schatten class Toeplitz operators on weighted Bergman spaces of the unit ball, *New York J. Math.* 13(2007), 299-316.