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MINIMAL BASE LOCI FOR ZERO-DIMENSIONAL SUBSCHEMES OF $\mathbb{P}^r$

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Abstract: Fix a zero-dimensional scheme $E \subset \mathbb{P}^r$ and an integer $t > 0$ such that $h^1(\mathcal{I}_E(t)) = 0$. Here we study the set of all $P \in \mathbb{P}^r \setminus E_{red}$ such that $h^0(\mathcal{I}_{E\cup\{P\}}(t)) = h^0(\mathcal{I}_E(t))$ and $h^0(\mathcal{I}_{E'\cup\{P\}}(t)) < h^0(\mathcal{I}_{E'}(t))$ for all $E' \subsetneq E$.

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1. Introduction

Fix a zero-dimensional scheme $E \subset \mathbb{P}^r$ and an integer $t > 0$ such that $h^1(\mathcal{I}_E(t)) = 0$. In the first part of this paper we study the set $\mathcal{A}(r, E, t)$ of all $P \in \mathbb{P}^r \setminus E_{red}$ such that $h^0(\mathcal{I}_{E\cup\{P\}}(t)) = h^0(\mathcal{I}_E(t))$ and $h^0(\mathcal{I}_{E'\cup\{P\}}(t)) < h^0(\mathcal{I}_{E'}(t))$ for all $E' \subsetneq E$. Then for every integer $b > 0$ we consider the set $\mathcal{B}(r, E, t, b)$ of all finite sets $S \subset \mathbb{P}^r \setminus E_{red}$ such that $\sharp(S) = b$, $h^1(\mathcal{I}_{E\cup S}(t)) > 0$ and $h^1(\mathcal{I}_F(t)) = 0$ for any scheme $F \subsetneq E \cup S$. Obviously $\mathcal{B}(r, E, t, 1) = \mathcal{A}(r, E, t)$ for any E. We work over an arbitrary algebraically closed field. The motivation comes from the evaluation codes associated to the complete linear system $|\mathcal{I}_E(t))|$. Here there is one of our results.

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Theorem 1. *Fix positive integers t, z, b, r such that $r \geq 2$, $z \leq 2t$ and $z + b < 3t$. Let $E \subset \mathbb{P}^r$ be a zero-dimensional scheme. If $z \geq t + 2$ assume $h^1(\mathcal{I}_E(t)) = 0$. We have $\mathcal{B}(r, E, t, b) \neq \emptyset$ and $S \in \mathcal{B}(r, E, t, b)$ if and only if (E, b, s) is in one of the following cases:*

(i) $z + b = t + 2$ *and there is a line* $L \subset \mathbb{P}^r$ *such that* $L \supset E \cup S$*; in this case we may take as* L *an arbitrary line of* $\mathbb{P}^r$*, as* E *any zero-dimensional subscheme of* L *with degree* z *and as* S *any subset of* $L \setminus E_{red}$ *with cardinality* $t + 2 - z$*;*

(ii) $z + b = 2t + 2$ *and there is a smooth conic* $D \subset \mathbb{P}^r$ *such that* $D \supset E \cup S$*; in this case we may take as* D *an arbitrary smooth conic of* $\mathbb{P}^r$*, as* E *any zero-dimensional subscheme of* D *with degree* z *and as* S *any subset of* $D \setminus E_{red}$ *with cardinality* $2t + 2 - z$*;*

(iii) $z+b = 2t+2$ *and the are lines* $L_1, L_2 \subset \mathbb{P}^r$ *such that* $L_1 \neq L_2$, $L_1 \cap L_2 \neq \emptyset$, $L_1 \cup L_2 \supset E \cup S$, $L_1 \cap L_2 \notin (E \cup S)_{red}$ *and* $\deg((E \cup S) \cap L_1)) = \deg((E \cup S) \cap L_2)) = t+1$*; in this case we may take as* $L_1 \cup L_2$ *an arbitrary reduced and reducible conic of* $\mathbb{P}^r$*, as* E *any zero-dimensional subscheme of* $L_1 \cup L_2$ *such that* $\deg(E) = z$, $L_1 \cap L_2 \notin E_{red}$, $\deg(L_1 \cap E) \leq t + 1$, $\deg(L_2 \cap E) \leq t+1$ *and as* S *any subset of* $L_1 \cup L_2 \setminus E_{red}$ *such that* $L_1 \cap L_2 \notin S$, $\sharp(S \cap L_1) = t + 1 - \deg(E \cap L_1)$ *and* $\sharp(S \cap L_2) = t + 1 - \deg(E \cap L_2)$*.*

2. The Other Results

Remark 1. Assume $\mathcal{B}(r, E, t, b) \neq \emptyset$ and fix $S \in \mathcal{B}(r, E, t, b)$. Since $h^1(\mathcal{I}_E(t)) = 0$, we have $b > 0$. Since $h^1(\mathcal{I}_{E \cup S'}(t)) = 0$ for any $S' \subset S$ such that $\sharp(S') = b-1$, we have $h^1(\mathcal{I}_{E \cup S}(t)) = 1$ and $h^0(\mathcal{I}_{E \cup S}(t)) = \binom{t+r}{r} - \deg(E) - b + 1$.

Lemma 1. *$\mathcal{B}(r, E, t, b) = \emptyset$ for every $b \geq \binom{t+r}{r} - \deg(E) + 2$.*

Proof. Apply Remark 1. □

Proposition 1. *Fix an integer $t > 0$ and a zero-dimensional scheme $E \subset \mathbb{P}^r$ such that $h^1(\mathcal{I}_E(t)) = 0$ and E has only finitely many subschemes of degree $\deg(E) - 1$. Set $b := \binom{r+t}{r} - \deg(E) + 1$. Then $\mathcal{B}(r, E, t, b) \neq \emptyset$ and $S \in \mathcal{B}(r, E, t, b)$ for a general $S \subset \mathbb{P}^r$.*

Proof. We have $h^1(\mathcal{I}_{E \cup S}(t)) = 1$. Since $E \cup S$ has only finitely many subschemes, while the base field is infinite, for general S we have $h^i(\mathcal{I}_F(t)) = 0$, $i = 0, 1$, for all $F \subset S$ such that $\deg(F) = \deg(E) + b - 1$. □

Corollary 1. *Fix an integer $t > 0$ and a zero-dimensional scheme and curvilinear $E \subset \mathbb{P}^r$ such that $h^1(\mathcal{I}_E(t)) = 0$ and E has only finitely many subschemes of degree $\deg(E)-1$. Set $b := \binom{r+t}{r} - \deg(E^) - 1$. Then $\mathcal{B}(r, E, t, b) \neq \emptyset$ and $S \in \mathcal{B}(r, E, t, b)$ for a general $S \subset \mathbb{P}^r$.*

Proof. A zero-dimensional and curvilinear subscheme has only finitely many subschemes. Apply Proposition 1. □

Proposition 2. *Fix integer $r \geq 2$ and $t \geq 1$ and a zero-dimensional scheme $E \subset \mathbb{P}^r$ such that $h^1(\mathcal{I}_E(t)) = 0$. If E is not Gorenstein, then $\mathcal{B}(r, E, t, b) = \emptyset$ for all $b > 0$.*

Proof. Fix an integer $b \geq 1$ and assume $\mathcal{B}(r, E, t, b) \neq \emptyset$. Fix $S \in \mathcal{B}(r, E, t, b)$ and $P \in S$. Set $E_1 := E \cup (S \setminus \{P\})$. Notice that E is Gorenstein if and only if E is Gorenstein. Let $\nu_{r,t} : \mathbb{P}^r \to \mathbb{P}^N$, $N := \binom{r+t}{t} - 1$ denote the order t Veronese embedding. Since $S \in \mathcal{B}(r, E, t, b)$, $P \in \mathcal{A}(r, E_1, t)$. Since $\mathcal{B}(r, E_1, t, 1) = \mathcal{A}(r, E_1, t)$, we have $P \in \mathcal{A}(r, E_1, t)$, i.e. there is no $E' \subsetneq E_1$ such that $\nu_{r,t}(P)$ is in the linear span of $\nu_{r,t}(E')$. Hence E_1 is Gorenstein (see [2], Lemma 2.4). □

Proposition 3. *Fix integers $r \geq 2$, $t \geq 1$ and z, b such that $0 \leq z \leq \binom{r+t}{t} - 1$, $b > 0$, and $\binom{r+t}{r} - r + 2 \leq z + b \leq \binom{r+t}{r} + 1$. Let $A \subset \mathbb{P}^r$ be a general set such that $\sharp(A) = z$. Then $\mathcal{B}(r, A, t, b) \neq \emptyset$ and the set of all $S \in \mathcal{B}(r, A, t, b)$ contains an algebraic family of dimension $r(b-1)$.*

Proof. Fix a general $S_1 \subset \mathbb{P}^r$ such that $\sharp(S_1) = b - 1$. It is sufficient to prove $\mathcal{A}(r, A \cup S_1, t) \neq \emptyset$. The set $S \cup A_1$ is a general subset of $\mathbb{P}^r$ with cardinality $z + b - 1$. Since $\binom{r+t}{r} - r \leq z + b - 1 \leq \binom{r+t}{r}$ and the set $S \cup A_1$ is general, we have $h^1(\mathcal{I}_{A \cup S_1}(t)) = 0$ and the base locus $\mathbb{B}$ of $|\mathcal{I}_{A \cup S_1}(t)|$ is an integral variety of dimension $\binom{r+t}{r} - z - b + 1 \geq 1$. In all cases we may find $P \in \mathbb{B}$ not contained in the base locus of any $|\mathcal{I}_F(t)|$ for any $F \subsetneq A \cup S_1$. Hence $P \in \mathcal{A}(r, A \cup S_1, t)$. □

Proposition 4. *Assume $r = 2$ and fix an integer $t > 0$ and a zero-dimensional scheme $E \subset \mathbb{P}^2$ such that $h^1(\mathcal{I}_E(t)) = 0$ and $\mathcal{A}(2, E, t)) \neq \emptyset$. Set $z := \deg(E)$ and assume $\mathcal{A}(2, E, t) \neq \emptyset$. We have $z \geq t + 1$. Assume the existence of an integer $s > 0$ such that $s^2 \leq z + 1$ and $t \geq s - 3 + (z+1)/s$. Then either $z = s(t + 3 - s)$, $\mathcal{A}(2, E, t) = \{P\}$, $P \notin E_{red}$ and $E \cup \{P\}$ is the complete intersection of a curve of degree s and a curve of degree t or there is an integer y such that $1 \leq y \leq s - 1$, $ty(t + (5-t)/2) \geq z + 1 \geq y(t - y + 3)$*

and E is contained in a curve C' of degree y; for any $P \in \mathcal{A}(2, E, t)$ we may find C' with the additional condition $C' \supset E \cup \{P\}$.

Proof. Fix any $P \in \mathcal{A}(2, E, t)$. Since $h^1(\mathcal{I}_{E\cup\{P\}}(t)) > 0$, we have $z \geq t+1$ (see [1], Lemma 34). Since $h^1(\mathcal{I}_E(t)) = 0$, $P \notin E_{red}$ and $h^0(\mathcal{I}_{E\cup\{P\}}(t)) = h^0(\mathcal{I}_E(t))$, we have $h^1(\mathcal{I}_{E\cup\{P\}}(t)) = 1$. Hence t is the maximal integer such that $h^1(\mathcal{I}_{E\cup\{P\}}(x)) > 0$. Since $\deg(E\cup\{P\}) = z+1 \geq s^2$ and $t \geq s-3+(z+1)/s$, we may apply to $E\cup\{P\}$ [3], Corollaire 2, and get that either $t = s-3+(z+1)/s$ and $E\cup\{P\}$ is the complete intersection of a plane curve C_1 of degree $(z+1)/s$ and a plane curve C_2 of degree s or there is an integer $y \in \{1, \dots, s-1\}$ and a plane curve C' of degree t such that $y(t+(5-y)/2) \geq \deg(C \cap (E \cup \{P\}) \geq y(t-y+3)$. In the latter case the definition of numerical character and the proof of [3], Corollary 2, also gives $h^1(\mathcal{I}_{C'\cap(E\cup\{P\}}(t)) > 0$. Since $P \in \mathcal{A}(2, E, t)$, we get $E \cup \{P\} \subset C'$. □

Remark 2. Proposition 4 is very weak, because it does not give a necessary condition. It is easy to fill in the cases to show that in these interval we have some E, but even in the complete intersection case $(z+1) = s(t+3-s)$ it strongly depends from E: take a degree s smooth plane curve and a complete intersection $F \subset C$ of C with a degree $t+3-s$ curve. If F has a reduced connected component, P, then $\mathcal{A}(2, F \setminus \{P\}, t) = \{P\}$. If F has no reduced connected component, then $\mathcal{A}(2, E, t) = \emptyset$ for every $E \subset F$ with $\deg(E) = \deg(F) - 1$. In most cases Bezout theorem gives that in the case $1 \leq y \leq s-1$ we may find a degree y curve C' containing $E \cup \mathcal{A}(2, E, t)$.

Lemma 2. *Fix integers $t > 0$ and $r \geq 2$. Let $E \cup S \subset \mathbb{P}^r$ be a zero-dimensional scheme such that $h^1(\mathcal{I}_{E\cup S}(t)) > 0$, $h^1(\mathcal{I}_W(t)) = 0$ for all $W \subsetneq E \cup S$, S is reduced, $E \cap S = \emptyset$ and* $\deg(E) \leq 2t$ *Set $Z := E \cup S$. Then Z is as in one of the following cases:*

- *(i)* $\deg(Z) = t+2$ *and there is a line $L \subset \mathbb{P}^r$ such that $L \supset E \cup S$; in this case we may take as L an arbitrary line of $\mathbb{P}^r$, as E any zero-dimensional subscheme of L with degree z and as S any subset of $L \setminus E_{red}$ with cardinality $t+2-z$;*
- *(ii)* $\deg(Z) = 2t+2$ *and there is a smooth conic $D \subset \mathbb{P}^r$ such that $D \supset E \cup S$;*
- *(iii)* $\deg(Z) = 2t+2$ *and the are lines $L_1, L_2 \subset \mathbb{P}^r$ such that $L_1 \neq L_2$, $L_1 \neq L_2 \neq \emptyset$, $L_1 \cup L_2 \supset E \cup S$, $L_1 \cap L_2 \notin (E \cup S)_{red}$ and* $\deg((E \cup S) \cap L_1)) = \deg((E \cup S) \cap L_2)) = t+1$.

Proof. If $r = 2$ (or if $r \geq 3$, but Z is contained in a plane), then we may apply Proposition 4. Now assume $r \geq 3$ and that Z is not contained in a plane. By induction on r we may also assume that the lemma in true in $\mathbb{P}^r$ and that Z spans $\mathbb{P}^r$. Set $Z_0 := Z$ and $w_0 := \deg(Z)$. For any hyperplane $H \subset \mathbb{P}^r$ and any zero-dimensional scheme $F \subset \mathbb{P}^r$ let $\mathrm{Res}_H(F)$ denote the residual scheme of F with respect to H, i.e. the closed subscheme of $\mathbb{P}^r$ with $\mathcal{I}_F : \mathcal{I}_H$ as its ideal sheaf. We have $\mathrm{Res}_H(F) \subseteq F$ and $\deg(F) = \deg(\mathrm{Res}_H(F)) + \deg(F \cap H)$. Notice that if F is zero-dimensional $G \subseteq F$ and $h^1(\mathcal{I}_G(x)) > 0$, then $h^1(\mathcal{I}_F(x)) > 0$ and that if F is contained in a linear subspace V, then $h^1(\mathcal{I}_F(x)) = h^1(V, \mathcal{I}_{F,V}(x))$. Let $H_1 \subset \mathbb{P}^r$ be a hyperplane such that $b_1 := \deg(Z \cap H_1)$ is maximal. Set $Z_1 := \mathrm{Res}_{H_1}(Z)$ and $z_1 = z_0 - b_1$. Notice that $z_1 = \deg(Z_1)$. For every integer $i \geq 2$ define recursively the integers z_i, b_i, the hyperplane $H_i \subset \mathbb{P}^r$ and the scheme $Z_i \subseteq Z_{i-1}$ in the following way. Let $H_i \subset \mathbb{P}^r$ be any hyperplane such that $b_i := \deg(Z_{i-1} \cap H_i)$ is maximal. Set $Z_i := \mathrm{Res}_{H_i}(Z_{i-1})$ and $z_i := z_{i-1} - b_i = \deg(Z_i)$. The sequence $\{b_i\}_{i \geq 1}$ is non-decreasing. Since any zero-dimensional subscheme $G \subset \mathbb{P}^r$ with $\deg(G) \leq r$ is contained in a hyperplane, we have $Z_i = \emptyset$ if $b_i \leq r - 1$. Since $r \geq 3$ and $\deg(Z) < 3t$, we get $Z_i = \emptyset$ for all $i \geq t$. For every integer $i \geq 1$ we have an exact sequence

$$0 \to \mathcal{I}_{Z_i}(t-i) \to \mathcal{I}_{Z_{i-1}}(t-i+1) \to \mathcal{I}_{Z_{i-1} \cap H_i, H_i}(t-i+1) \to 0 \qquad (1)$$

Since $h^1(\mathcal{I}_{Z_0}(t)) > 0$, there is as integer $i \geq 1$ such that $h^1(H_i, \mathcal{I}_{Z_{i-1} \cap H_i, H_i}(t - i+1)) > 0$. We call e the minimal such an integer. If $e = 1$, then we get $Z \subset H_1$ and hence we conclude by induction on r.

(a) Here we assume $e \geq 3$. Since $Z_i = \emptyset$ for all $i \geq t$, we have $e \leq t$. Since $h^1(H_e, \mathcal{I}_{Z_{e-1} \cap H_e, H_e}(t-e+1)) > 0$ and $t-e+1 \geq 0$, we have $b_e \geq t-e+3$. Since $b_i \geq b_e$ for $i < e$, we get $z_0 \geq e(t-e+3)$. Since $\deg(Z) < 3t$ and the function $x \mapsto x(t-x+3)$ is increasing if $x \leq (t+3)/2$ and decreasing if $x > (t+3)/2$, while $3 \leq e \leq t$, we get a contradiction.

(b) Here we assume $e = 2$. Since $b_1 \geq b_2$ and $b_1 + b_2 < 3t$, we have $b_2 \leq 2(t-1)+1$. Since $h^1(H_1, \mathcal{I}_{Z_1 \cap H_1, H_1}(t-1)) > 0$ we have $b_2 \geq t+1$ and there is a line $R \subset H_1$ such that $\deg(R \cap H_1) \geq t+1$ (see [1], Lemma 34). If $\deg(R \cap Z) \geq t+2$, then we easily see that we are in case (i). Hence $\deg(Z \cap R) = t+1$. Let $H \subset \mathbb{P}^r$ be a hyperplane containing R and such that $c := \deg(H \cap Z)$ is maximal. If $h^1(H, \mathcal{I}_{Z \cap H}(t)) > 0$, then we get $Z \subset H$. Hence we may assume $h^1(H, \mathcal{I}_{Z \cap H, H}(t)) = 0$. From the residual exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_H(Z)}(t-1) \to \mathcal{I}_Z(t) \to \mathcal{I}_{Z \cap H, H}(t)) \to 00 \qquad (2)$$

we get $h^1(\mathcal{I}_{\mathrm{Res}_H(Z)}(t-1)) > 0$. Since $\deg(\mathrm{Res}_H(Z) \leq 3t-1-t-1 \leq 2(t-1)$, there is a line $T \subset \mathbb{P}^r$ such that $\deg(T \cap \mathrm{Res}_H(Z) \geq t+1$.

(b1) In this step we assume $T \neq R$. Hence $\deg((T \cup R) \cap Z)) \geq 2t+2$. First assume $T \cap R \neq \emptyset$. Since $h^0(T \cup R, \mathcal{O}_{T\cup R}(t)) = 2t+1$, we get $Z \subset T \cup R$. Since we assumed $\deg(Z \cap L) \leq t+1$, for every line L, we are in case (iii). Now assume $T \cap R = \emptyset$. Since $\deg(L \cap Z) \leq t+1$ for every line R, we get $\deg(Z \cap (L \cup R)) = 2t+2$ and $h^1(\mathcal{I}_{Z\cup(R\cup T)}(t)) = 0$. First assume $r = 3$. Let $Q \subset \mathbb{P}^3$ be a general quadric surface containing $R \cup T$. Since $\mathrm{Res}_Q(Z)$ has degree $\leq 3t-1-2t-2 \leq t-1$, we have $h^1(\mathcal{I}_{\mathrm{Res}_Q(Z)}(t-2)) = 0$. Hence an exact sequence similar to (2) with $t-2$ instead of $t-1$ and Q instead of H gives $h^1(Q, \mathcal{I}_{Q\cap Z,Z}(t)) > 0$. Hence we get $Z \subset Q$. Write $Q = \mathbb{P}^1 \times \mathbb{P}^1$ with R, T of type $(1,0)$. Since $\deg(\mathrm{Res}_{R\cup T}(Z)) \leq 3t-1-2t-2 \leq t-1$, we have $h^1(Q, \mathcal{I}_{\mathrm{Res}_{R\cup T}(Z)}(t-2,t)) = 0$. Since $\deg(Z \cap R) = \deg(Z \cap T) = t+1$, we have $h^1(R \cup T, \mathcal{I}_{Z\cap(R\cup T)}(t)) = 0$ Hence another residual exact sequence on Q with $\mathcal{O}_Q(t-2,t)$ instead of $\mathcal{O}_{\mathbb{P}^3}(t-1)$ and $R \cup T$ instead of H gives $h^1(Q, \mathcal{I}_{Z\cap Q,Q}(t)) = 0$, a contradiction. Now assume $r \geq 4$. Hence there is a hyperplane M containing $L \cup R$. Since $\deg(\mathrm{Res}_M(Z)) \leq 3t-1-2t-2 \leq t$, we have $h^1(\mathcal{I}_{\mathrm{Res}_M(Z)}(t-1)) = 0$. By the exact sequence (2) with M instead of H we get $h^1(M, \mathcal{I}_{M\cap Z,M}(t)) > 0$. We conclude by induction on r.

(b2) In this step we assume $T = R$. Let $2H_1$ denote the degree 2 divisor with H_1 as its support. Since $T = R$, we have $\deg(Z \cap 2H_1) \geq 2t+2$. Hence $\deg(\mathrm{Res}_{2H_1}(Z)) \leq 3t-1-2t-2 \leq t-1$. Hence $h^1(\mathcal{I}_{\mathrm{Res}_{2H_1}(Z)}(t-2)) = 0$. Hence a residual exact sequence similar to (2) with $2H_1$ instead of H and $t-2$ instead of $t-1$ gives $h^1(2H_1, \mathcal{I}_{Z\cap 2H_1, 2H_1}(t)) > 0$. Hence $Z \subset 2H_1$. Hence $Z_{red} \subset H_1$, i.e. $E_{red} \cup S \subset H_1$. We get $H_1 \supset \mathrm{Res}_{H_1}(Z) = \mathrm{Res}_{H_1}(E)$. Since $\deg(\mathrm{Res}_{H_1}(E)) = \deg(\mathrm{Res}_{H_1}(E) \cap H_1) \leq \deg(E \cap H_1)$, we get $\deg(\mathrm{Res}_{H_1}(Z)) \leq \deg(E)/2 \leq t$, contradiction. □

Proof of Theorem 1. Take (E, S) as in one of the cases (i), (ii) and (iii). It is easy to check that $h^1(\mathcal{I}_{E\cup S}(t)) = 1$ and $h^1(\mathcal{I}_W(t)) = 0$ for every scheme $W \subsetneq E \cup S$. Hence if (E, S, b) is as in one of the cases (i), (ii) and (iii), then $S \in \mathcal{B}(r, E, t, b)$. Now take z, b, E, S such that $S \in \mathcal{B}(r, E, t, b)$. Hence $h^1(\mathcal{I}_{E\cup S}(t)) = 0$ and $h^1(\mathcal{I}_W(t)) = 0$ for all $W \subsetneq E \cup S$. Since $z + b < 3t$, Lemma 2 gives we are in one of the cases (i), (ii) or (iii). □

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