WEAK STABILITY OF A CRITICAL TWISTED NEMATIC
STATE UNDER THE MAGNETIC FIELD
IN LIQUID CRYSTALS

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Abstract: We consider the weak stability of a critical twisted nematic state under the magnetic field in the theory of liquid crystals. We treat the Oseen-Frank energy functional with the strong anchoring condition, that is to say, the Dirichlet boundary condition for the director field.

1. Introduction

It is well known that the stability of a liquid crystal configuration is influenced by the external magnetic field. In this paper, we shall consider the stability of a critical twisted nematic configuration. For such a theory in nematic liquid crystal, we follow the Oseen-Frank energy given by Ericksen and Leslie [7]. The bulk energy is given by

$$W[n] = \int_{\Omega} W(\nabla n, n) dx$$

where $n = n(x)$ is the unit vector field called the director field, $\Omega \subset \mathbb{R}^3$ is the domain occupied by the material. And $W(\nabla n, n)$ is the Oseen-Frank energy.
density:

\[ W(\nabla \mathbf{n}, \mathbf{n}) = K_1 (\text{div} \mathbf{n})^2 + K_2 (\mathbf{n} \cdot \mathbf{curl} \mathbf{n})^2 + K_3 |\mathbf{n} \times \mathbf{curl} \mathbf{n}|^2 + (K_2 + K_4)(\text{tr}(\nabla \mathbf{n})^2 - (\text{div} \mathbf{n})^2) \]

where \( K_i \ (i = 1, 2, 3) \) are positive constants which are called the elastic coefficients and \( K_4 \) is a real constant.

We consider the case where equilibrium molecular configurations are local minima of \( W[\mathbf{n}] \) and satisfy the strong anchoring condition, mathematically the Dirichlet boundary condition:

\[ \mathbf{n}(x) = \mathbf{n}_0(x), \quad x \in \partial \Omega \]

where \( \partial \Omega \) is the boundary of \( \Omega \). Under the strong anchoring condition,

\[ \int_{\Omega} \{\text{tr}(\nabla \mathbf{n})^2 - (\text{div} \mathbf{n})^2\}dx \]

represents a surface energy and so we can neglect the term for finding energy minimizing configurations (cf. Hardt et al. [8]).

In the situation where the liquid crystal material is subject to a static magnetic field \( \mathbf{H} \), we must add the magnetic energy contribution \(-\chi_a (\mathbf{H} \cdot \mathbf{n})^2\) to the energy density (cf. de Gennes and Prost [6], Lin and Pan [9], Cohen and Luskin [4], Pan [10] and Aramaki [2, 1, 3]). For brevity, we assume that \( \mathbf{H} = \sigma \mathbf{h} \) where \( \mathbf{h} \) is a unit vector field and \( \sigma \) is a constant for which \(|\sigma|\) is the intensity of \( \mathbf{H} \) and \( \Omega \) is a bounded smooth domain in \( \mathbb{R}^3 \) with boundary \( \partial \Omega \). Thus we shall consider the energy functional:

\[ \mathcal{F}_{\sigma \mathbf{h}}[\mathbf{n}] = \int_{\Omega} \{K_1 (\text{div} \mathbf{n})^2 + K_2 (\mathbf{n} \cdot \mathbf{curl} \mathbf{n})^2 + K_3 |\mathbf{n} \times \mathbf{curl} \mathbf{n}|^2\}dx - \chi_a \sigma^2 \int_{\Omega} (\mathbf{h} \cdot \mathbf{n})^2 dx \quad (1.1) \]

Let \( H^1(\Omega, \mathbb{R}^3) \) and \( H^1_0(\Omega, \mathbb{R}^3) \) be the usual Sobolev space of vector valued functions and define

\[ H^1(\Omega, \mathbb{S}^2) = \{ \mathbf{n} \in H^1(\Omega, \mathbb{R}^3); |\mathbf{n}(x)| = 1 \text{ a.e. } x \in \Omega \}. \]

Moreover, for given smooth vector field \( \mathbf{n}_0 : \partial \Omega \to \mathbb{S}^2 \), we define the space of admissible director fields

\[ H^1(\Omega, \mathbb{S}^2, \mathbf{n}_0) = \{ \mathbf{n} \in H^1(\Omega, \mathbb{S}^2); \mathbf{n} = \mathbf{n}_0 \text{ on } \partial \Omega \}. \]
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We note that if $n_0$ is Lipschitzian, then $H^1(\Omega, S^2, n_0) \neq \emptyset$ (cf. [8, Lemma 1.1].

In the present paper, we give a criterion of the weak stability of a critical twisted nematic state $n^q = (\cos(qx_3), \sin(qx_3), 0)$ and $n_0 = n^q|_{\partial \Omega}$. [4] considered the weak stability of $n^q$ without magnetic field and $\Omega = (0, 1) \times (0, 1) \times (0, 1)$. We shall extend the result to the general case.

2. Critical Points and Weak Stability

In this section, we give the notations of critical points of $F_{\sigma h}$ and weak stabilities of critical points.

For $n \in H^1(\Omega, S^2)$, $v \in H^1_0(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$ and small $t \in \mathbb{R}$, define

$$n_t = \frac{n + tv}{|n + tv|}.$$  \hspace{1cm} (2.1)

Then we can write

$$n_t = n + tn_1 + t^2n_2 + O(t^3)$$

where

$$n_1 = v - (v \cdot n)n,$$

$$n_2 = -(v \cdot n)v + \frac{1}{2}(3(v \cdot n)^2 - |v|^2)n.$$  \hspace{1cm} (2.2)

We shall expand $F_{\sigma h}[n_t]$ with respect to powers of $t$ (cf. [2] and [9]).

Proposition 2.1. We can write

$$F_{\sigma h}[n_t] = F_{\sigma h}[n] + 2t\left\{ A(n; v) - \chi_a \sigma^2 \int_\Omega (h \cdot n)(h \cdot n_1) dx \right\}$$

$$+ t^2 \left\{ B(n; v) - \chi_a \sigma^2 \int_\Omega \left\{ (h \cdot n_1)^2 + 2(h \cdot n)(h \cdot n_2) \right\} dx \right\} + O(t^3)$$

where

$$A(n; v) = \int_\Omega \{ K_1(\text{div } n)(\text{div } n_1) + K_2(n \cdot \text{curl } n)(n_1 \cdot \text{curl } n + n \cdot \text{curl } n_1)$$

$$+ K_3(n \times \text{curl } n) \cdot (n_1 \times \text{curl } n + n \times \text{curl } n_1) \} dx,$$  \hspace{1cm} (2.3)

$$B(n; v) = \int_\Omega \left\{ K_1((\text{div } n_1)^2 + 2(\text{div } n)(\text{div } n_1)) + K_2\{ (n_1 \cdot \text{curl } n + n \cdot \text{curl } n_1)^2$$

$$+ 2(n \cdot \text{curl } n)(n_2 \cdot \text{curl } n + n_1 \cdot \text{curl } n_1 + n \cdot \text{curl } n_2) \} \right\}$$
\begin{align*}
+ K_3 \{|n_1 \times \text{curl } n + n \times \text{curl } n_1|^2 \\
+ 2(n \times \text{curl } n) \cdot (n_2 \times \text{curl } n + n_1 \times \text{curl } n_1 + n \times \text{curl } n_2)\} \right\} dx.
\end{align*}
(2.4)

**Definition 2.2.** (i) We say that \(n \in H^1(\Omega, S^2, n_0)\) is a critical point of \(F_{\sigma h}\), if and only if for any \(v \in H_0^1(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)\),
\[
\frac{d}{dt} \bigg|_{t=0} F_{\sigma h}[n_t] = 0.
\]
That is to say,
\[
\mathcal{A}(n; v) - \chi_a \sigma^2 \int_{\Omega} (h \cdot n) (h \cdot n_1) dx = 0.
\] (2.5)

(ii) We say that a critical point \(n\) of \(F_{\sigma h}\) is weakly stable (local minimizer), if and only if for any \(v \in H_0^1(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)\), there exists \(T = T(v) > 0\) such that for all \(0 < t < T\),
\[
F_{\sigma h}[n] \leq F_{\sigma h}[n_t].
\]

From Proposition 2.1 and Definition 2.2, we have (cf. [4, Corollary 2.4])

**Corollary 2.3.** Assume that \(n \in H^1(\Omega, S^2, n_0)\) be a critical point of \(F_{\sigma h}\). If there exists \(v \in H^1_0(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)\) such that
\[
\mathcal{B}(n; v) - \chi_a \sigma^2 \int_{\Omega} \{(h \cdot n_1)^2 + 2(h \cdot n) (h \cdot n_2)\} dx < 0,
\]
then \(n\) is not weak stable. Conversely, if
\[
\mathcal{B}(n; v) - \chi_a \sigma^2 \int_{\Omega} \{(h \cdot n_1)^2 + 2(h \cdot n) (h \cdot n_2)\} dx > 0
\]
for all \(v \in H^1_0(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)\) such that \(v\) is not parallel to \(n\) on a set of positive measure, then \(n\) is weakly stable.

### 3. The Weak Stability of a Twisted Nematic State

A twisted nematic configuration is defined by
\[
n^q = (\cos(qx_3), \sin(qx_3), 0)
\] (3.1)
and we define
\[
n_0 = n^q \bigg|_{\partial \Omega}.
\] (3.2)
In this section, we assume that the exterior magnetic field $h$ is perpendicular to $n^q$, that is to say,
\[ h \cdot n^q = 0 \quad \text{in } \Omega. \]  
(3.3)

By a simple computations, we have
\[
\begin{align*}
\text{curl } n^q &= -qn^q, \quad \text{div } n^q = 0 \\
n_1 \cdot n^q &= 0, \quad n_2 \cdot n^q = -\frac{1}{2} |n_1|^2 \quad \text{in } \Omega.
\end{align*}
\]  
(3.4)

**Lemma 3.1.** $n^q$ is a critical point of $F_{\sigma h}$.

**Proof.** Using (3.4), we have
\[
A(n^q; v) = K_2 \int_{\Omega} n^q \cdot \text{curl } n_1 dx.
\]
Since $n_1 = 0$ on $\partial \Omega$, it follows that
\[
A(n^q; v) = K_2 \int_{\Omega} \text{curl } n^q \cdot n_1 dx = -K_2q \int_{\Omega} n^q \cdot n_1 dx = 0.
\]
Since $h \cdot n^q = 0$, we have (2.5) with $n = n^q$. Thus $n^q$ is a critical point of $F_{\sigma h}$. \hfill \Box

We compute $B(n^q; v)$. Using (3.4) and the formula:
\[
(A \times B) \cdot (C \times D) = (A \cdot C)(B \cdot D) - (B \cdot C)(A \cdot D)
\]
for vectors $A, B, C, D$, we have
\[
B(n^q; v) = \int_{\Omega} \left[ K_1 \left( \text{div } n_1 \right)^2 + (K_2 - K_3) \left\{ (n^q \cdot \text{curl } n_1)^2 - 2q(n_1 \cdot \text{curl } n_1) \right\} \\
+ (K_3 - 2K_2)q^2 |n_1|^2 + K_3 |\text{curl } n_1|^2 \right] dx.
\]

From now on, we assume that
\[ K_2 = K_3. \]  
(3.5)

Then we can write
\[
B(n^q; v) = \int_{\Omega} \left\{ K_1 \left( \text{div } n_1 \right)^2 + K_2 |\text{curl } n_1|^2 - K_2q^2 |n_1|^2 \right\} dx. \]  
(3.6)
We define
\[ c_0(\Omega) = c_0(\Omega, K_1, K_2) = \inf \left\{ \frac{\mathcal{F}[w]}{\int_\Omega |w|^2 dx} ; \ w \in H^1_0(\Omega, \mathbb{R}^3), \ w \neq 0, \ w \cdot n^q = 0 \text{ in } \Omega \right\} \]
(3.7)
where
\[ \mathcal{F}[w] = \int_\Omega \left( K_1 (\text{div } w)^2 + K_2 |\text{curl } w|^2 \right) dx. \]

Since
\[ \int_\Omega |\nabla n|^2 dx = \int_\Omega \{(\text{div } n)^2 + |\text{curl } n|^2\} dx \]
for all \( n \in H^1_0(\Omega, \mathbb{R}^3) \), we see that
\[ c_0(\Omega) \geq \min(K_1, K_2)c_1(\Omega) \]
where
\[ c_1(\Omega) = \inf \left\{ \frac{\int_\Omega |\nabla w|^2 dx}{\int_\Omega |w|^2 dx} ; \ w \in H^1_0(\Omega, \mathbb{R}^3), \ w \neq 0 \right\}. \]

Here we note that since \( c_1(\Omega) \) is the lowest eigenvalue of the Laplace operator with the Dirichlet condition, \( c_1(\Omega) > 0 \).

**Lemma 3.2.** Under the hypothesis (3.5), \( c_0(\Omega, K_1, K_2) \) is achieved.

**Proof.** Let \( \{w_j\} \) be a minimizing sequence of \( c_0(\Omega) \). By the homogeneity of the definition of \( c_0(\Omega) \), we may assume that \( \|w_j\|_{L^2(\Omega, \mathbb{R}^3)} = 1 \). Then
\[ \int_\Omega \left( K_1 (\text{div } w_j)^2 + K_2 |\text{curl } w_j|^2 \right) dx = (c_0(\Omega) + o(1)) \leq C. \]
Thus \( \{\text{div } w_j\} \) is bounded in \( L^2(\Omega) \) and \( \{\text{curl } w_j\} \) is bounded in \( L^2(\Omega, \mathbb{R}^3) \). Since \( w_j = 0 \) on \( \partial \Omega \), it follows from Dautray and Lions [5] (cf. Temam [11]) that \( \{w_j\} \) is bounded in \( H^1_0(\Omega, \mathbb{R}^3) \). Passing to a subsequence, we may assume that \( w_j \rightarrow w_0 \) weakly in \( H^1_0(\Omega, \mathbb{R}^3) \), strongly in \( L^2(\Omega, \mathbb{R}^3) \) and a.e. in \( \Omega \). Therefore we see that \( \|w\|_{L^2(\Omega, \mathbb{R}^3)} = 1 \), \( w_0 \cdot n^q = 0 \) a.e. in \( \Omega \). By the fundamental theory of functional analysis, we have
\[ \int_\Omega \left( K_1 (\text{div } w_0)^2 + K_2 |\text{curl } w_0|^2 \right) dx \leq \liminf_{j \rightarrow \infty} \int_\Omega \left( K_1 (\text{div } w_j)^2 + K_2 |\text{curl } w_j|^2 \right) dx = c_0(\Omega). \]
Moreover, when \( c_0(\Omega) - K_2q^2 > 0 \) and (3.5) holds, we define
\[
H^2_{sh} = \inf_{\chi_a} \left\{ \frac{\mathcal{F}[w] - K_2q^2\|w\|_{L^2(\Omega, \mathbb{R}^3)}}{\int_{\Omega} (h \cdot w)^2 \, dx} : \right. \\
\left. w \in H^1_0(\Omega, \mathbb{R}^3), w \cdot n^q = 0 \text{ a.e. in } \Omega, h \cdot w \neq 0 \right\}
\]
Then we have

**Proposition 3.3.** When \( c_0(\Omega) - K_2q^2 > 0 \), we see that \( H_{sh} > 0 \) and it is achieved.

**Proof.** As a test function, choose \( w = \phi(x)h \), \( 0 \neq \phi \in C_0^\infty(\Omega) \), we see that \( H_{sh} < \infty \). Let \( \{w_j\} \subset H^1_0(\Omega, \mathbb{R}^3) \) satisfying \( w_j \cdot n^q = 0 \) a.e. in \( \Omega \) and \( \|h \cdot w_j\|_{L^2(\Omega, \mathbb{R}^3)} = 1 \) be a minimizing sequence of \( H_{sh} \). Then
\[
\mathcal{F}[w_j] - K_2q^2\|w_j\|_{L^2(\Omega, \mathbb{R}^3)} = \chi_a(H^2_{sh} + o(1)) \leq C.
\]
From this, we have \( (c_0(\Omega) - K_2q^2)\|w_j\|_{L^2(\Omega, \mathbb{R}^3)}^2 \leq C \). By the hypothesis, \( \|w_j\|_{L^2(\Omega, \mathbb{R}^3)} \leq C_1 \). Since
\[
\min(K_1, K_2) \int_\Omega |\nabla w_j|^2 \, dx \leq \mathcal{F}[w_j] \leq K_2q^2C_1 + C.
\]
Therefore \( \{w_j\} \) is bounded in \( H^1_0(\Omega, \mathbb{R}^3) \). Passing to a subsequence, we may assume that \( w_j \to w_0 \) weakly in \( H^1_0(\Omega, \mathbb{R}^3) \), strongly in \( L^2(\Omega, \mathbb{R}^3) \) and a.e. in \( \Omega \). Hence \( w_0 \cdot n^q = 0 \) a.e. in \( \Omega \) and \( \|h \cdot w_0\|_{L^2(\Omega)} = 1 \). As in the proof of Lemma 3.2, we have
\[
\mathcal{F}[w_0] - K_2q^2\|w_0\|_{L^2(\Omega, \mathbb{R}^3)}^2 \leq \liminf_{j \to \infty} \{\mathcal{F}[w_j] - K_2q^2\|w_j\|_{L^2(\Omega, \mathbb{R}^3)}\} = \chi_a H^2_{sh}.
\]

Thus we can write
\[
\mathcal{F}_{\sigma h}[n_t] = \mathcal{F}_{\sigma h}[n^q] + t^2 \left\{ B(n^q; v) - \chi_a \sigma^2 \int_\Omega (h \cdot n_1)^2 \, dx \right\} + O(t^3)
\]
\[
= \mathcal{F}_{\sigma h}[n^q] + t^2 \left\{ \mathcal{F}[n_1] - K_2q^2\|n_1\|_{L^2(\Omega, \mathbb{R}^3)} - \chi_a \sigma^2 \int_\Omega (h \cdot n_1)^2 \, dx \right\}
\]
\[
+ O(t^3).
\]
(3.10)
We note that \( v \in H^1_0(\Omega, \mathbb{R}^3) \) is not parallel to \( n^q \) on a set of positive measure if and only if \( n_1 = v - (v \cdot n^q)n^q \neq 0 \) on a set of positive measure.

We give the main theorem.
Theorem 3.4. Assume that (3.5) holds. Then we have the following.

(i) When \(c_0(\Omega) - K_2 q^2 < 0\), \(n^q\) is not weakly stable for any \(\sigma\).
(ii) When \(c_0(\Omega) - K_2 q^2 > 0\), if \(|\sigma| < H_{sh}\), then \(n^q\) is weakly stable, and if \(|\sigma| > H_{sh}\), then \(n^q\) is not weakly stable.

Proof. (i) Choose a minimizer \(w\) of \(c_0(\Omega)\). Since \(w \cdot n^q = 0\), if we put \(v = w\), we see that \(n_1 = w\) where \(n_1\) is defined in (2.2). Then we have

\[
B(n^q; v) - \chi_a \sigma^2 \int_\Omega (h \cdot n_1)^2 dx \leq B(n^q; v)
= \mathcal{F}[w] - K_2 q^2 \|w\|_{L^2(\Omega, \mathbb{R}^3)}^2
= (c_0(\Omega) - K_2 q^2) \|w\|_{L^2(\Omega, \mathbb{R}^3)}^2 < 0.
\]

From Corollary 2.3, \(n^q\) is not weakly stable for any \(\sigma\).

(ii) When \(c_0(\Omega) - K_2 q^2 > 0\) and \(|\sigma| < H_{sh}\), let \(v \in H^1_0(\Omega, \mathbb{R}^3)\) be not parallel to \(n^q\). Then \(n_1 = v - (v \cdot n^q) n^q \neq 0\) and \(n_1 \cdot n^q = 0\) a.e. in \(\Omega\). Then

\[
B(n^q; v) - \chi_a \sigma^2 \int_\Omega (h \cdot n_1)^2 dx
= \mathcal{F}[n_1] - K_2 q^2 \|n_1\|_{L^2(\Omega, \mathbb{R}^3)}^2 - \chi_a \sigma^2 \int_\Omega (h \cdot n_1)^2 dx.
\]

If \(h \cdot n_1 \equiv 0\),

\[
B(n^q; v) - \chi_a \sigma^2 \int_\Omega (h \cdot n_1)^2 dx
= \mathcal{F}[n_1] - K_2 q^2 \|n_1\|_{L^2(\Omega, \mathbb{R}^3)}^2
\geq (c_0(\Omega) - K_2 q^2) \|n_1\|_{L^2(\Omega, \mathbb{R}^3)}^2 > 0.
\]

If \(h \cdot n_1 \neq 0\),

\[
B(n^q; v) - \chi_a \sigma^2 \int_\Omega (h \cdot n_1)^2 dx
= \mathcal{F}[n_1] - K_2 q^2 \|n_1\|_{L^2(\Omega, \mathbb{R}^3)}^2 - \chi_a \sigma^2 \int_\Omega (h \cdot n_1)^2 dx
\geq \chi_a (H_{sh}^2 - \sigma^2) \int_\Omega (h \cdot n_1)^2 dx > 0.
\]

Thus in this case where \(c_0(\Omega) - K_2 q^2 > 0\) and \(|\sigma| < H_{sh}\), it follows from Corollary 2.3 that \(n^q\) is weakly stable.
When $|\sigma| > H_{sh}$, choose a minimizer $w$ of $H_{sh}$. Then
\[
\mathcal{F}[w] - K_2q^2\|w\|_{L^2(\Omega, \mathbb{R}^3)}^2 = \chi_a H_{sh}^2 \int_\Omega (h \cdot w)^2 dx.
\]
If we put $v = w$ in (2.3), we see that
\[
\mathcal{F}[n_1] - K_2q^2\|n_1\|_{L^2(\Omega, \mathbb{R}^3)}^2 - \chi_a \sigma^2 \int_\Omega (h \cdot n_1)^2 dx
= \chi_a (H_{sh}^2 - \sigma^2) \int_\Omega (h \cdot n_1)^2 dx < 0.
\]
Since $H^1(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$ is dense in $H^1(\Omega, \mathbb{R}^3)$, it follows from Corollary 2.3 that $n^q$ is not weakly stable.

**Remark 3.5.** When $K_1 = K_2$ and $\Omega = (0, 1) \times (0, 1) \times (0, 1)$, if we remember
\[
c_1(\Omega) = \inf \left\{ \frac{\int_\Omega |\nabla n|^2 dx}{\int_\Omega |n|^2 dx} ; n \in H^1_0(\Omega, \mathbb{R}^3), n \neq 0 \right\},
\]
we can see that $c_1(\Omega) = 3\pi^2$ (cf. [4]). In this case,
\[
c_0(\Omega) = \inf \left\{ \frac{\int_\Omega |\nabla n|^2 dx}{\int_\Omega |n|^2 dx} ; n \in H^1_0(\Omega, \mathbb{R}^3), n \neq 0, n \cdot n^q = 0 \text{ a.e. in } \Omega \right\},
\]
Then $c_1(\Omega) \leq c_0(\Omega)$. If we choose a test field $n = (0, 0, \psi)$ of $c_0(\Omega)$ where $\psi = \sqrt{8} \sin(\pi x_1) \sin(\pi x_2) \sin(\pi x_3)$, then $n \cdot n^q = 0$ and $n \neq 0$. Thus we see that $c_0(\Omega) \leq 3\pi^2$. Therefore, we have $c_0(\Omega) = c_1(\Omega) = 3\pi^2$.

**References**


