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SYMMETRIC TENSOR RANK OF POINTS IN THE LINEAR SPAN OF SUITABLE ZERO-DIMENSIONAL SUBSCHEMES

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Abstract: Let $\nu_d : \mathbb{P}^m \to \mathbb{P}^n$ the d-Veronese embedding. Fix $P \in \mathbb{P}^n$ such that $P \in \langle \nu_d(Z) \rangle$ with Z in linearly general position (or in t-uniform position). Here we give lower bounds on the symmetric rank of P in terms of $\deg(Z)$.

AMS Subject Classification: 14N05
Key Words: symmetric tensor rank, cactus rank, zero-dimensional scheme

1. Introduction

Let $\nu_d : \mathbb{P}^m \to \mathbb{P}^n$, $n := \binom{n+d}{n} - 1$, denote the degree d Veronese embedding of $\mathbb{P}^m$. For any zero-dimensional scheme $A \subset \nu_d(\mathbb{P}^m)$ let $\langle A \rangle$ denote the intersections of all hyperplanes of $\mathbb{P}^n$ containing A, with the convention $\langle A \rangle = \emptyset$ if no such hyperplane exists. For any $P \in \mathbb{P}^n$ the *symmetric rank* or the *symmetric tensor rank* of P is the minimal cardinality $sr(P)$ of a finite subset $S \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(S) \rangle$ (see [5], [4], [3], [2]). The *cactus rank* of P is the minimal degree of a zero-dimensional scheme $Z \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(Z) \rangle$. In many situations we look at some $P \in \mathbb{P}^n$ for which there is a small degree zero-dimensional $Z \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(Z) \rangle$ and $P \notin \langle \nu_d(Z') \rangle$. We may also now some geometric or cohomological properties of Z. Can we find a lower

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bount for $sr(P)$? Of course, if Z is reduced, then $sr(P) \le \deg(Z)$. Even in the case in which Z is reduced, it may useful to know the non-existence of set $S \subset \mathbb{P}^m$ with $S \not\supseteq Z$ and $P \in \langle \nu_d(S) \rangle$ and low $\sharp(S)$.

Fix an integer $t \ge 1$. A zero-dimensional scheme $Z \subset \mathbb{P}^m$ is said to be *in linearly general position* if for any linear subspace $L \subsetneq \mathbb{P}^m$ we have $\deg(Z \cap L) \le \dim(L) + 1$. Z is said to be in *weakly t-general position* if $\deg(Z \cap T) \le \binom{m+t}{m} - 1$ for every degree t hypersurface of $\mathbb{P}^m$. Z is said be in *t-general position* if it is in linearly general position and in weakly s-general position for each integer $s \in \{2, \dots, t\}$. Z is said to be in *deep t-general position* if it is in t-uniform position and for every hypersurface $L \subset \mathbb{P}^m$ with $\deg(L) = t$ we have $h^1(\mathcal{I}_{Z \cap L}(t)) = 0$. We prove the following results.

Theorem 1. *Fix integers $t \ge 2$, $d \ge 3t$, $m \ge 2$, and a zero-dimensional scheme $Z \subset \mathbb{P}^m$ in deep t-general position. Set $k := \lfloor d/t \rfloor$, $x := d - kt$, $z := \deg(Z)$, $\delta := \lceil /(\binom{m+t}{m} - 1) \rceil$. Assume the existence of $P \in \langle \nu_d(Z) \rangle$ such that $P \notin \langle \nu_d(Z') \rangle$ for any scheme $Z' \subsetneq Z$ and of a finite set $S \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(S) \rangle$ and $S \not\supseteq Z$. Set $s := \sharp(S)$ and $\alpha := \sharp(S \cap Z)$. Then $s - \alpha \ge 1 + (\delta - 1)(\binom{m+t-1}{m-1} - 1)$.*

This result is weak, but in the case $t = 1$, i.e. if Z is in linearly general position, we may drastically improve Theorem 1 in the following way.

Theorem 2. *Fix integers $m \ge 2$, $d \ge 6$. Let $Z \subset \mathbb{P}^m$ be a zero-dimensional scheme in linearly general position in $\mathbb{P}^m$. Set $z := \deg(Z)$ and assume $m < z \le (m-1)d + 1$. Set $\gamma := \lceil z/m \rceil$. Assume the existence of $P \in \langle \nu_d(Z) \rangle$ such that $P \notin \langle \nu_d(Z') \rangle$ for any scheme $Z' \subsetneq Z$ and of a finite set $S \subset \mathbb{P}^m$ such that $P \in \langle \nu_d(S) \rangle$ and $S \nsubseteq Z$. Set $s := \sharp(S)$ and $\alpha := \sharp(S \cap Z)$. Then $s - \alpha \ge \min\{md + 1 - z, 2 - m + \gamma(d - \gamma + 1)\}$.*

We may exchange the role of Z and S in the statements of Theorems 1 and 2 and get a new, but equivalent, statement. We work over an algebraically closed field with characteristic 0.

2. Lemmas

Let X be any projective scheme and any effective Cartier divisor D of X. For any zero-dimensional scheme $Z \subset X$ let $\mathrm{Res}_D(Z)$ denote the residual scheme of Z with respect to D, i.e. the closed subscheme of X with $\mathcal{I}_Z : \mathcal{I}_D$ as its ideal sheaf. We have $\mathrm{Res}_D(Z) \subseteq Z$ and $\deg(Z) = \deg(Z \cap D) + \deg(\mathrm{Res}_D(Z))$. For any

Lemma 1. *Fix integers $k \geq 2$, $x > 0$ and a zero-dimensional scheme $E \subset \mathbb{P}^k$ such that $h^1(\mathcal{I}_E(x)) > 0$. Let η be the dimension of the linear subspace of $\mathbb{P}^k$ spanned by E.*

(a) We have $\deg(E) \geq x + 1 + \eta$.

(b) If $\deg(E) \leq 2x + \eta$, then there is a line $D \subset \mathbb{P}^k$ such that $\deg(D \cap E) \geq x + 2$.

Proof. If $\eta = 1$, then the result is obvious. Hence we may assume $\eta \geq 2$. Decreasing if necessary k we may assume $k = \eta$. We may also use induction on k. Part (a) follows from [2], Lemma 34. Assume $\deg(E) \leq 2x + \eta$. Let $H \subset \mathbb{P}^k$ be a hyperplane such that $\deg(E \cap H)$ is maximal. Set $G := \mathrm{Res}_H(E)$. Since E spans $\mathbb{P}^k$, we have $\deg(E \cap H) \geq k - 1$ and $G \neq \emptyset$. First assume $h^1(H, \mathcal{I}_{H \cap E}(x)) > 0$. The inductive assumption gives $\deg(E \cap H \geq x + 1 + k - 1$. Hence $\deg(E) \geq x + 1 + \eta$. If $\deg(E \cap H) \leq 2x + k - 1$, then the inductive assumption gives the existence of a line $D \subset H$ such that $\deg(D \cap H) \geq x + 2$, proving also part (b) in this case. Now assume $h^1(H, \mathcal{I}_{E \cap H}(x)) = 0$. Let η' be the dimension of the linear span of G. From the exact sequence

$$0 \to \mathcal{I}_G(x-1) \to \mathcal{I}_E(x) \to \mathcal{I}_{E \cap H}(x) \to 0 \tag{1}$$

we get $h^1(\mathcal{I}_G(x-1)) > 0$. By induction on x we get $\deg(G) \geq x + \eta'$ if $x \geq 2$ and $\deg(G) \geq 2$ if $x = 1$. In both cases we get $\deg(E) \geq x + 1 + k$. Now assume $\deg(E) \leq 2x + k$. If $\deg(G) \leq 2(x-1) + \eta'$, then there is a line $D \subset \mathbb{P}^m$ such that $\deg(D \cap G) \geq x + 1$. Hence $\deg(D \cap E) \geq x + 1$. The maximality property of H and the assumption that E spans $\mathbb{P}^k$ gives $\deg(E \cap H) \geq x + k$. Since $\deg(G) \geq x + 1$, we get $\deg(E) \geq 2x + 1 + k$, a contradiction. □

Lemma 2. *Fix integers $x > 0$, $k \geq 2$, a hyperplane $H \subset \mathbb{P}^k$ and a zero-dimensional scheme $E \subset \mathbb{P}^k$ such that $\deg(E) - \deg(E \cap H) \leq x$. Then $h^1(\mathcal{I}_E(x)) = h^1(\mathcal{I}_{E \cap H}(x)) = h^1(H, \mathcal{I}_{E \cap H, H}(x))$.*

Proof. Since $E \cap H \subseteq E$ and E is zero-dimensional, we have $h^1(\mathcal{I}_E(x)) \geq h^1(\mathcal{I}_{E \cap H}(x))$. Since the restriction map $H^0(\mathcal{O}_{\mathbb{P}^k}(x)) \to H^0(H, \mathcal{O}_H(x))$ is surjective, we have $h^1(\mathcal{I}_{E \cap H}(x)) = h^1(H, \mathcal{I}_{E \cap H, H}(x))$. Hence it is sufficient to prove $h^1(\mathcal{I}_E(x)) \leq h^1(H, \mathcal{I}_{E \cap H, H}(x))$. The zero-dimensional scheme $\mathrm{Res}_H(E)$ has degree $\deg(E) - \deg(E \cap H) \leq x$. Hence $h^1(\mathcal{I}_{\mathrm{Res}_H(E)}(x)) = 0$ (see [2], Lemma 34). Use the cohomology exact sequence of the exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_H(E)}(x-1) \to \mathcal{I}_{E \cap H, H}(x) \to 0 \tag{2}$$

□

Lemma 3. *Fix integers $y \ge 4$, $v \ge 1$ and a zero-dimensional scheme $E \subset \mathbb{P}^v$ such that $h^1(\mathcal{I}_E(y)) > 0$ and $h^1(H, \mathcal{I}_{E\cap H,H}(y)) = 0$ for every a hyperplane $H \subset \mathbb{P}^v$. Assume $E = E_1 \sqcup E_2$ with E_1 linearly independent and E_2 reduced and spanning $\mathbb{P}^v$. Then $\deg(E_2) \ge v + y$.*

Proof. Since E_1 is linearly independent, $\deg(E_1) \le v+1$ and E_1 spans a linear subspace of dimension $\deg(E_1) - 1$. Since $\deg(E_1) \le v+1$, the lemma is true for $v = 1$. Hence we may assume $v \ge 2$. Fix a hyperplane $H \subset \mathbb{P}^v$ such that $\deg(E_2 \cap H)$ is maximal. By assumption $h^1(H, \mathcal{I}_{E\cap H,H}(y)) = 0$. We have $\mathrm{Res}_H(E) = \mathrm{Res}_H(E_1) \sqcup \mathrm{Res}_H(E_2)$. The case $y = x$ of (2) gives $h^1(\mathcal{I}_{\mathrm{Res}_{H(E)}}(y - 1) > 0$. Since $\mathrm{Res}_H(E_1) \subseteq E_1$ is linearly independent, and $y - 1 \ge 1$, we have $h^1(\mathcal{I}_{\mathrm{Res}_{H(E_1)}}(y-1)) = 0$. Hence the case $y = x$ of (2) gives $\deg \mathrm{Res}_H(E_2)) > 0$. Since we imposed that $\deg(E_2 \cap H)$ is maximal, we get that E_2 spans $\mathbb{P}^v$ and $\deg(E_2 \cap H) \ge v$. Hence it is sufficient to prove $\deg(\mathrm{Res}_H(E_2)) \ge y$. Lemma 1 for $x := y - 1$ gives $\deg(\mathrm{Res}_H(E_2)) \ge y - 1$ and $\deg(\mathrm{Res}_H(E_2)) \ge y$, unless $\mathrm{Res}_H(E_2)$ is contained in a line D and $\deg(D \cap \mathrm{Res}_H(E_2)) = y - 1$. Assume the existence of such a line D. Since E_2 spans $\mathbb{P}^v$, the maximality property of $\deg(E_2 \cap H)$ gives $\deg(E_2 \cap H) \ge (y-1)+v-2$. Hence $\deg(E_2) \ge 2y-2+v-2$. We get $\deg(E_2) \ge v + y$, because $y \ge 4$. □

3. Proofs of Theorems 1 and 2

Proof of Theorem 2. We have $2 \le \gamma \le \lceil (d+1)/2 \rceil$. We may assume $s - \alpha \le md + 1 - z$, because if this inequality is not satisfied, then Theorem 2 is true. Since $P \notin \nu_d(Z'))$ for any $Z' \subsetneq Z$, $\nu_d(Z)$ is linearly independent, i.e. $h^1(\mathcal{I}_Z(d)) = 0$. Taking a proper subset of S if necessary, we may assume $P \in\notin \langle \nu_d(S') \rangle$ for any $S' \subsetneq S$. Hence $h^1(\mathcal{I}_S(d)) > 0$. By [1], Lemma 1, we have $h^1(\mathcal{I}_{Z\cup S}(d)) > 0$, i.e. $\langle \nu_d(Z) \rangle \cap \langle \nu_d(S) \rangle \supsetneq \langle \nu_d(Z \cap S) \rangle$. Hence there is $P_1 \in \langle \nu_d(Z) \rangle \cap \langle \nu_d(S \setminus S \cap Z) \rangle$. Taking P_1 instead of P and $S \setminus S \cap Z$ instead of S we reduce to the case $Z \cap S = \emptyset$. We also reduce to the case in which the linear space $\langle \nu_d(Z) \rangle \cap \langle \nu_d(S) \rangle$ is a unique point. In this case we have $h^1(\mathcal{I}_{W'}(d)) = 0$ for all $W' \subsetneq Z \cup S$. Set $W := Z \cup S$.

Let $a_1 := \sharp(T \cap S)$ be the maximal integers among all hyperplanes $T \subset \mathbb{P}^m$. Among all hyperplanes with $\sharp(S \cap T_1) = a_1$, choose one such that the integer $b_1 := \deg(W_0 \cap T_1)$ is maximal. Set $W_1 := \mathrm{Res}_{T_1}(W)$ and $S_1 := S \setminus S \cap T_1$. For each integer $i \in \{2, \dots, d\}$ define recursively the integers a_i, b_i, hyperplane $T_i \subset \mathbb{P}^m$, the scheme W_i and the set $S_i \subseteq Z_i$ in the following way. Let a_i be the maximal integer $\sharp(S_{i-1} \cap T)$ for some hyperplane $T \subset \mathbb{P}^m$. Among the hyperplanes T such that $a_i = \sharp(S_{i-1} \cap T)$ choose one, T_i, for which $b_i := \deg(W_{i-1} \cap T_i)$ is maximal. Set $W_i := \mathrm{Res}_{T_i}(W_{i-1})$ and $S_i := S_{i-1} \setminus S_{I-1} \cap T_i$. For each integer $x \geq 1$ set $T[x] := \cup_{i=1}^{x} T_i$ with the convention that a hyperplane H appears with multiplicity a in $T[x]$ if $T_i = T$ for a integers $i \in \{1, \dots, x\}$. Notice that the sequence $\{a_i\}_{1 \leq i \leq d}$ is non-decreasing and that if $a_i \leq m-1$, then $S_{i-1} \subset T_i$ and hence $S_i = \emptyset$. Obviously $S_i \subseteq W_i$ and $a_i \leq b_i$. If $b_i \leq m-1$, then $W_{i-1} \subset T_i$ and hence $W_i = \emptyset$. Since $\deg(W) \leq md+1$, we get $b_{d+1} = 0$ (i.e. $W_{d+1} = \emptyset$, i.e. $W \subset T[d+1]$) and $\deg(W_d) \leq 1$. The latter inequality implies $h^1(\mathcal{I}_{W_d}) = 0$. Since $h^1(\mathcal{I}_W(d)) > 0$ and $W \subset T[d+1]$, there is an integer $i \in \{1, \dots, d+1\}$ such that $h^1(\mathcal{I}_{T[i] \cap W}(d)) > 0$. Let g be the minimal such an integer. Since $h^1(\mathcal{I}_{W'}(d)) = 0$ for all $W' \subsetneq W$, we get $W \subset T[g]$. Since Z spans $\mathbb{P}^m$++ ,we have $g \geq 2$.

Notice that $\mathrm{Res}_{T[x]}(W) = W_x$ for all $x > 0$. The minimality of the integer g implies $h^1(\mathcal{I}_{T[g-1] \cap W}(d)) = 0$. Since the restriction map $H^0(\mathcal{O}_{\mathbb{P}^m}(d)) \to H^0(T[g-1], \mathcal{O}_{T[g-1]}(d))$ is surjective, we get $h^1(T[g-1], \mathcal{I}_{T[g-1] \cap W, T[g-1]}(d)) = 0$ From the exact sequence

$$0 \to \mathcal{I}_{W_{g-1}}(d-g+1) \to \mathcal{I}_W(d) \to \mathcal{I}_{T[g-1] \cap W, T[g-1]}(d) \to 0 \tag{3}$$

we get $h^1(\mathcal{I}_{W_{g-1}}(d-g+1)) > 0$. Since $W \subset T[g]$, we have $W_{g-1} \subset T_g$. Hence $0 < h^1(\mathcal{I}_{W_{g-1}}(d-g+1)) = h^1(T_g, \mathcal{I}_{W_{g-1}}(d-g+1))$. Since $h^1(\mathcal{I}_{W_d}) = 0$, we get $g \leq d$. Set $F_1 := Z_{g-1}$ and $F_2 := S_{g-1}$. We have $Z_{g-1} = Z_{g-1} \cap T_g$ and $S_{g-1} = S_{g-1} \cap T_g$. Hence $\deg(F_1) = b_g - a_g$ and $\sharp(F_2) = a_g$. Let $V \subseteq T_g$ be a minimal linear subspace of T_g with the property $h^1(V, \mathcal{I}_{V \cap (F_1 \cup F_2)}(d-g+1)) > 0$. Set $E_i := F_i \cap V$. Since F_1 is in linearly general position, Lemma 2 implies that E_2 spans V. First assume $d-g+1 \geq 4$. Since E_1 is linearly independent in V, we may apply Lemma 2 for the integer $y := d-g+1$ and get $a_g \geq v+(d-g+1)$. Since $S_g \neq \emptyset$, $S_{g-1} \cap T_{g-1}$ spans T_{g-1}. Notice that $S_g \cap T_{g-1} = \emptyset$, because S is reduced. Hence there is a hyperplane M of $\mathbb{P}^m$ containing V and $m-v$ further points of S_{g-1}. The maximality property of the integer a_{g-1} gives $a_{g-1} \geq m + (d-g+1)$. Since $a_i \geq a_{g-1}$ for all $i \leq g-1$, we get $s \geq v + (g-1)m + g(d-g+1)$. Even in the case $v = 1$ we have $a_g \geq 2$. Hence $s \geq 2 + (g-1)m + g(d-g+1)$. Let $\psi : \mathbb{R} \to \mathbb{R}$ be the function $t \to t(d+1+m-t)$. The function ψ is increasing in the interval $[0, (d+m+1)/2]$ and decreasing for $t > (d+m+1)/2$. Notice that $2 \leq \gamma \leq (d+m+1)/2$ and that $\psi(d-2) = (d-3)m+3(d-2)$. First assume $d-2 \geq (d+m+1)/2$, i.e. $d \geq m+5$. We get $s \geq \min\{2-m+\gamma(d-\gamma+1), 2+(d-3)m+3(d-2)\}$. Since $d \geq 6$ and $\gamma \leq \lceil (d+1)/2 \rceil$, If $d-2 < (d+m+1)/2$, then we get $s \geq -m+\gamma(d-\gamma+1)$.

Now assume $d-g+1 \leq 3$, i.e. $g \in \{d-2, d-1, d\}$. Since Z_g is linearly independent, we have $a_g \geq 2$. We have $a_i \geq m$ for all $i < g$. Hence $s - \alpha \geq$

$(d-2)m+2$. □

Proof of Theorem 1. Set $W := Z\cup S$. As in the proof of Theorem 2 we reduce to the case $\alpha = 0$ and the case in which $h^1(\mathcal{I}_{W'}(d)) = 0$ for all $W' \subsetneq W$. We have $h^1(\mathcal{I}_W(d)) > 0$.

We have $x \in \{0,\dots,k-1\}$.

(a) In this step we assume $x = 0$. Set $W_0 = W = Z\cup S$, $W_0 := S$. Let $a_1 := \sharp(T\cap S)$ be the maximal integers among all degree t hypersurfaces $T \subset \mathbb{P}^m$ with degree t (even the reducible ones). Among all degree t hypersurfaces with $\sharp(S\cap T_1) = a_1$, choose one (not necessarily reduced or irreducible) such that the integer $b_1 := \deg(W_0\cap T_1)$ is maximal. Set $W_1 := \mathrm{Res}_{T_1}(W)$ and $S_1 := S\setminus S\cap T_1$. For each integer $i \in \{2,\dots,k\}$ (if any) define recursively the integers a_i, b_i, the degree t hypersurface $T_i \subset \mathbb{P}^m$, the scheme W_i and the set $S_i \subseteq Z_i$ in the following way. Let a_i be the maximal integer $\sharp(S_{i-1}\cap T)$ for some degree t hypersurface $T \subset \mathbb{P}^m$. Among the degree t hypersurfaces T such that $a_i = \sharp(S_{i-1}\cap T)$ choose one, T_i, for which $b_i := \deg(W_{i-1}\cap T_i)$ is maximal. Set $W_i := \mathrm{Res}_{T_i}(W_{i-1})$ and $S_i := S_{i-1}\setminus S_{i-1}\cap T_i$. For any $a \in \{1,\dots,k\}$ set $T[a] := \cup_{i=1}^{a} T_i$, with the convention that a degree t hypersurface T appears with multiplicity b in the effective divisor $T[a] \subset \mathbb{P}^m$ if there are b integers $i \in \{1,\dots,a\}$ with $T_i = T$. Notice that if $b_i \le \binom{m+t-1}{m-1} - 1$ and $i \le k-1$, then $W_{i-1} \subset T_i$ and hence $W_i = \emptyset$. Since $z+s \le k(\binom{m+t-1}{m-1} - 1)$, we have $W \subset T[k]$. Since $h^1(\mathcal{I}_W(d)) > 0$, there is a minimal integer $g \in \{1,\dots,k\}$ such that $h^1(\mathcal{I}_{W\cap T[g]}(d)) > 0$. The minimality condition implies $W \subset T[g]$. Hence $s = \sum_{i=1}^{g} a_i$. Since Z is in t-general position, we have $g \ge \delta$. Look at the residual exact sequence

$$0 \to \mathcal{I}_{W_g}(d-(g-1)t) \to \mathcal{I}_W(d) \to \mathcal{I}_{W\cap T[g-1],T[g-1]}(d) \to 0 \tag{4}$$

We have $h^1(T[g-1], \mathcal{I}_{W\cap T[g-1],T[g-1]}(d)) = h^1(\mathcal{I}_{W\cap T[g-1]}(d)) = 0$. Hence (4) gives $h^1(W_g, \mathcal{I}_{W_g}(d-(g-1)t) > 0$. Since $d-(g-1)t \ge t$, $Z_g \subseteq Z$ and Z is in deep t-linear position, we have $h^1(\mathcal{I}_{Z_g}(d-(g-1)t)) = 0$. Hence (4) gives $S_g \ne \emptyset$. Notice that the sequence $\{a_i\}_{1\le i\le k}$ is non-decreasing and that if $a_i \le \binom{m+t}{m} - 1$, then $S_i \subset T_i$ and hence $\mathrm{Res}_{T_i}(S_i) = \emptyset$. Hence $s - \alpha = \sum_{i=1}^{g} a_i \ge 1 + (g-1)(\binom{m+t}{m} - 1) \ge 1 + (\delta-1)(\binom{m+t}{m} - 1)$.

(b) Now assume $x > 0$. We repeat the previous construction with the following modifications. Let $T_{k+1} \subset \mathbb{P}^m$ be any degree x hypersurface containing W_x (it exists, because $z+s-\alpha \le k(\binom{m+t}{m} - 1) + \binom{m+x}{m} - 1$. Set $T[k+1] := T[k]\cup T_{k+1}$. Let g be the minimal integer $\le k+1$ such that $h^1(\mathcal{I}_{T[g]\cap W}(d)) > 0$. If $g \le k$, then the proof of part (a) works. Now assume

$g = k+1$. Now we only get $S_{g-1} \neq \emptyset$. Hence $a_i \geq (\binom{m+t}{m} - 1)$ for all $i \leq k-1$ and $a_k > 0$. Hence $s - \alpha \geq 1 + (k-1)(\binom{m+t}{m} - 1)$. $\square$

Lemma 4. *Let $T \subset \mathbb{P}^4$ be the cone with vertex O and as a basis rational normal curve $D \subset \mathbb{P}^3$. Let $Y \subset T$ be any curve such that $O \notin T$. Then* $\deg(Y) \equiv 0 \pmod 3$, *Y is the complete intersection of T and a hypersurface of degree $a := \deg(Y)/3$ and $p_a(Y') = 1 + a(9a-15)/2$.*

Proof. Let $u : F_3 \to \mathbb{P}^4$ be the morphism induced by the complete linear system $|h + 3f|$. The morphism u is an embedding outside h and contracts h to a point. Up to a projective transformation we may assume $T = u(F_3)$. In this case $O = u(h)$. Since $O \notin Y$, there is a curve $Y' \subset F_3$ such that $Y' \cap h = \emptyset$ and u' induces an isomorphism $Y' \to Y$. Take integers a, b' such that $Y' \in |ah + b'f|$. Since $O \notin Y$, we have $h \cap Y' = \emptyset$. Hence $b' = 3a$. We have $b' = 3a$. Since $\deg(Y) = (ah + b'f) \cdot (h + 3f)$, we get $\deg(Y) = 3a$. Hence Y is scheme-theoretically the zero-locus of a section of $H^0(T, \mathcal{O}_T(a))$. Since T is arithmetically Cohen-Macaulay, the restriction map $H^0(\mathbb{P}^4, \mathcal{O}_{\mathbb{P}^4}(a)) \to H^0(T, \mathcal{O}_T(a))$ is surjective. Hence Y is the complete intersection of T and a degree a hypersurface. Since $Y \cong Y'$, have $p_a(Y) = p_a(Y')$. Since $\omega_{F_3} \cong \mathcal{O}_{F_3}(-2h - 5f)$ ++, the adjunction formula gives Hence $2p_a(Y') - 2 = (ah + 3af) \cdot ((a-2)h + (3a-5)f) = 3a(3a-5)$. $\square$

Lemma 5. *Let $T \subset \mathbb{P}^4$ be the cone surface with vertex O and as a bases a singular integral curve $D \subset \mathbb{P}^3$ with degree 4. Let $Y \subset T$ be any curve such that $O \notin Y$. Then Y is singular.*

Proof. Assume that Y is smooth. Looking separately at each connected component of Y we reduce to the case in which Y is connected. Notice that D is a rational curve with a unique singular point. Let $\alpha : \mathbb{B} \to \mathbb{P}^4$ be the blowing up of O and $\square$

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