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CARTESIAN CLOSED ALGEBRAIC CATEGORIES

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Abstract: It is well known that the category $\boldsymbol{Alg(1)}$ of all universal algebras with one unary operation is cartesian closed. We extend this result for the category $\boldsymbol{Alg(\Sigma)}$ of universal algebras of a fixed type Σ all of whose operations are unary. That is, we show that the category $\boldsymbol{Alg(\Sigma)}$ is cartesian closed. In fact, we characterize some cartesian closed full isomorphism closed subcategories of $\boldsymbol{Alg(\Sigma)}$ using a concept called T-friendly.

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1. Introduction

The category $\boldsymbol{Top}$ of topological spaces and continuous maps is observed to be inadequate (for example, products and quotients in $\boldsymbol{Top}$ do not commute) and is replaced by some convenient topological categories which happen to be cartesian closed (see [4], [16]). An extensive study has been made on carte-

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sian closed topological categories (see, for example, [4], [16], [1], [3], [9], [10], [11], [12], [13], [14], and [15]), in particular, some characterizations are obtained for cartesian closedness of topological categories (see [9]) and monotopological categories (see [13]). Cartesian closed topological categories are especially important in the theory of topological algebra for the construction of free algebras over topological objects and, obviously, for the construction of quotients (see [14] and [15]).

The present author studied cartesian closedness of topologically algebraic categories of universal algebras. It turned out that the category of universal algebras has to be cartesian closed whenever the associated category of topological algebras is cartesian closed (see [8]). This created an interest in finding cartesian closed categories of universal algebras. Even though, there has been an ample work found in the literature on monoidal closed categories of universal algebras (see, for example, [7] and [6]), it seems a little has been done about cartesian closed algebraic categories.

It is well known that the category $\boldsymbol{Alg(1)}$ of all universal algebras with one unary operation is cartesian closed (see, [2] page 408). We extend this result for the category $\boldsymbol{Alg(\Sigma)}$ of universal algebras of a fixed type Σ all of whose operations are unary. That is, we show that the category $\boldsymbol{Alg(\Sigma)}$ is cartesian closed. In fact, we characterize some cartesian closed full isomorphism closed subcategories of $\boldsymbol{Alg(\Sigma)}$ using a concept called T-friendly. It turns out that a full isomorphism closed subcategory $\boldsymbol{A}$ of $\boldsymbol{Alg(\Sigma)}$ containing a free $\boldsymbol{A}$-object T on a singleton is cartesian closed iff the set $\hom_{\boldsymbol{A}}(A \times T, B)$ of all $\boldsymbol{A}$-morphisms from $A \times T$ to B is an $\boldsymbol{A}$-object with a certain Σ-structure for any two $\boldsymbol{A}$-objects A and B.

We will also describe full isomorphism closed subcategories of the category $\boldsymbol{Alg(\Omega)}$ of universal algebras of a fixed type Ω that have canonical function spaces. It turns out that any object A in a category of universal algebras with one n-ary operation (n is any fixed positive integer) with canonical function spaces has a simple algebraic structure. Namely, there exist n universal algebras $A_1, \dots, A_n$ such that n-ary operation on A_i is the i-th canonical projection $A_i^n \to A_i$ $(1 \leq i \leq n)$ and A is isomorphic to the product $A_1 \times \cdots \times A_n$.

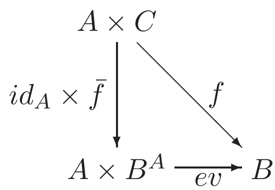
2. Preliminaries

A family $\Omega = (n_j)_{j \in J}$ of natural numbers indexed by some set J is called a **type**. The index set J is called the **order** of Ω. In the following, we let a type $\Omega = (n_j)_{j \in J}$ be fixed. A pair $(A, (\omega_j)_{j \in J})$ of a set A and a family $\omega_j : A^{n_j} \to A$

($j \in J$) of mappings is called an **Ω-algebra** (see, for example, [5]). For the sake of simplicity, we write A instead of of the pair $(A, (\omega_j)_{j \in J})$ and $\omega_{j,A}$ for the **n_j-ary operation** ω_j on A. If the Ω-algebra A is clear from the context, we drop the suffix A in denoting its n_j-ary ($j \in J$) operation. Moreover, the symbol Σ is used instead of Ω in case $n_j = 1$ for each $j \in J$. A mapping $f : A \to B$ between two Ω-algebras A and B is said to be an **Ω-homomorphism** iff for each $j \in J$, $f \circ \omega_{j,A} = \omega_{j,B} \circ f^n$ where $n = n_j$ and $f^n : A^n \to B^n$ is the mapping with the obvious definition $(a_1, \ldots, a_n) \to (fa_1, \ldots, fa_n)$.

The symbol $\boldsymbol{Alg(\Omega)}$ denotes the category whose objects are Ω-algebras and whose morphisms are Ω-homomorphisms.

A category $\boldsymbol{A}$ is called **cartesian closed** iff it has finite products and for each $\boldsymbol{A}$-object A the functor $(A \times -) : \boldsymbol{A} \to \boldsymbol{A}$ is co-adjoint (see, for example, [2] page 407). Equivalently, a category with finite products is cartesian closed iff for any two $\boldsymbol{A}$-objects A and B there exists an object B^A and a $\boldsymbol{A}$-morphism $ev : A \times B^A \to B$ such that for each $\boldsymbol{A}$-morphism $f : A \times C \to B$ there exists a unique $\boldsymbol{A}$-morphism $\bar{f} : C \to B^A$ so that the diagram



(id_A is the identity map on A) commutes. B^A is called a **power object** and $\bar{f} : C \to B^A$ is called the **exponential morphism** for $f : A \times C \to B$.

A construct (i.e., a concrete category over $\boldsymbol{Set}$) $\boldsymbol{A}$ is said to have **canonical function spaces** iff $\boldsymbol{A}$ has finite concrete products, $\boldsymbol{A}$ is cartesian closed, and $ev : A \times B^A \to B$ can be chosen such a way that the underlying set for the power object B^A is the set of all $\boldsymbol{A}$-morphisms from A to B and ev is the restriction of the canonical evaluation map in $\boldsymbol{Set}$.

Equivalently, a construct $\boldsymbol{A}$ has **canonical function spaces** iff $\boldsymbol{A}$ has finite concrete products, $\boldsymbol{A}$ is cartesian closed, and each constant function between $\boldsymbol{A}$-objects is an $\boldsymbol{A}$-morphism (see [2], page 415).

In this work, we assume that all subcategories are full and isomorphism closed. The fact that the most of the natural subcategories fall into this class justifies our assumption.

3. Cartesian Closed Subcategories of Alg(Σ)

In the following discussion, we reserve the symbol T for a Σ-algebra generated by a distinguished member $\mathbf{0}$ in T (i.e., for any $t \in T$ there exist $j_1 \in J, \ldots, j_n \in J$ such that $\omega_{j_n,T} \circ \ldots \circ \omega_{j_1,T}(\mathbf{0}) = t$), equipped with a family $(\delta_j)_{j\in J}$ of Σ-homomorphisms $\delta_j : T \to T$ satisfying

$$\delta_j(\mathbf{0}) = \omega_{j,T}(\mathbf{0}). \tag{3.0}$$

A Σ-algebra A is said to be **T-friendly** iff for each $t \in T$ there exists a function $A_t : A \to A$, such that

$$A_{\mathbf{0}} = id_A, \tag{3.1}$$

$$\omega_{j,A} \circ A_t = A_{\omega_{j,T}(t)}, \tag{3.2}$$

and

$$A_t \circ \omega_{j,A} = A_{\delta_j(t)} \tag{3.3}$$

for all $j \in J$ and for all $t \in T$.

A subcategory $\boldsymbol{A}$ of $\boldsymbol{Alg(\Sigma)}$ is called **T-friendly** iff each $\boldsymbol{A}$-object is T-friendly. Consider the following examples.

Example 3.1. Equip the set $\mathbf{N}$ of nonnegative integers with the unary operation $\omega : \mathbf{N} \to \mathbf{N}$ that maps any integer n to its successor $n+1$ and take $\delta = \omega$. (The distinguished member of $\mathbf{N}$ is the integer zero.) For any object A in $\boldsymbol{Alg(1)}$ with a unary operation u, define $A_n := u^n$ to be the composition of u to itself n times. It is straight forward to see that $\boldsymbol{Alg(1)}$ is $\mathbf{N}$-friendly.

Example 3.2. Let T be the set of all n-tuples ($n \geq 1$) of members of J together with a distinguished point $\mathbf{0}$. For each $j \in J$, define

$$\omega_j(\mathbf{0}) = (j), \omega_j(j_1, \ldots, j_n) = (j_1, \ldots, j_n, j),$$

$$\delta_j(\mathbf{0}) = (j), \text{ and } \delta_j(j_1, \ldots, j_n) = (j, j_n, \ldots, j_1).$$

Then T is a Σ-algebra with the unary operations ω_j (T is the Σ-word algebra on the alphabet $\{\mathbf{0}\}$) and δ_j's are Σ-homomorphisms. For any Σ-algebra A, define

$$A_{\mathbf{0}} := id_A$$

and

$$A_{(j_1,\ldots,j_n)} = \omega_{j_n,A} \circ \ldots \circ \omega_{j_1,A}.$$

With these definitions, $\boldsymbol{Alg(\Sigma)}$ is T-friendly.

Example 3.3. If $(J, +, \mathbf{0})$ is a monoid, then the subcategory $\boldsymbol{A}$ of $\boldsymbol{Alg(\Sigma)}$ consisting of all J-sets (A Σ-algebra A is said to be a **J-set** iff $\omega_{\mathbf{0},A} = id_A$ and $\omega_{j,A} \circ \omega_{j',A} = \omega_{j+j',A}$ for any $j \in J$, $j' \in J$.) is J-friendly: Indeed, the unary operations $\omega_{j,J}(j \in J)$ on J are given by $\omega_{j,J}(j') = j + j'$ $(j' \in J)$ and the homomorphisms δ_j $(j \in J)$ on J are defined by $\delta_j(j') = j' + j$ $(j' \in J)$. For any $\boldsymbol{A}$-object A and any $j \in J$, A_j is the n_j-ary operation $\omega_{j,A}$ on A.

If A and B are any two Σ-algebras, then we write $\hom_{\boldsymbol{A}}(A \times T, B)$ for the Σ-algebra whose underlying set is the set of all Σ-homomorphisms from $A \times T$ to B and whose n_j-ary (unary) operation ω_j assigns the Σ-homomorphism $f \circ (id_A \times \delta_j)$ for any morphism $f : A \times T \to B$ in $\hom_{\boldsymbol{A}}(A \times T, B)$, i.e.,

$$\omega_j f = f \circ (id_A \times \delta_j). \tag{3.4}$$

If A is T-friendly and $a \in A$, then define $g_{A,a} : T \to A$ by

$$g_{A,a}(t) := A_t a \tag{3.5}$$

for any $t \in T$.

Lemma 3.4. *If A is T-friendly, then $g_{A,a}$ is a Σ-homomorphism and*

$$g_{A,\omega_j a} = g_{A,a} \circ \delta_j \tag{3.6}$$

for any $a \in A$ and $j \in J$.

Proof. $g_{A,a}$ is a Σ-homomorphism by (3.2) and (3.6) follows from (3.3). □

Proposition 3.5. *Suppose $\boldsymbol{A}$ is a subcategory of $\boldsymbol{Alg(\Sigma)}$ closed under finite products such that $\boldsymbol{A}$ is T-friendly and $\hom_{\boldsymbol{A}}(A \times T, B)$ is an $\boldsymbol{A}$-object for any two $\boldsymbol{A}$-objects A and B, then $\boldsymbol{A}$ is cartesian closed.*

The power object B^A associated with A and B is $\hom_{\boldsymbol{A}}(A \times T, B)$. The exponential morphism $\bar{f} : C \to B^A$ for the $\boldsymbol{A}$-morphism $f : A \times C \to B$ is given by

$$\bar{f}(c) := f \circ (id_A \times g_{C,c}) \tag{3.7}$$

for any $c \in C$. The evaluation morphism $ev : A \times B^A \to B$ is defined by

$$ev(a, f) := f(a, \mathbf{0}) \tag{3.8}$$

for any $a \in A$ and $f \in B^A$.

Proof. It is enough to show that $\bar{f}$ and ev are $\boldsymbol{A}$-morphisms such that $ev \circ (id_A \times \bar{f} = f$ and that $\bar{f}$ is unique with this property (see [2], pages 407, 408).

$\bar{f}$ is an $\boldsymbol{A}$-morphism because, for any $c \in C$ and $j \in J$,

$$\begin{aligned} \bar{f}(\omega_j c) &= f \circ (id_A \times g_{C,\omega_j c}) \text{ by } (3.7) \\ &= f \circ (id_A \times g_{C,c} \circ \delta_j) \text{ by } (3.6) \\ &= \omega_j[f \circ (id_A \times g_{C,c})] \text{ by } (3.4) \\ &= \omega_j[\bar{f}(c)] \text{ by } (3.7). \end{aligned}$$

ev is an $\boldsymbol{A}$-morphism:

$$\begin{aligned} ev[\omega_j(a,f)] &= ev(\omega_j a, \omega_j f) \\ &= (\omega_j f)(\omega_j a, \mathbf{0}) \text{ by } (3.8) \\ &= f(\omega_j a, \delta_j \mathbf{0}) \text{ by } (3.4) \\ &= f(\omega_j a, \omega_j \mathbf{0}) \text{ by } (3.0) \\ &= \omega_j f(a, \mathbf{0}) \text{ since } f \text{ is an } \boldsymbol{A}\text{-morphism} \\ &= \omega_j[ev(a,f)] \text{ by } (3.8). \end{aligned}$$

$$\begin{aligned} ev \circ (id_A \times \bar{f})(a,c) &= ev(a, \bar{f}(c)) \\ &= \bar{f}(c)(a, \mathbf{0}) \text{ by } (3.8) \\ &= f(a, C_{\mathbf{0}} c) \text{ by } (3.7) \\ &= f(a,c) \text{ by } (3.1). \end{aligned}$$

Thus $ev \circ (id_A \circ \bar{f}) = f$. It remains to show that $\bar{f}$ is unique with this property. Suppose $g : C \to B^A$ is any $\boldsymbol{A}$-morphism such that $ev \circ (id_A \circ g) = f$. Let $c \in C$, $a \in A$, and $t \in T$. Choose indices $j_1 \in J, \ldots, j_n \in J$ such that

$$\omega_{j_n} \circ \ldots \circ \omega_{j_1}(\mathbf{0}) = t.$$

Using the fact that δ_j's are Σ-homomorphisms agreeing with ω_j's at $\mathbf{0}$ (by (3.0)), we can see that

$$\delta_{j_1} \circ \ldots \circ \delta_{j_n}(\mathbf{0}) = t.$$

$$\begin{aligned} g(\omega_{j_n} \circ \ldots \circ \omega_{j_1}(c))(a,0) &= [(\omega_{j_n} \circ \ldots \circ \omega_{j_1})(g(c))](a,0) \\ &= g(c)(a, \delta_{j_1} \circ \ldots \circ \delta_{j_n}(\mathbf{0})) \end{aligned}$$

$$= \quad g(c)(a,t).$$

Thus

$$g(\omega_{j_n} \circ \ldots \circ \omega_{j_1}(c))(a,0) = g(c)(a,t).$$

Similarly,

$$\bar{f}(\omega_{j_n} \circ \ldots \circ \omega_{j_1}(c))(a,0) = \bar{f}(c)(a,t).$$

However, the left hand side of the above two equalities is the same as $f(a, \omega_{j_n} \circ \ldots \circ \omega_{j_1}(c))$ because $ev \circ (id_A \times g) = f$ and $ev \circ (id_A \times \bar{f}) = f$. Thus $\bar{f}(c)(a,t) = g(c)(a,t)$. This means $\bar{f} = g$. □

Corollary 3.6. *$\boldsymbol{Alg}(\boldsymbol{\Sigma})$ is cartesian closed. If $(J, +, \mathbf{0})$ is a monoid, then the category of J-sets is cartesian closed.*

Proof. This follows from the above proposition and the discussion in examples (3.2) and (3.3). Note that, if A and B are J-sets then $\hom_{\boldsymbol{A}}(A \times J, B)$ is a J-set because $\delta_{j+j'} = \delta_{j'} \circ \delta_j$. □

We now establish the converse of Proposition (3.5). Namely:

Proposition 3.7. *Any cartesian closed subcategory $\boldsymbol{A}$ of $\boldsymbol{Alg}(\boldsymbol{\Sigma})$ containing an object T such that $\boldsymbol{A}$ is T-friendly contains $\hom_{\boldsymbol{A}}(A \times T, B)$ for any two $\boldsymbol{A}$-objects A and B.*

Moreover, the power objects, the exponential morphisms, and the evaluation morphisms in $\boldsymbol{A}$ are described as in Proposition (3.5).

Proof. Suppose A and B are any two $\boldsymbol{A}$-objects. Write D for the power object B^A in $\boldsymbol{A}$ and let $EV : A \times D \to B$ be the evaluation morphism in $\boldsymbol{A}$. Define $\varphi : D \to \hom_{\boldsymbol{A}}(A \times T, B)$ by

$$\varphi(d) = EV \circ (id_A \times g_{D,d})$$

($g_{D,d}$, defined in (3.5), is the exponential morphism for $\varphi(d)$) for any $d \in D$.

We show that φ is an isomorphism. If $\varphi(d) = \varphi(d')$, then $g_{D,d} = g_{D,d'}$ by the uniqueness of an exponential morphism. In particular, $d = g_{D,d}(\mathbf{0}) = g_{D,d'}(\mathbf{0}) = d'$. Thus φ is one-one.

If $f \in \hom_{\boldsymbol{A}}(A \times T, B)$, then there exists a unique $\boldsymbol{A}$-morphism $\bar{f} : T \to D$ such that $EV \circ (id_A \times \bar{f}) = f$. Writing $\bar{f}(\mathbf{0}) = d$, we see that $\bar{f} = g_{D,d}$ since these two $\boldsymbol{A}$-morphisms agree at the generator $\mathbf{0}$ of T. Hence $\varphi(d) = f$. Therefore, φ is onto. Suppose $d \in D$ and $j \in J$. Then

$$\varphi(\omega_j d) \quad = \quad EV \circ (id_A \times g_{D,\omega_j d})$$

$$\begin{aligned} &= EV \circ (id_A \times g_{D,d} \circ \delta_j) hboxby (3.6) \\ &= \omega_j [EV \circ (id_A \times g_{D,d})] \text{ by } (3.4) \\ &= \omega_j \varphi(d). \end{aligned}$$

Thus φ is an isomorphism and clearly $EV(a,d) = ev(a, \varphi(d))$, where ev is defined as in (3.8). Since $\boldsymbol{A}$ is full isomorphism closed subcategory of $\boldsymbol{Alg(\Sigma)}$, $\hom_{\boldsymbol{A}}(A \times T, B)$ can be taken as a power object associated with A and B, ev as the evaluation morphism and the exponential morphism $\bar{f} : C \to D$ for $f : A \times C \to B$ such that $EV \circ (id_A \times \bar{f}) = f$ may be replaced by $\varphi \circ \bar{f}$. □

Corollary 3.8. *A subcategory $\boldsymbol{A}$ of $\boldsymbol{Alg(\Sigma)}$ closed under finite products containing an object T such that $\boldsymbol{A}$ is T-friendly is cartesian closed iff $\hom_{\boldsymbol{A}}(A \times T, B)$ is an $\boldsymbol{A}$-object for any two $\boldsymbol{A}$-objects A and B.*

Corollary 3.9. *A subcategory $\boldsymbol{A}$ of $\boldsymbol{Alg(\Sigma)}$ closed under finite products containing a free $\boldsymbol{A}$-object T on a singleton is cartesian closed iff $\hom_{\boldsymbol{A}}(A \times T, B)$ is an $\boldsymbol{A}$-object for any two $\boldsymbol{A}$-objects A and B.*

Proof. It is sufficient to prove that $\boldsymbol{A}$ is T-friendly. Assume that T is a free $\boldsymbol{A}$-object on the set $\{\mathbf{0}\}$. Then clearly $\delta_j : T \to T$ is the unique Σ-homomorphism that extends the association $\delta_j(\mathbf{0}) = \omega_{j,T}(\mathbf{0})$ required by (3.0). Suppose A is an $\boldsymbol{A}$-object. We show that A is T-friendly. For any $a \in A$, write $f_{A,a}$ for the unique Σ-homomorphism $f_{A,a} : T \to A$ such that

$$f_{A,a}(\mathbf{0}) = a. \tag{3.9}$$

For any $t \in T$, define $A_t : A \to A$ by

$$A_t(a) = f_{A,a}(t) \tag{3.10}$$

for any $a \in A$.

(3.1) is clear from (3.9).

For any $j \in J$, $t \in T$, and $a \in A$,

$$\begin{aligned} A_{\omega_{j,T}(t)}(a) &= f_{A,a}(\omega_{j,T}(t)) \text{ by } (3.10) \\ &= \omega_{j,A} \circ f_{A,a}(t) \text{ since } f_{A,a} \text{ is a } \Sigma\text{-homomorphism} \\ &= \omega_{j,A} \circ A_t(a) \text{ by } (3.10), \end{aligned}$$

which implies that (3.2) is valid. To verify (3.3), first note that

$$f_{A,\omega_{j,A}(a)}(\mathbf{0}) = \omega_{j,A}(a) \text{ by } (3.9)$$

$$
\begin{aligned}
&= \omega_{j,A} \circ f_{A,a}(\mathbf{0}) \text{ by (3.9)} \\
&= f_{A,a} \circ \omega_{j,T}(\mathbf{0}) \text{ since } f_{A,a} \text{ is a } \Sigma\text{-homomorphism} \\
&= f_{A,a} o \delta_j(\mathbf{0}) \text{ by (3.0).}
\end{aligned}
$$

Since $f_{A,\omega_{j,A}(a)}$ and $f_{A,a} \circ \delta_j$ are Σ-homomorphisms agreeing at $\mathbf{0}$, we have

$$f_{A,\omega_{j,A}(a)} = f_{A,a} \circ \delta_j. \tag{3.11}$$

Hence,

$$
\begin{aligned}
A_t \circ \omega_{j,A}(a) &= A_t(\omega_{j,A}(a)) \\
&= f_{A,\omega_{j,A}(a)}(t) \text{ by (3.10)} \\
&= f_{A,a} \circ \delta_j(t) \text{ by (3.11)} \\
&= A_{\delta_j(t)}(a) \text{ by (3.10).}
\end{aligned}
$$

Thus (3.3) is valid. □

4. Subcategories of Alg(Ω) with Canonical Function Spaces

For any set A and for any positive integer n, $d_{n,A}$ denotes the map from A to A^n that assigns to each member a of A the n-tuple $(a, \dots, a)$ each of whose coordinates are equal to a. The following proposition shows that any subcategory of ***Alg*(Σ)** with canonical function spaces is simply ***Set***.

Proposition 4.1. *Suppose **A** is a subcategory of **Alg(Ω)** that is closed under finite products. Then*

*(a) constant functions between any two **A**-objects are **A**-morphisms iff for any **A**-object A and for any $j \in J$,*

$$\omega_{j,A} \circ d_{n_j,A} = id_A; \tag{4.1}$$

*(b) **A** has canonical function spaces iff equation (4.1) holds and for any two **A**-objects A and B, the set $H := \hom_{\boldsymbol{A}}(A, B)$ of all **A**-morphisms from A to B is an **A**-object such that*

$$\omega_{j,H}(f_1, \dots, f_n) = \omega_{j,B} \circ (f_1 \times \cdots \times f_n) \circ d_{n,A}, \tag{4.2}$$

$$\omega_{j,B}[\omega_{j,B}(f_1 a_1, \dots, f_1 a_n), \dots, \omega_{j,B}(f_n a_1, \dots, f_n a_n)] = \omega_{j,B}(f_1 a_1, \dots, f_n a_n) \tag{4.3}$$

where $j \in J$, $n = n_j$, $f_1 \in H, \ldots, f_n \in H$, $a_1 \in A, \ldots, a_n \in A$.

In this case exponential object B^A *in* $\boldsymbol{A}$ *is* $\hom_{\boldsymbol{A}}(A, B)$*; the exponential morphism* $\bar{f} : C \to B^A$ *in* $\boldsymbol{A}$*, associated with an* $\boldsymbol{A}$*-morphism* $f : A \times C \to B$*, is defined by*

$$\bar{f}(c)(a) = f(a, c); \tag{4.4}$$

and the evaluation morphism $ev : A \times B^A \to B$ *in* $\boldsymbol{A}$ *is a restriction of the canonical evaluation map in* ***Set***.

Proof. Note that a constant function $f : B \to A$ with the constant value $a \in A$ is an $\boldsymbol{A}$-morphism iff for any $j \in J$ $(n = n_j)$, $\omega_{j,A} \circ f^n = f \circ \omega_{j,B}$, or, $\omega_{j,A} \circ d_{n,A}(a) = a$. Thus statement (a) is valid. Suppose $\boldsymbol{A}$ has canonical function spaces. Then clearly (4.1) holds since constant functions are $\boldsymbol{A}$-morphisms. $\boldsymbol{A}$ is closed under $\hom_{\boldsymbol{A}}$ functor by the definition. Assume that A and B are any two $\boldsymbol{A}$-objects and let $H := \hom_{\boldsymbol{A}}(A, B)$. If $f_1 \in H, \ldots, f_n \in H$, and $a \in A$, then, since the restriction $ev : A \times H \to B$ of the canonical evaluation map in ***Set*** is an evaluation morphism in $\boldsymbol{A}$, we have

$$\begin{aligned}
[\omega_{j,H}(f_1, \ldots, f_n)](a) &= ev[a, \omega_{j,H}(f_1, \ldots, f_n)] \\
&= ev[\omega_{j,A}(a, \ldots, a), \omega_{j,H}(f_1, \ldots, f_n)] \text{ by (4.1)} \\
&= ev[\omega_{j,A \times H}((a, f_1), \ldots, (a, f_n))] \\
&= \omega_{j,B}[ev(a, f_1), \ldots, ev(a, f_n)], \\
&\qquad\qquad \text{since } ev \text{ is an } \boldsymbol{A} - \text{morphism}, \\
&= \omega_{j,B}(f_1 a, \ldots, f_n a).
\end{aligned}$$

Thus (4.2) holds. Moreover, if $a_1 \in A, \ldots, a_n \in A$, then

$$\begin{aligned}
\omega_{j,B}(f_1 a_1, \ldots, f_n a_n) &= \omega_{j,B}(ev(a_1, f_1), \ldots, ev(a_n, f_n)) \\
&= ev[\omega_{j,A}(a_1, \ldots, a_n), \omega_{j,H}(f_1, \ldots, f_n)], \\
&\qquad\qquad \text{since } ev \text{ is an } \boldsymbol{A} - \text{morphism}, \\
&= [\omega_{j,H}(f_1, \ldots, f_n)](\omega_{j,A}(a_1, \ldots, a_n)) \\
&= \omega_{j,B}[f_1(\omega_{j,A}(a_1, \ldots, a_n)), \ldots, f_n(\omega_{j,A}(a_1, \ldots, a_n))] \\
&\qquad\qquad \text{by (4.2)} \\
&= \omega_{j,B}(\omega_{j,B}(f_1 a_1, \ldots, f_1 a_n), \ldots, \omega_{j,B}(f_n a_1, \ldots, f_n a_n)).
\end{aligned}$$

Conversely, assume that $\boldsymbol{A}$ is closed under $\hom_{\boldsymbol{A}}$ and satisfies (4.1) - (4.3). Assume A, B, and C are $\boldsymbol{A}$-objects, and $f : A \times C \to B$ is an $\boldsymbol{A}$-morphism.

Write $H := \hom_{\boldsymbol{A}}(A, B)$. Since $\boldsymbol{A}$ is closed under the formation of finite products, $A \times C$ has the obvious algebraic structure. Since the diagram

$$\begin{array}{ccc} A \times C & & \\ {\scriptstyle id_A \times \bar{f}} \downarrow & \searrow^{f} & \\ A \times H & \xrightarrow[ev]{} & B \end{array}$$

commutes ($\bar{f}$ is defined by (4.4)), it is enough to show that $\bar{f}$ and ev are $\boldsymbol{A}$-morphisms. Let $j \in J$, $n = n_j$, and $c_1, \ldots c_n \in C$. For any $a \in A$,

$$\begin{aligned} \bar{f}(\omega_{j,C}(c_1, \ldots, c_n))(a) &= f(a, \omega_{j,C}(c_1, \ldots, c_n)) \text{ by (4.4)} \\ &= f(\omega_{j,A}(a, \ldots, a), \omega_{j,C}(c_1, \ldots, c_n)) \text{ by (4.1)} \\ &= f(\omega_{j,A\times C}((a, c_1), \ldots, (a, c_n))) \\ &= \omega_{j,B}(f(a, c_1), \ldots, f(a, c_n)) \\ &= \omega_{j,B}(\bar{f}(c_1)a, \ldots, \bar{f}(c_n)a) \\ &= \omega_{j,H}(\bar{f}(c_1), \ldots, \bar{f}(c_n))(a) \text{ by (4.2).} \end{aligned}$$

Thus $\bar{f} \circ \omega_{j,C} = \omega_{j,H} \circ \bar{f}^n$. Hence $\bar{f}$ is an $\boldsymbol{A}$-morphism. Similarly, if $a_1, \ldots, a_n \in A$ and $f_1, \ldots, f_n \in H$, then

$$\begin{aligned} & ev[\omega_{j,A\times H}((a_1, f_1), \ldots, (a_n, f_n))] = ev[\omega_{j,A}(a_1, \ldots, a_n), \omega_{j,H}(f_1, \ldots, f_n)] \\ = \; & [\omega_{j,H}(f_1, \ldots, f_n)](\omega_{j,A}(a_1, \ldots, a_n)) \\ = \; & \omega_{j,B}[f_1(\omega_{j,A}(a_1, \ldots, a_n)), \ldots, f_n(\omega_{j,A}(a_1, \ldots, a_n))] \text{ by (4.2)} \\ = \; & \omega_{j,B}(\omega_{j,B}(f_1a_1, \ldots, f_1a_n), \ldots, \omega_{j,B}(f_na_1, \ldots, f_na_n)) \\ = \; & \omega_{j,B}(f_1a_1, \ldots, f_na_n) \text{ by (4.3)} \\ = \; & \omega_{j,B}(ev(a_1, f_1), \ldots, ev(a_n, f_n)). \end{aligned}$$

Thus ev is an $\boldsymbol{A}$-morphism. □

Remarks 4.2. If $\boldsymbol{A}$ has canonical function spaces, then for any two $\boldsymbol{A}$-objects A and B, $j \in J$, $n = n_j$, $H := \hom_{\boldsymbol{A}}(A, B)$, and $f_1 \in H, \ldots, f_n \in H$, the map $\omega_{j,B} \circ (f_1 \times \cdots \times f_n) \circ d_{n,A}$, being equal to $\omega_{j,H}(f_1, \ldots, f_n)$, is an $\boldsymbol{A}$-morphism. Consequently, for any $i \in J$, $m = n_i$, $a_1 \in A, \ldots, a_m \in A$, we have

$$\begin{aligned} & \omega_{j,B}[\omega_{i,B}(f_1a_1, \ldots, f_1a_m), \ldots, \omega_{i,B}(f_na_1, \ldots, f_na_m)] \\ & = \omega_{i,B}[\omega_{j,B}(f_1a_1, \ldots, f_na_1), \ldots, \omega_{j,B}(f_1a_m, \ldots, f_na_m)]. \qquad (4.5) \end{aligned}$$

5. A Structure Theorem

We conclude our work with a structure theorem for some categories of universal algebras with one operation. First we have a result as a consequence of Proposition (4.1).

Lemma 5.1. *Suppose* $\boldsymbol{A}$ *is a subcategory of* $\boldsymbol{Alg(\Omega)}$ *closed under finite products, admitting free objects on sets with* n_j $(j \in J)$ *elements. Then* $\boldsymbol{A}$ *has canonical function spaces iff* $\boldsymbol{A}$ *is closed under* $\hom_{\boldsymbol{A}}$ *(i.e., for any two* $\boldsymbol{A}$*-objects* A *and* B*, the set* $H := \hom_{\boldsymbol{A}}(A, B)$ *of all* $\boldsymbol{A}$*-morphisms from* A *to* B *is an* $\boldsymbol{A}$*-object whose* n_j*-ary* $(j \in J)$ *operation is given by (4.2)), and for any* $\boldsymbol{A}$*-object* A*,* $i \in J$*,* $j \in J$*,* $n = n_j$*,* $m = n_i$*,* $a \in A$*,* $a_{rs} \in A$ $(1 \le r \le m,$ $1 \le s \le n)$*,*

$$\omega_j(a, \ldots, a) = a, \tag{5.1}$$

$$\begin{aligned}\omega_j(\omega_i(a_{11}, \ldots, a_{m1}), \ldots, \omega_i(a_{1n}, \ldots, a_{mn}))\\ = \omega_i(\omega_j(a_{11}, \ldots, a_{1n}), \ldots, \omega_j(a_{m1}, \ldots, a_{mn})),\end{aligned} \tag{5.2}$$

and, for $i = j$*,*

$$\omega_j(\omega_j(a_{11}, \ldots, a_{n1}), \ldots, \omega_j(a_{1n}, \ldots, a_{nn})) = \omega_j(a_{11}, \ldots, a_{nn}). \tag{5.3}$$

In particular, the subcategory of $\boldsymbol{Alg(\Omega)}$ *defined by the above three identities has canonical function spaces.*

Proof. Suppose $\boldsymbol{A}$ has canonical function spaces. Note that (5.1) is just a restatement of (4.1). Let $i \in J$, $j \in J$, $m = n_i$, $n = n_j$, A be an $\boldsymbol{A}$-object, and $a_{rs} \in A$ $(1 \le r \le m,\ 1 \le s \le n)$. Let B be the free $\boldsymbol{A}$-object on $\mathbf{N}_m := \{1, \ldots, m\}$. Let $f_s : B \to A$ be the $\boldsymbol{A}$-morphism that extends the association $r \to a_{rs}$ $(r \in \mathbf{N}_m)$ for $1 \le s \le n$. With these $\boldsymbol{A}$-morphisms equations (4.5) and (4.3) imply (5.2), and (5.3).

Conversely, assume that $\boldsymbol{A}$ is closed under $\hom_{\boldsymbol{A}}$ satisfying (5.1) - (5.3). For any two $\boldsymbol{A}$-objects A and B, $j \in J$, $n = n_j$, $\boldsymbol{A}$-morphisms $f_1 : A \to B, \ldots, f_n : A \to B$, members $a_1, \ldots, a_n$ in A, write $a_{rs} = f_s a_r$ for $1 \le r, s \le n$. Then (5.3) implies (4.3). Hence by Lemma (4.1), $\boldsymbol{A}$ has canonical function spaces.

To prove the last statement, assume that $\boldsymbol{A}$ is the subcategory of $\boldsymbol{Alg(\Omega)}$ defined by the identities (5.1), (5.2), and (5.3). In order to show that $\boldsymbol{A}$ has canonical function spaces, it remains to prove that $\boldsymbol{A}$ is closed under $\hom_{\boldsymbol{A}}$ functor. Suppose A and B are any two $\boldsymbol{A}$-objects and write $H := \hom_{\boldsymbol{A}}(A, B)$. First we show that H is closed under $\omega_{j,H}$ $(j \in J)$ defined in (4.2).

Let $n = n_j$, $f_1, \ldots, f_n \in H$. Define $g : A \to B$ by

$$g(a) = \omega_{j,B}(f_1 a, \ldots, f_n a)$$

for any $a \in A$. For any $i \in J$, $m = n_i$, $a_1 \in A, \ldots, a_m \in A$,

$$\begin{aligned} g(\omega_{i,A}(a_1, \ldots, a_m)) &= \omega_{j,B}[f_1\omega_{i,A}(a_1, \ldots, a_m), \ldots, f_n\omega_{i,A}(a_1, \ldots, a_m)] \\ &= \omega_{j,B}[\omega_{i,B}(f_1 a_1, \ldots, f_1 a_m), \ldots, \omega_{i,B}(f_n a_1, \ldots, f_n a_m)] \\ &= \omega_{i,B}[\omega_{j,B}(f_1 a_1, \ldots, f_n a_1), \ldots, \omega_{j,B}(f_1 a_m, \ldots, f_n a_m)] \\ &\qquad \text{by (5.2)} \\ &= \omega_{i,B}(g a_1, \ldots, g a_m). \end{aligned}$$

Thus $g \in H$. Moreover, since (5.1) - (5.3) hold in B, they are also valid in H by the definition of operations on H. Hence H is an $\boldsymbol{A}$-object. □

The following example illustrates some objects of $\boldsymbol{Alg(\Omega)}$ satisfying the equations (5.1), (5.2), and (5.3).

Example 5.2. Let I be the set $\mathbf{N}$ of all positive integers or the set $\mathbf{N}_k$ of first k positive integers. Suppose $(C_i)_{i\in I}$ is a family of sets and A is the product of $(C_i)_{i\in I}$ in $\boldsymbol{Set}$. If $j \in J$, $n = n_j$, $a_1 \in A, \ldots, a_n \in A$, then $\omega_j(a_1, .., a_n)$ is a member of A defined by

$$\omega_j(a_1, \ldots, a_n)(i) = \begin{cases} a_i(i) & \text{if } i \le n, \\ a_1(i) & \text{otherwise,} \end{cases}$$

for any $i \in I$. Then A is an Ω-algebra satisfying (5.1), (5.2), and (5.3).

For brevity, let us write $\boldsymbol{Alg(n)}$ for the category of all universal algebras with one n-ary operation. We describe objects of an essentially algebraic subcategory of $\boldsymbol{Alg(n)}$ having function spaces. They are precisely the ones of the type explained in the above example where $I = N_n$.

Proposition 5.3. *Suppose $\boldsymbol{A}$ is a subcategory of $\boldsymbol{Alg}(n)$ with canonical function spaces that contain the free object on $\mathbf{N}_n$. Then for any $\boldsymbol{A}$-object A there exist n $\boldsymbol{Alg(n)}$-objects $A_1, \ldots, A_n$ such that n-ary operation on A_i is the i-th canonical projection $A_i^n \to A_i$, and A is isomorphic to the product $A_1 \times \cdots \times A_n$ in $\boldsymbol{Alg(n)}$ whose n-ary operation ω' is given by*

$$\omega'(a_1, a_2, \ldots, a_n) = (a_{11}, a_{22}, \ldots, a_{nn}), \tag{5.4}$$

where

$$a_i = (a_{i1}, a_{i2}, \ldots, a_{in})$$

for $1 \le i \le n$.

Proof. By Lemma (5.1), equations (5.1) and (5.3) hold in A. For any $a \in A$ and $i \in \mathbf{N}_n$, write $P_i a$ for the image of the product of A's with $\{a\}$ at the i-th place under the n-ary operation ω on A, i.e.,

$$P_i a := \{\omega(z_1, \ldots, z_n) : z_1 \in A, \ldots, z_n \in A, z_i = a\}. \tag{5.5}$$

Let A_i be the set of all $P_i a$ where $a \in A$, i.e.,

$$A_i := \{P_i a : a \in A\}, \tag{5.6}$$

whose n-ary operation is the i-th canonical projection, and let B be the product of $A_1, \ldots, A_n$ in $\boldsymbol{Alg(n)}$.

We show that A is isomorphic to B. Define $\varphi : B \to A$ by

$$\varphi(P_1 a_1, \ldots, P_n a_n) := \omega(a_1, \ldots, a_n) \tag{5.7}$$

for any $a_1 \in A, \ldots, a_n \in A$.

To prove that φ is well defined, note that

$$P_1 a_1 \cap \ldots \cap P_n a_n = \{\omega(a_1, \ldots, a_n)\}. \tag{5.8}$$

Indeed, clearly the set on the right is a subset of the set on the left, and if z is a member of the set on the left then there exist members z_{rs} $(1 \leq r, s \leq n)$ in A such that

$$z_{rr} = a_r$$

and

$$z = \omega(z_{1r}, z_{2r}, .., z_{nr})$$

for $1 \leq r \leq n$. Now

$$\begin{aligned} \omega(a_1, \ldots, a_n) &= \omega(z_{11}, \ldots, z_{nn}) \\ &= \omega_j(\omega_j(z_{11}, \ldots, z_{n1}), \ldots, \omega_j(z_{1n}, \ldots, z_{nn})) \text{ by (5.3)} \\ &= \omega(z, \ldots, z) = z, \text{ by (5.1)}. \end{aligned}$$

Note that from (5.7),

$$\varphi \circ (P_1 \times \cdots \times P_n) = \omega, \tag{5.9}$$

and hence by (4.1),

$$\varphi \circ (P_1 \times \cdots \times P_n) \circ d_{n,A} = id_A. \tag{5.10}$$

In particular φ is onto.

To show that φ is one-one, let us assume that $a_i \in A$, $e_i \in A$ $(1 \leq i \leq n)$, and

$$\omega(a_1, \ldots, a_n) = \omega(e_1, \ldots, e_n). \tag{5.11}$$

Fix i, $1 \leq i \leq n$. Assume $z_1 \in A, \ldots, z_n \in A$ such that $z_i = a_i$. Write

$$z_{rs} = \begin{cases} a_r, & \text{if } s = i \\ z_s, & \text{if } s \neq i \end{cases}$$

and

$$y_{rs} = \begin{cases} e_r, & \text{if } s = i \\ z_s, & \text{if } s \neq i \end{cases}$$

for $1 \leq r \leq n$, $1 \leq s \leq n$. Then

$$\begin{aligned} \omega(z_1, \ldots, z_n) &= \omega(z_{11}, \ldots, z_{nn}) \\ &= \omega(\omega(z_{11}, \ldots, z_{n1}), \ldots, \omega(z_{1n}, \ldots, z_{nn})) \text{ by (5.3)} \\ &= \omega(\omega(y_{11}, \ldots, y_{n1}), \ldots, \omega(y_{1n}, \ldots, y_{nn})) \text{ by (5.11)} \\ &= \omega(y_{11}, \ldots, y_{nn}) \text{ by (5.3),} \end{aligned}$$

which belongs to $P_i e_i$. This shows that $P_i a_i$ is a subset of $P_i e_i$. By interchanging the roles of a_r's and e_r's, we see that

$$P_i a_i = P_i e_i.$$

This being true for any i, φ is one-one. By the definition of n-ary operation on B (see equation (5.4)) and by (5.3), it is easy to see that φ is a Ω-homomorphism. □

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