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WEAK STABILITY OF A CRITICAL TWISTED NEMATIC STATE UNDER THE MAGNETIC FIELD IN LIQUID CRYSTALS

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Abstract: We consider the weak stability of a critical twisted nematic state under the magnetic field in the theory of liquid crystals. We treat the Oseen-Frank energy functional with the strong anchoring condition, that is to say, the Dirichlet boundary condition for the director field.

1. Introduction

It is well known that the stability of a liquid crystal configuration is influenced by the external magnetic field. In this paper, we shall consider the stability of a critical twisted nematic configuration. For such a theory in nematic liquid crystal, we follow the Oseen-Frank energy given by Ericksen and Leslie [7]. The bulk energy is given by

$$\mathcal{W}[\boldsymbol{n}] = \int_\Omega W(\nabla \boldsymbol{n}, \boldsymbol{n}) dx$$

where $\boldsymbol{n} = \boldsymbol{n}(x)$ is the unit vector field called the director field, $\Omega \subset \mathbb{R}^3$ is the domain occupied by the material. And $W(\nabla \boldsymbol{n}, \boldsymbol{n})$ is the Oseen-Frank energy

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density:

$$W(\nabla \boldsymbol{n}, \boldsymbol{n}) = K_1(\operatorname{div} \boldsymbol{n})^2 + K_2(\boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{n})^2 + K_3|\boldsymbol{n} \times \operatorname{curl} \boldsymbol{n}|^2 \\ + (K_2 + K_4)(\operatorname{tr}(\nabla \boldsymbol{n})^2 - (\operatorname{div} \boldsymbol{n})^2)$$

where K_i $(i = 1, 2, 3)$ are positive constants which are called the elastic coefficients and K_4 is a real constant.

We consider the case where equilibrium molecular configurations are local minima of $\mathcal{W}[\boldsymbol{n}]$ and satisfy the strong anchoring condition, mathematically the Dirichlet boundary condition:

$$\boldsymbol{n}(x) = \boldsymbol{n}_0(x), \quad x \in \partial\Omega$$

where $\partial\Omega$ is the boundary of Ω. Under the strong anchoring condition,

$$\int_\Omega \{\operatorname{tr}(\nabla \boldsymbol{n})^2 - (\operatorname{div} \boldsymbol{n})^2\} dx$$

represents a surface energy and so we can neglect the term for finding energy minimizing configurations (cf. Hardt et al. [8]).

In the situation where the liquid crystal material is subject to a static magnetic field $\boldsymbol{H}$, we must add the magnetic energy contribution $-\chi_a(\boldsymbol{H} \cdot \boldsymbol{n})^2$ to the energy density (cf. de Gennes and Prost [6], Lin and Pan [9], Cohen and Luskin [4], Pan [10] and Aramaki [2, 1, 3]). For brevity, we assume that $\boldsymbol{H} = \sigma \boldsymbol{h}$ where $\boldsymbol{h}$ is a unit vector field and σ ia a constant for which $|\sigma|$ is the intensity of $\boldsymbol{H}$ and Ω is a bounded smooth domain in $\mathbb{R}^3$ with boundary $\partial\Omega$. Thus we shall consider the energy functional:

$$\mathcal{F}_{\sigma \boldsymbol{h}}[\boldsymbol{n}] = \int_\Omega \{K_1(\operatorname{div} \boldsymbol{n})^2 + K_2(\boldsymbol{n} \cdot \operatorname{curl} \boldsymbol{n})^2 + K_3|\boldsymbol{n} \times \operatorname{curl} \boldsymbol{n}|^2\} dx \\ - \chi_a \sigma^2 \int_\Omega (\boldsymbol{h} \cdot \boldsymbol{n})^2 dx \quad (1.1)$$

Let $H^1(\Omega, \mathbb{R}^3)$ and $H^1_0(\Omega, \mathbb{R}^3)$ be the usual Sobolev space of vector valued functions and define

$$H^1(\Omega, \mathbb{S}^2) = \{\boldsymbol{n} \in H^1(\Omega, \mathbb{R}^3); |\boldsymbol{n}(x)| = 1 \text{ a.e. } x \in \Omega\}.$$

Moreover, for given smooth vector field $\boldsymbol{n}_0 : \partial\Omega \to \mathbb{S}^2$, we define the space of admissible director fields

$$H^1(\Omega, \mathbb{S}^2, \boldsymbol{n}_0) = \{\boldsymbol{n} \in H^1(\Omega, \mathbb{S}^2); \boldsymbol{n} = \boldsymbol{n}_0 \text{ on } \partial\Omega\}.$$

We note that if $\boldsymbol{n}_0$ is Lipschitzian, then $H^1(\Omega, \mathbb{S}^2, \boldsymbol{n}_0) \neq \emptyset$ (cf. [8, Lemma 1.1].

In the present paper, we give a criterion of the weak stability of a critical twisted nematic state $\boldsymbol{n}^q = (\cos(qx_3), \sin(qx_3), 0)$ and $\boldsymbol{n}_0 = \boldsymbol{n}^q\big|_{\partial\Omega}$. [4] considered the weak stability of $\boldsymbol{n}^q$ without magnetic field and $\Omega = (0,1) \times (0,1) \times (0,1)$. We shall extend the result to the general case.

2. Critical Points and Weak Stability

In this section, we give the notations of critical points of $\mathcal{F}_{\sigma \boldsymbol{h}}$ and weak stabilities of critical points.

For $\boldsymbol{n} \in H^1(\Omega, \mathbb{S}^2)$, $\boldsymbol{v} \in H^1_0(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$ and small $t \in \mathbb{R}$, define

$$\boldsymbol{n}_t = \frac{\boldsymbol{n} + t\boldsymbol{v}}{|\boldsymbol{n} + t\boldsymbol{v}|}. \tag{2.1}$$

Then we can write

$$\boldsymbol{n}_t = \boldsymbol{n} + t\boldsymbol{n}_1 + t^2\boldsymbol{n}_2 + O(t^3)$$

where

$$\begin{aligned} \boldsymbol{n}_1 &= \boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{n}, \\ \boldsymbol{n}_2 &= -(\boldsymbol{v}\cdot\boldsymbol{n})\boldsymbol{v} + \frac{1}{2}[3(\boldsymbol{v}\cdot\boldsymbol{n})^2 - |\boldsymbol{v}|^2]\boldsymbol{n}. \end{aligned} \tag{2.2}$$

We shall expand $\mathcal{F}_{\sigma \boldsymbol{h}}[\boldsymbol{n}_t]$ with respect to powers of t (cf. [2] and [9]).

Proposition 2.1. *We can write*

$$\begin{aligned} \mathcal{F}_{\sigma \boldsymbol{h}}[\boldsymbol{n}_t] &= \mathcal{F}_{\sigma \boldsymbol{h}}[\boldsymbol{n}] + 2t\left\{\mathcal{A}(\boldsymbol{n};\boldsymbol{v}) - \chi_a\sigma^2\int_\Omega (\boldsymbol{h}\cdot\boldsymbol{n})(\boldsymbol{h}\cdot\boldsymbol{n}_1)dx\right\} \\ &\quad + t^2\left\{\mathcal{B}(\boldsymbol{n};\boldsymbol{v}) - \chi_a\sigma^2\int_\Omega \{(\boldsymbol{h}\cdot\boldsymbol{n}_1)^2 + 2(\boldsymbol{h}\cdot\boldsymbol{n})(\boldsymbol{h}\cdot\boldsymbol{n}_2)\}dx\right\} + O(t^3) \end{aligned}$$

where

$$\begin{aligned} \mathcal{A}(\boldsymbol{n};\boldsymbol{v}) = \int_\Omega &\{K_1(\operatorname{div}\boldsymbol{n})(\operatorname{div}\boldsymbol{n}_1) + K_2(\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})(\boldsymbol{n}_1\cdot\operatorname{curl}\boldsymbol{n} + \boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n}_1) \\ &+ K_3(\boldsymbol{n}\times\operatorname{curl}\boldsymbol{n})\cdot(\boldsymbol{n}_1\times\operatorname{curl}\boldsymbol{n} + \boldsymbol{n}\times\operatorname{curl}\boldsymbol{n}_1)\}dx, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \mathcal{B}(\boldsymbol{n};\boldsymbol{v}) = \int_\Omega &\big[K_1\{(\operatorname{div}\boldsymbol{n}_1)^2 + 2(\operatorname{div}\boldsymbol{n})(\operatorname{div}\boldsymbol{n}_1)\} + K_2\{(\boldsymbol{n}_1\cdot\operatorname{curl}\boldsymbol{n} + \boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n}_1)^2 \\ &+ 2(\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n})(\boldsymbol{n}_2\cdot\operatorname{curl}\boldsymbol{n} + \boldsymbol{n}_1\cdot\operatorname{curl}\boldsymbol{n}_1 + \boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{n}_2)\} \end{aligned}$$

$$+ K_3\{|\boldsymbol{n}_1 \times \operatorname{curl}\boldsymbol{n} + \boldsymbol{n} \times \operatorname{curl}\boldsymbol{n}_1|^2$$
$$+ 2(\boldsymbol{n} \times \operatorname{curl}\boldsymbol{n}) \cdot (\boldsymbol{n}_2 \times \operatorname{curl}\boldsymbol{n} + \boldsymbol{n}_1 \times \operatorname{curl}\boldsymbol{n}_1 + \boldsymbol{n} \times \operatorname{curl}\boldsymbol{n}_2)\}\big]dx. \tag{2.4}$$

Definition 2.2. (i) We say that $\boldsymbol{n} \in H^1(\Omega, \mathbb{S}^2, \boldsymbol{n}_0)$ is a critical point of $\mathcal{F}_{\sigma\boldsymbol{h}}$, if and only if for any $\boldsymbol{v} \in H_0^1(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$,

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}_t] = 0.$$

That is to say,

$$\mathcal{A}(\boldsymbol{n}; \boldsymbol{v}) - \chi_a\sigma^2 \int_\Omega (\boldsymbol{h} \cdot \boldsymbol{n})(\boldsymbol{h} \cdot \boldsymbol{n}_1)dx = 0. \tag{2.5}$$

(ii) We say that a critical point $\boldsymbol{n}$ of $\mathcal{F}_{\sigma\boldsymbol{h}}$ is weakly stable (local minimizer), if and only if for any $\boldsymbol{v} \in H_0^1(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$, there exists $T = T(\boldsymbol{v}) > 0$ such that for all $0 < t < T$,

$$\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}] \le \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}_t].$$

From Proposition 2.1 and Definition 2.2, we have (cf. [4, Corollary 2.4])

Corollary 2.3. *Assume that $\boldsymbol{n} \in H^1(\Omega, \mathbb{S}^2, \boldsymbol{n}_0)$ be a critical point of $\mathcal{F}_{\sigma\boldsymbol{h}}$. If there exists $\boldsymbol{v} \in H_0^1(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$ such that*

$$\mathcal{B}(\boldsymbol{n}; \boldsymbol{v}) - \chi_a\sigma^2 \int_\Omega \{(\boldsymbol{h} \cdot \boldsymbol{n}_1)^2 + 2(\boldsymbol{h} \cdot \boldsymbol{n})(\boldsymbol{h} \cdot \boldsymbol{n}_2)\}dx < 0,$$

then $\boldsymbol{n}$ is not weak stable. Conversely, if

$$\mathcal{B}(\boldsymbol{n}; \boldsymbol{v}) - \chi_a\sigma^2 \int_\Omega \{(\boldsymbol{h} \cdot \boldsymbol{n}_1)^2 + 2(\boldsymbol{h} \cdot \boldsymbol{n})(\boldsymbol{h} \cdot \boldsymbol{n}_2)\}dx > 0$$

for all $\boldsymbol{v} \in H_0^1(\Omega, \mathbb{R}^3) \cap L^\infty(\Omega, \mathbb{R}^3)$ such that $\boldsymbol{v}$ is not parallel to $\boldsymbol{n}$ on a set of positive measure, then $\boldsymbol{n}$ is weakly stable.

3. The Weak Stability of a Twisted Nematic State

A twisted nematic configuration is defined by

$$\boldsymbol{n}^q = (\cos(qx_3), \sin(qx_3), 0) \tag{3.1}$$

and we define

$$\boldsymbol{n}_0 = \left.\boldsymbol{n}^q\right|_{\partial\Omega}. \tag{3.2}$$

In this section, we assume that the exterior magnetic field $\boldsymbol{h}$ is perpendicular to $\boldsymbol{n}^q$, that is to say,

$$\boldsymbol{h} \cdot \boldsymbol{n}^q = 0 \quad \text{in } \Omega. \tag{3.3}$$

By a simple computations, we have

$$\begin{aligned} &\operatorname{curl} \boldsymbol{n}^q = -q\boldsymbol{n}^q, \quad \operatorname{div} \boldsymbol{n}^q = 0 \\ &\boldsymbol{n}_1 \cdot \boldsymbol{n}^q = 0, \quad \boldsymbol{n}_2 \cdot \boldsymbol{n}^q = -\frac{1}{2}|\boldsymbol{n}_1|^2 \quad \text{in } \Omega. \end{aligned} \tag{3.4}$$

Lemma 3.1. *$\boldsymbol{n}^q$ is a critical point of $\mathcal{F}_{\sigma \boldsymbol{h}}$.*

Proof. Using (3.4), we have

$$\mathcal{A}(\boldsymbol{n}^q; \boldsymbol{v}) = K_2 \int_\Omega \boldsymbol{n}^q \cdot \operatorname{curl} \boldsymbol{n}_1 dx.$$

Since $\boldsymbol{n}_1 = 0$ on $\partial\Omega$, it follows that

$$\mathcal{A}(\boldsymbol{n}^q; \boldsymbol{v}) = K_2 \int_\Omega \operatorname{curl} \boldsymbol{n}^q \cdot \boldsymbol{n}_1 dx = -K_2 q \int_\Omega \boldsymbol{n}^q \cdot \boldsymbol{n}_1 dx = 0.$$

Since $\boldsymbol{h} \cdot \boldsymbol{n}^q = 0$, we have (2.5) with $\boldsymbol{n} = \boldsymbol{n}^q$. Thus $\boldsymbol{n}^q$ is a critical point of $\mathcal{F}_{\sigma \boldsymbol{h}}$. □

We compute $\mathcal{B}(\boldsymbol{n}^q; \boldsymbol{v})$. Using (3.4) and the formula:

$$(\boldsymbol{A} \times \boldsymbol{B}) \cdot (\boldsymbol{C} \times \boldsymbol{D}) = (\boldsymbol{A} \cdot \boldsymbol{C})(\boldsymbol{B} \cdot \boldsymbol{D}) - (\boldsymbol{B} \cdot \boldsymbol{C})(\boldsymbol{A} \cdot \boldsymbol{D})$$

for vectors $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}$, we have

$$\begin{aligned} \mathcal{B}(\boldsymbol{n}^q; \boldsymbol{v}) = \int_\Omega \big[&K_1 (\operatorname{div} \boldsymbol{n}_1)^2 + (K_2 - K_3)\{(\boldsymbol{n}^q \cdot \operatorname{curl} \boldsymbol{n}_1)^2 - 2q(\boldsymbol{n}_1 \cdot \operatorname{curl} \boldsymbol{n}_1)\} \\ &+ (K_3 - 2K_2) q^2 |\boldsymbol{n}_1|^2 + K_3 |\operatorname{curl} \boldsymbol{n}_1|^2 \big] dx. \end{aligned}$$

From now on, we assume that

$$K_2 = K_3. \tag{3.5}$$

Then we can write

$$\mathcal{B}(\boldsymbol{n}^q; \boldsymbol{v}) = \int_\Omega \{K_1 (\operatorname{div} \boldsymbol{n}_1)^2 + K_2 |\operatorname{curl} \boldsymbol{n}_1|^2 - K_2 q^2 |\boldsymbol{n}_1|^2\} dx. \tag{3.6}$$

We define

$$c_0(\Omega) = c_0(\Omega, K_1, K_2) = \inf\left\{ \frac{\mathcal{F}[\boldsymbol{w}]}{\int_\Omega |\boldsymbol{w}|^2 dx}; \boldsymbol{w} \in H_0^1(\Omega, \mathbb{R}^3), \boldsymbol{w} \neq 0, \boldsymbol{w}\cdot\boldsymbol{n}^q = 0 \text{ in } \Omega \right\} \tag{3.7}$$

where

$$\mathcal{F}[\boldsymbol{w}] = \int_\Omega \{K_1(\operatorname{div}\boldsymbol{w})^2 + K_2|\operatorname{curl}\boldsymbol{w}|^2\}dx.$$

Since

$$\int_\Omega |\nabla \boldsymbol{n}|^2 dx = \int_\Omega \{(\operatorname{div}\boldsymbol{n})^2 + |\operatorname{curl}\boldsymbol{n}|^2\}dx$$

for all $\boldsymbol{n} \in H_0^1(\Omega, \mathbb{R}^3)$, we see that

$$c_0(\Omega) \geq \min(K_1, K_2) c_1(\Omega)$$

where

$$c_1(\Omega) = \inf\left\{ \frac{\int_\Omega |\nabla \boldsymbol{w}|^2 dx}{\int_\Omega |\boldsymbol{w}|^2 dx}; \boldsymbol{w} \in H_0^1(\Omega, \mathbb{R}^3), \boldsymbol{w} \neq 0 \right\}.$$

Here we note that since $c_1(\Omega)$ is the lowest eigenvalue of the Laplace operator with the Dirichlet condition, $c_1(\Omega) > 0$.

Lemma 3.2. *Under the hypothesis* (3.5), $c_0(\Omega, K_1, K_2)$ *is achieved.*

Proof. Let $\{\boldsymbol{w}_j\}$ be a minimizing sequence of $c_0(\Omega)$. By the homogeneity of the definition of $c_0(\Omega)$, we may assume that $\|\boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{R}^3)} = 1$. Then

$$\int_\Omega \{K_1(\operatorname{div}\boldsymbol{w}_j)^2 + K_2|\operatorname{curl}\boldsymbol{w}_j|^2\}dx = (c_0(\Omega) + o(1)) \leq C.$$

Thus $\{\operatorname{div}\boldsymbol{w}_j\}$ is bounded in $L^2(\Omega)$ and $\{\operatorname{curl}\boldsymbol{w}_j\}$ is bounded in $L^2(\Omega, \mathbb{R}^3)$. Since $\boldsymbol{w}_j = 0$ on $\partial\Omega$, it follows from Dautray and Lions [5] (cf. Temam [11]) that $\{\boldsymbol{w}_j\}$ is bounded in $H_0^1(\Omega, \mathbb{R}^3)$. Passing to a subsequence, we may assume that $\boldsymbol{w}_j \to \boldsymbol{w}_0$ weakly in $H_0^1(\Omega, \mathbb{R}^3)$, strongly in $L^2(\Omega, \mathbb{R}^3)$ and a.e. in Ω. Therefore we see that $\|\boldsymbol{w}\|_{L^2(\Omega,\mathbb{R}^3)} = 1$, $\boldsymbol{w}_0 \cdot \boldsymbol{n}^q = 0$ a.e. in Ω. By the fundamental theory of functional analysis, we have

$$\int_\Omega \{K_1(\operatorname{div}\boldsymbol{w}_0)^2 + K_2|\operatorname{curl}\boldsymbol{w}_0|^2 dx \\ \leq \liminf_{j\to\infty} \int_\Omega \{K_1(\operatorname{div}\boldsymbol{w}_j)^2 + K_2|\operatorname{curl}\boldsymbol{w}_j|^2\}dx = c_0(\Omega).$$

□

Moreover, when $c_0(\Omega) - K_2q^2 > 0$ and (3.5) holds, we define

$$H_{sh}^2 = \frac{1}{\chi_a} \inf\Big\{ \frac{\mathcal{F}[\boldsymbol{w}] - K_2q^2\|\boldsymbol{w}\|_{L^2(\Omega,\mathbb{R}^3)}}{\int_\Omega (\boldsymbol{h}\cdot\boldsymbol{w})^2dx}; \quad \boldsymbol{w} \in H_0^1(\Omega,\mathbb{R}^3), \boldsymbol{w}\cdot\boldsymbol{n}^q = 0 \text{ a.e. in } \Omega, \boldsymbol{h}\cdot\boldsymbol{w} \neq 0 \Big\} \tag{3.9}$$

Then we have

Proposition 3.3. *When $c_0(\Omega) - K_2q^2 > 0$, we see that $H_{sh} > 0$ and it is achieved.*

Proof. As a test function, choose $\boldsymbol{w} = \phi(x)\boldsymbol{h}$, $0 \neq \phi \in C_0^\infty(\Omega)$, we see that $H_{sh} < \infty$. Let $\{\boldsymbol{w}_j\} \subset H_0^1(\Omega,\mathbb{R}^3)$ satisfying $\boldsymbol{w}_j\cdot\boldsymbol{n}^q = 0$ a.e. in Ω and $\|\boldsymbol{h}\cdot\boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{R}^3)} = 1$ be a minimizing sequence o H_{sh}. Then

$$\mathcal{F}[\boldsymbol{w}_j] - K_2q^2\|\boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{R}^3)} = \chi_a(H_{sh}^2 + o(1)) \leq C.$$

From this, we have $(c_0(\Omega) - K_2q^2)\|\boldsymbol{w}_j\|^2_{L^2(\Omega,\mathbb{R}^3)} \leq C$. By the hypothesis, $\|\boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{R}^3)} \leq C_1$. Since

$$\min(K_1,K_2)\int_\Omega |\nabla\boldsymbol{w}_j|^2dx \leq \mathcal{F}[\boldsymbol{w}_j] \leq K_2q^2C_1 + C.$$

Therefore $\{\boldsymbol{w}_j\}$ is bounded in $H_0^1(\Omega,\mathbb{R}^3)$. Passing to a subsequence, we may assume that $\boldsymbol{w}_j \to \boldsymbol{w}_0$ weakly in $H_0^1(\Omega,\mathbb{R}^3)$, strongly in $L^2(\Omega,\mathbb{R}^3)$ and a.e. in Ω. Hence $\boldsymbol{w}_0\cdot\boldsymbol{n}^q = 0$ a.e. in Ω and $\|\boldsymbol{h}\cdot\boldsymbol{w}_0\|_{L^2(\Omega)} = 1$. As in the proof of Lemma 3.2, we have

$$\mathcal{F}[\boldsymbol{w}_0] - K_2q^2\|\boldsymbol{w}_0\|^2_{L^2(\Omega,\mathbb{R}^3)} \leq \liminf_{j\to\infty}\{\mathcal{F}[\boldsymbol{w}_j] - K_2q^2\|\boldsymbol{w}_j\|_{L^2(\Omega,\mathbb{R}^3)}\} = \chi_aH_{sh}^2.$$

□

Thus we can write

$$\begin{aligned}\mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}_t] &= \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}^q] + t^2\Big\{\mathcal{B}(\boldsymbol{n}^q;\boldsymbol{v}) - \chi_a\sigma^2\int_\Omega(\boldsymbol{h}\cdot\boldsymbol{n}_1)^2dx\Big\} + O(t^3)\\ &= \mathcal{F}_{\sigma\boldsymbol{h}}[\boldsymbol{n}^q] + t^2\Big\{\mathcal{F}[\boldsymbol{n}_1] - K_2q^2\|\boldsymbol{n}_1\|_{L^2(\Omega,\mathbb{R}^3)} - \chi_a\sigma^2\int_\Omega(\boldsymbol{h}\cdot\boldsymbol{n}_1)^2dx\Big\}\\ &\quad + O(t^3).\end{aligned} \tag{3.10}$$

We note that $\boldsymbol{v} \in H_0^1(\Omega,\mathbb{R}^3)$ is not parallel to $\boldsymbol{n}^q$ on a set of positive measure if and only if $\boldsymbol{n}_1 = \boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{n}^q)\boldsymbol{n}^q \neq 0$ on a set of positive measure.

We give the main theorem.

Theorem 3.4. *Assume that* (3.5) *holds. Then we have the following.*

(i) *When* $c_0(\Omega) - K_2q^2 < 0$, $\boldsymbol{n}^q$ *is not weakly stable for any* σ.

(ii) *When* $c_0(\Omega) - K_2q^2 > 0$, *if* $|\sigma| < H_{sh}$, *then* $\boldsymbol{n}^q$ *is weakly stable, and if* $|\sigma| > H_{sh}$, *then* $\boldsymbol{n}^q$ *is not weakly stable.*

Proof. (i) Choose a minimizer $\boldsymbol{w}$ of $c_0(\Omega)$. Since $\boldsymbol{w}\cdot\boldsymbol{n}^q = 0$, if we put $\boldsymbol{v} = \boldsymbol{w}$, we see that $\boldsymbol{n}_1 = \boldsymbol{w}$ where $\boldsymbol{n}_1$ is defined in (2.2). Then we have

$$\begin{aligned}\mathcal{B}(\boldsymbol{n}^q;\boldsymbol{v}) - \chi_a\sigma^2\int_\Omega(\boldsymbol{h}\cdot\boldsymbol{n}_1)^2dx &\le \mathcal{B}(\boldsymbol{n}^q;\boldsymbol{v})\\ &= \mathcal{F}[\boldsymbol{w}] - K_2q^2\|\boldsymbol{w}\|^2_{L^2(\Omega,\mathbb{R}^3)}\\ &= (c_0(\Omega) - K_2q^2)\|\boldsymbol{w}\|^2_{L^2(\Omega,\mathbb{R}^3)} < 0.\end{aligned}$$

From Corollary 2.3, $\boldsymbol{n}^q$ is not weakly stable for any σ.

(ii) When $c_0(\Omega)-K_2q^2 > 0$ and $|\sigma| < H_{sh}$, let $\boldsymbol{v}\in H^1_0(\Omega,\mathbb{R}^3)$ be not parallel to $\boldsymbol{n}^q$. Then $\boldsymbol{n}_1 = \boldsymbol{v} - (\boldsymbol{v}\cdot\boldsymbol{n}^q)\boldsymbol{n}^q \neq 0$ and $\boldsymbol{n}_1\cdot\boldsymbol{n}^q = 0$ a.e. in Ω. Then

$$\begin{aligned}&\mathcal{B}(\boldsymbol{n}^q;\boldsymbol{v}) - \chi_a\sigma^2\int_\Omega(\boldsymbol{h}\cdot\boldsymbol{n}_1)^2dx\\ &\qquad = \mathcal{F}[\boldsymbol{n}_1] - K_2q^2\|\boldsymbol{n}_1\|^2_{L^2(\Omega,\mathbb{R}^3)} - \chi_a\sigma^2\int_\Omega(\boldsymbol{h}\cdot\boldsymbol{n}_1)^2dx.\end{aligned}$$

If $\boldsymbol{h}\cdot\boldsymbol{n}_1 \equiv 0$,

$$\begin{aligned}&\mathcal{B}(\boldsymbol{n}^q;\boldsymbol{v}) - \chi_a\sigma^2\int_\Omega(\boldsymbol{h}\cdot\boldsymbol{n}_1)^2dx\\ &\qquad = \mathcal{F}[\boldsymbol{n}_1] - K_2q^2\|\boldsymbol{n}_1\|^2_{L^2(\Omega,\mathbb{R}^3)}\\ &\qquad\qquad \ge (c_0(\Omega) - K_2q^2)\|\boldsymbol{n}_1\|^2_{L^2(\Omega,\mathbb{R}^3)} > 0.\end{aligned}$$

If $\boldsymbol{h}\cdot\boldsymbol{n}_1 \not\equiv 0$,

$$\begin{aligned}&\mathcal{B}(\boldsymbol{n}^q;\boldsymbol{v}) - \chi_a\sigma^2\int_\Omega(\boldsymbol{h}\cdot\boldsymbol{n}_1)^2dx\\ &\qquad = \mathcal{F}[\boldsymbol{n}_1] - K_2q^2\|\boldsymbol{n}_1\|^2_{L^2(\Omega,\mathbb{R}^3)} - \chi_a\sigma^2\int_\Omega(\boldsymbol{h}\cdot\boldsymbol{n}_1)^2dx\\ &\qquad\qquad \ge \chi_a(H^2_{sh} - \sigma^2)\int_\Omega(\boldsymbol{h}\cdot\boldsymbol{n}_1)^2dx > 0.\end{aligned}$$

Thus in this case where $c_0(\Omega) - K_2q^2 > 0$ and $|\sigma| < H_{sh}$, it follows from Corollary 2.3 that $\boldsymbol{n}^q$ is weakly stable.

When $|\sigma| > H_{sh}$, choose a minimizer $\boldsymbol{w}$ of H_{sh}. Then

$$\mathcal{F}[\boldsymbol{w}] - K_2 q^2 \|\boldsymbol{w}\|^2_{L^2(\Omega,\mathbb{R}^3)} = \chi_a H^2_{sh} \int_\Omega (\boldsymbol{h}\cdot\boldsymbol{w})^2 dx.$$

If we put $\boldsymbol{v} = \boldsymbol{w}$ in (2.3), we see that

$$\mathcal{F}[\boldsymbol{n}_1] - K_2 q^2 \|\boldsymbol{n}_1\|^2_{L^2(\Omega,\mathbb{R}^3)} - \chi_a \sigma^2 \int_\Omega (\boldsymbol{h}\cdot\boldsymbol{n}_1)^2 dx$$
$$= \chi_a (H^2_{sh} - \sigma^2) \int_\Omega (\boldsymbol{h}\cdot\boldsymbol{n}_1)^2 dx < 0.$$

Since $H^1(\Omega,\mathbb{R}^3) \cap L^\infty(\Omega,\mathbb{R}^3)$ is dense in $H^1(\Omega,\mathbb{R}^3)$, it follows from Corollary 2.3 that $\boldsymbol{n}^q$ is not weakly stable. □

Remark 3.5. When $K_1 = K_2$ and $\Omega = (0,1)\times(0,1)\times(0,1)$, if we remember

$$c_1(\Omega) = \inf\left\{ \frac{\int_\Omega |\nabla \boldsymbol{n}|^2 dx}{\int_\Omega |\boldsymbol{n}|^2 dx}; \boldsymbol{n} \in H^1_0(\Omega,\mathbb{R}^3), \boldsymbol{n} \neq 0 \right\},$$

we can see that $c_1(\Omega) = 3\pi^2$ (cf. [4]). In this case,

$$c_0(\Omega) = \inf\left\{ \frac{\int_\Omega |\nabla \boldsymbol{n}|^2 dx}{\int_\Omega |\boldsymbol{n}|^2 dx}; \boldsymbol{n} \in H^1_0(\Omega,\mathbb{R}^3), \boldsymbol{n} \neq 0, \boldsymbol{n}\cdot\boldsymbol{n}^q = 0 \text{ a.e. in } \Omega \right\},$$

Then $c_1(\Omega) \le c_0(\Omega)$. If we choose a test field $\boldsymbol{n} = (0,0,\psi)$ of $c_0(\Omega)$ where $\psi = \sqrt{8}\sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3)$, then $\boldsymbol{n}\cdot\boldsymbol{n}^q = 0$ and $\boldsymbol{n} \neq 0$. Thus we see that $c_0(\Omega) \le 3\pi^2$. Therefore, we have $c_0(\Omega) = c_1(\Omega) = 3\pi^2$.

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