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CLASSIFICATION OF REGULAR DIGRAPHS, NORMALLY REGULAR DIGRAPHS, AND STRONGLY REGULAR DIGRAPHS

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Abstract: We use an exhaustive generation with isomorph-rejection to classify three types of structured digraphs. The first type is the class of regular digraphs where each vertex has the same number of out-neighbors and in-neighbors. The second type is the class of normally regular digraphs introduced by Jørgensen. In these digraphs, the number of common out-neighbors (or in-neighbors) of vertices x and y depends only on whether they are adjacent. We observe some remarks on the order of the automorphism group on some of those digraphs. Also, we make some improvements in the table of results of Jørgensen's search. The thrid type is the class of strongly regular digraphs introduced by Duval. In those digraphs, the number of directed paths of length 2 from vertex x to vertex y depends only on whether x dominates y. Our results on those digraphs were on full agreement with those of Duval and Jørgensen.

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1. Introduction

Let G be a group acting on a finite non-empty set X of structures. An algo-

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rithm for *exhaustive generation with isomorph-rejection* has two fundamental components. The first one is the generation of structures from a finite set X so that at least one representative from each G-orbit is obtained. The second component concerns the detection and the elimination of structures that are isomorphic to those already generated so that exactly one structure is generated from each G-orbit on X. The later component is the most time-consuming part in the algorithm. Different techniques were developed for that purpose by many authors, see [1], [9] for more details.

A *directed graph*, or just a *digraph*, $\mathcal{D}$ is a pair $(V(\mathcal{D}), E(\mathcal{D}))$. If $(x, y) \in E(\mathcal{D})$, then we say that x *dominates* y and that y is dominated by x. In this case, we write $x \to y$. Moreover, if there is a directed edge between x and y, we say that x and y are *adjacent*, denoted by $x \sim y$. Otherwise, they are *non-adjacent*. If we have both $x \to y$ and $y \to x$, then we speak of *undirected* edges.

Let $[m]$ denote the set of integers $\{1, 2, \cdots, m\}$. Two digraphs $\mathcal{D}_1 = (V_1, E_1)$ and $\mathcal{D}_2 = (V_2, E_2)$ are said to be *isomorphic* if there is a bijective map $\alpha : V_1 \to V_2$ so that $(x^\alpha, y^\alpha) \in E_2$ whenever $(x, y) \in E_1$. In that case, we say that α is an *isomorphism* mapping $\mathcal{D}_1$ onto $\mathcal{D}_2$. An isomorphism mapping a digraph $\mathcal{D}$ onto itself is called an *automorphism*. The set of all such automorphisms forms a group called the *automorphism group*, denoted by $\mathrm{Aut}(\mathcal{D})$. The automorphism group order of $\mathcal{D}$ is then denoted by $\big|Aut(\mathcal{D})\big|$.

A digraph $\mathcal{D} = (V, E)$ can be represented by an $n \times n$ $\{0, 1\}$-*adjacency* matrix $A = (a_{ij})$ so that for $i, j \in [n]$, $a_{ij} = 1$ if $x_i \to x_j$ (for $x_i, x_j \in V$), and $a_{ij} = 0$ otherwise.

A digraph $\mathcal{D}$ is said to be *regular* of degree k (or k-regular), if each vertex dominates k vertices, and is dominated by k vertices. A k-regular digraph is called *normally regular digraph* (NRD for short), if every pair of vertices dominate λ vertices if they are adjacent, and μ vertices otherwise, see [8] for more details. A *strongly regular digraph* (SRD for short) is a k-regular digraph with the properties: a) the number of undirected edges (i.e. 2-cycles) incident to every vertex is t, b) the number of directed paths of length 2 from vertex x to vertex y is λ if $x \to y$, and c) the number of directed paths of length 2 from vertex x to vertex y is μ if $x \not\to y$. Further details on SRDs can be found in [4].

In [7], Jørgensen considered the classes of NRDs and SRDs by an exhaustive search using an isomorph-rejection technique based on the notion of *orderly generation*, independently introduced by Faradžev [6] and Read [12]. In the presented paper, we reconsider the generation procedure of NRDs and SRDs by means of generation of canonical augmentation, introduced by McKay [11]. Our goal is to confirm and to extend the results in Jørgensen's search [7]. Some computer-search techniques (might differ from those of [7]) are discussed. Moreover, we discuss some properties related to the automorphism group orders for some classes of NRDs with a given value of μ. We also consider the (isomorph-free) generation of regular digraphs with small parameters.

2. The search Algorithm and Regular Digraphs

We carry out a *row-by-row* (or vertex-by-vertex) backtrack search over all adjacency matrices of k-regular digraphs on n vertices. Starting with the $n \times n$ *zero matrix*, we augment one row at a time subject to the conditions below. Assuming that rows $1, 2, \cdots, r-1$ of adjacency matrix A have been already augmented, then row r must satisfy

RD1. it has exactly k ones,

RD2. diagonal entry (r, r) must be zero (loops are not allowed),

RD3. for $i < r$, entry (r, i) must be zero if entry (i, r) is one (2-cycles are not allowed),

RD4. if $r < n$, then the sum of the ones in each column must be at most k, otherwise $(r = n)$ it must be exactly k,

RD5. row r must be canonical.

In each possible augmentation of row r, we repeat the search with r replaced by $r+1$ and so on. If there is no such row satisfying these conditions, the search backtracks, otherwise all n rows have been generated and the matrix satisfies the conditions a k-regular digraph.

The test in Condition **RD5** is called the *canonicity test.* Following the method of canonical augmentation, the canonicity test gurantees that the augmented object at hand follows the canonical construction path, see [11] for further details. In particular, if row s was the first row in the canonical form of A (which can be computed along with $\mathrm{Aut}(A)$ by *nauty* [10]), then row r is *canonical* (and accepted) if it was equivalent to row s under $\mathrm{Aut}(A)$. Otherwise, it is rejected and the search backtracks.

2.1. Bordering Technique

The canonicity test is the most *expensive* test (in the sense of time) in the search. For each row in the adjacency matrix of a digraph, one calls nauty to decide the canonicity of that row. Most of these calls would return back with a "NO". Moreover, these calls are also done for *partial* solution that would not be completed anyway. To avoid as many of these (unwanted) calls as possible, one can choose a reasonable level (row) that no more calls for nauty are done. In particular, we choose a level where we ignore (or switch off) the canonicity test. Such a level is called the *border* level (or border row). Indeed, once a partial solution is completed, we require the canonicity test for all rows that were ignored earlier, i.e. from the border row up to row n. We call this technique the *bordering technique*.

The benefits of applying the canonicity test are worth the effort in the early rows. In particular, such test (in the early stages of the search) keeps the number of possibilities down and thereby reduces the size of the search space. This simple technique was applied in the search for regular digraphs, normally regular digraphs, and strongly regular digraphs, and in all of the three cases it saves us a lot of time.

3. Normally Regular Digraphs

A normally regular digraph with parameters n, k, λ, and μ, denoted by NRD(n, k, λ, μ), is a digraph on n vertices without 2-cycles or multiple edges so that the following properties hold:

N1. every vertex has k out-neighbors,

N2. every pair of adjacent vertices has λ common out-neighbors,

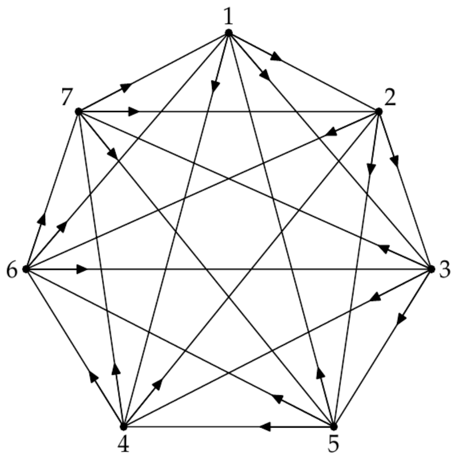
N3. every pair of non-adjacent vertices has μ common out-neighbors,

It is shown in [8] that the adjacency matrix of an NRD is normal, and thus the properties of out-neighbors hold for in-neighbors as well. In the literature, a *tournament* satisfying properties **N1**, **N2**, and **N3** is called *doubly regular tournament*, see [13] for further details.

An adjacency matrix A representing an NRD(n, k, λ, μ) satisfies

$$AA^t = kI + \lambda(A + A^t) + \mu(J - I - A - A^t),$$

where A^t, I, and J denote the transpose of A, $n \times n$ identity matrix, and the all-ones $n \times n$ matrix, respectively. In particular, if $V = \{x_1, x_2, \cdots, x_n\}$ is the vertex set of the NRD, then for $i, j \in [n]$ the (i, j)- entry of AA^t is (resp.) k, λ, or μ if (resp.) $i = j$, $i \neq j$ and $x_i \sim x_j$, or $i \neq j$ and $x_i \not\sim x_j$.



$A =$

	1	2	3	4	5	6	7
1		x	x	x			
2			x		x	x	
3				x	x		x
4		x				x	x
5	x			x		x	
6	x		x				x
7	x	x			x		

Figure 1: $NRD(7,3,1,0)$ with its adjacency matrix A

Figure 1 presents the unique NRD$(7,3,1,0)$ together with its adjacency matrix. For our conveninnce, we write 'x' marks for the ones in the matrix and empty square for the zero entries. This NRD is also the unique doubly regular tournament on 7 vertices, see Spence [14].

The structures of normally regular digraphs with $\mu \in \{\lambda, \lambda+1\}$ are related to some combinatorial structures including *symmetric 2-designs* and *Hadamard-designs*. A detailed discussion in this regard can be found in [8]. In the rest of this section, we consider the case of $\mu \in \{0, k\}$.

3.1. Normally Regular Digraphs with $\mu = 0$

It has been proved in [8] that a connected normally regular digraph with $\mu = 0$ must be one of the following three cases of Theorem 3.1.

Theorem 3.1. *A connected digraph is an NRD with $\mu = 0$ if and only if either:*

(1) *it is a directed cycle ($k = 1$),*

(2) *it is a doubly regular tournament, or*

(3) *it has an adjacency matrix of the following form*

$$\begin{pmatrix} 0 & 1 & \dots & 1 & 0 & 0 & \dots & 0 \\ 0 & & & & 1 & & & \\ \vdots & & A & & \vdots & & A^t & \\ 0 & & & & 1 & & & \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & 1 \\ 1 & & & & 0 & & & \\ \vdots & & A^t & & \vdots & & A & \\ 1 & & & & 0 & & & \end{pmatrix} \tag{1}$$

where A is an adjacency matrix of a doubly regular tournament.

Let T be a doubly regular tournament with vertex set $V(T) = \{x_1, x_2, \cdots, x_n\}$. A digraph $\mathcal{D}$ with vertex set

$$V(\mathcal{D}) = \{y_0, y_1, \cdots, y_n, y'_0, y'_1, \cdots, y'_n\}$$

and directed edges of the forms:

$$y_0 \to y_i \to y'_0 \to y'_i \to y_0, \quad \text{for } i \in [n],$$

and

$$y_i \to y_j \to y'_i \to y'_j \to y_i \quad \text{if } x_i \to x_j \text{ in } T \text{ for all } i, j \in [n],$$

is a non-tournament NRD$(2n + 2, n, k, 0)$.

The automorphism group of the resulted NRD is related to the automorphism group of the initial (tournament) NRD in the sense of group order as indicated in Proposition 3.2.

Proposition 3.2. *Given a doubly regular tournament T with parameters $(n, k, \lambda, 0)$, let $\mathcal{D}$ be the constructed NRD$(2n + 2, n, k, 0)$ as described in Theorem 3.1. Then,*

$$\big|Aut(\mathcal{D})\big| = n \cdot \big|Aut(T)\big|.$$

It is clear that any vertex y_i for $i \in [n]$ can be mapped under the automorphism group of $\mathcal{D}$ to y_0. Thus, y'_i must be mapped to y'_0 as well to preserve adjacencies. Since y_i can be chosen in n different ways, the result is proved.

We remark that in our search we skip the search for those kind of NRDs with $\mu = 0$ of the form (1) of Theorem 3.1. However, we are interested in doubly regular tournament which then can be used to construct those (non-tournament) NRDs with $\mu = 0$.

3.2. Normally Regular Digraphs with $\mu = k$

Theorem 3.3. (see Jørgensen [8]) *A directed graph $\mathcal{D}$ is an NRD(n, k, λ, k) if and only if there is a number s such that $\mathcal{D}$ is obtained from a doubly regular tournament T by replacing each vertex of x in T by a set V_x of s new vertices such that if $x \to y$ in T, then $u \to v$ in $\mathcal{D}$ for every $u \in V_x$ and for every $v \in V_y$. In particular, $s = k - 2\lambda = n - 2k$.*

An instance consequence of Theorem 3.3 is the non-existence result for NRDs with $\mu = k$ with a prime number of vertices.

Corollary 3.4. *A non-tournament normally regular digraph with parameters (n, k, λ, k) with a prime n does not exist.*

Clearly, if such an NRD exists, say $\mathcal{D}$, then there exist a tournament T with m vertices so that $\mathcal{D}$ is obtained from T as described in Theorem 3.3. That is, $n = s \cdot m$ for some $s > 1$ which is not possible.

The automorphism group orders of a tournament T and the resulted NRD $\mathcal{D}$ are related in the following way.

Proposition 3.5. *Let $\mathcal{D}$ be an NRD(n, k, λ, k) that was obtained from a tournament T as explained in Theorem 3.3, and let $s = n - 2k$. Then,*

$$\big|Aut(\mathcal{D})\big| = (\, s!\,)^{\left|V(T)\right|} \cdot \big|Aut(T)\big|.$$

Since each vertex in T is replaced by s new vertices, we can map (under Aut($\mathcal{D}$)) any vertex out of those s vertices to any other vertex within the same set, and since there are $\big|V(T)\big|$ new sets of vertices, the result follows.

3.3. The Search for Normally Regular Digraphs

We now describe the main framework of our search for normally regular digraphs, given a set of *feasible* parameters $n, k, \lambda,$ and μ. By feasible, we mean that $\mu \notin \{0, k\}$ and that these parameters satisfy the conditions of Theorem 3.6 which is discussed in more details in [7].

Theorem 3.6. *Suppose that there exists a normally regular digraph with parameters (n, k, λ, μ).*

(a) If n is even, then $\eta = k - \mu + (\mu - \lambda)^2$ is a square.

(b) If n is odd, then the equation $x^2 + (-1)^{\frac{v+1}{2}} \mu y^2 = \eta z^2$ has an integer solution $(x, y, z) \neq (0, 0, 0)$.

(c) If $2\mu > k + \lambda$, then $n - 2k$ divides n.

(d) If $\lambda = 0$, then $k > 2\mu + \frac{1}{2} + \sqrt{2\mu + \frac{1}{4}}$, unless $\mu = k$ or $\mu = 1$.

Following a row-by-row backtracking search, the construction procedure starts with the $n \times n$ adjacency matrix A whose entries are all zeros. Assuming that rows $1, 2, \ldots, r-1$ have been constructed, row r (for $r \leq n$) must satisfy the following properties:

NRD1. it has exactly k ones,

NRD2. diagonal entry (r, r) must be zero (loops are not allowed),

NRD3. for $i < r$, entry (r, i) must be zero if entry (i, r) is one (2-cycles are not allowed),

NRD4. if entry (r, i) and entry (i, r) are both zeros, then the dot product of row r and row i equals to μ, otherwise it equals to λ, for $i = 1, 2, \ldots, r-1$,

NRD5. row r must be canonical.

Property **NRD4** corresponds to the λ- and μ-condition in the corresponding NRD. Note that this property checks only on the out-degree conditions. One can also check on the in-degree to reduce the number of candidates of the search in the following way. The dot product of any two columns is at most the maximum of λ and μ at all times. If it is already known that the two columns correspond to adjacent (respectively, non-adjacent) vertices, then the dot product is at most λ (μ).

In Property **NRD5**, we accept row r if it was canonical. In addition, we can use the bordering technique to fasten the search.

4. Strongly Regular Digraphs

A strongly regular digraph, introduced by Duval [4], with parameters n, k, λ, μ, and t, denoted by SRD(n, k, λ, μ, t), is a digraph on n vertices without multiple edges and no loops so that the following properties are satisfied:

S1. every vertex has k out-neighbors,

S2. the number of 2-cycles incident with every vertex is t,

S3. the number of directed paths of length 2 from vertex x to vertex y is λ if x dominates y, and it is μ otherwise.

An adjacency matrix A of an SDR(n, k, λ, μ, t) satisfies

$$A^2 = tI + \lambda A + \mu(J - I - A), \text{ and } AJ = JA = kJ.$$

A row-by-row backtrack search is carried out to classify all non-isomorphic SRDs(n, k, λ, μ, t) in the following way. The search starts with the all-zeros adjacency matrix A, and we augment one row at a time without violating the properties of an SRD. Assuming that rows $1, 2, \cdots, r-1$ have been generated as desired, row r (for $r \leq n$) must satisfy the following conditions:

SRD1. it has exactly k ones,

SRD2. diagonal entry (r, r) must be zero (loops are not allowed),

SRD3. if row r and column r has been filled in with k ones, then the dot product of them must be t,

SRD4. row r must not violate the properties of λ- and μ-conditions,

SRD5. row r must be canonical.

The λ- and μ-condition tested in **SRD4** are basically checking if the number of directed paths of length two from vertex x_i to vertex x_j ($i \in [r]$ and $j \in [n]$) is at most λ if it is known that $x_i \to x_j$. It is otherwise ($x_i \not\to x_j$) at most μ. There are different ways to do such test. One particular way is presented in Jørgensen [7]. Another is to check the number of directed paths of length two each time a new row is augmented. In this case, a final check is required when the adjacency matrix A is completed with n rows. Moreover, the canonicity test of Condition **SRD5** can be decided by using the presented techniques in Section 2.

5. Results

The presented running times are all CPU times on a 3.0 GHz Intel Core2 Duo running on a Mac operating system machine. For large vertex numbers, the search was splitted into small parts and times are not given.

Table 1 presents the results of k-regular digraphs on v vertices where $k \leq \frac{v-1}{2}$. The search covers all k-regular digraphs on upto 13 vertices.

We remark that all of the results presented in Table 1 were confirmed with programs *directg* [11] and *water* [3]. Moreover, the results of 3- and 4-regular digraphs on 13 vertices could not be reached (in a reasonable time) by the

$v\backslash k$	1	2	3	4	5	6
3	1	0	0	0	0	0
4	1	0	0	0	0	0
5	1	1	0	0	0	0
6	2	4	0	0	0	0
7	2	9	3	0	0	0
8	3	55	26	0	0	0
9	4	453	1 547	15	0	0
10	5	4 357	146 487	3 987	0	0
11	6	47 598	16 018 987	5 104 171	1 223	0
12	9	569 769	1 985 766 270	7 353 314 011	8 636 086	0
13	10	7 371 560	278 601 715 904	11 545 150 361 450	?	1 495 297

Table 1: All k-regular digraphs with at most 13 vertices (with some gaps)

$v\backslash k$	2	3	4	5
8	0	0		
9	10 (4)	93 (51)	0	
10	87 (59)	2123 (1672)	1545 (1184)	
11	235 (170)	95345	54629	2840

Table 2: Running times (to the nearest seconds) for some of k-regular digraphs. The bordering technique was applied on some of the cases and the times are presented in brackets

approach presented in this paper. It was computed with *directg* and *water*. Many thanks to Gunnar Brinkmann in this regard.

NRD(n,k,λ,μ):time	isot.	NRD(n,k,λ,μ):time	isot.
	$\mu=\lambda$		
NRD$(13,4,1,1)$:3	4	NRD$(25,9,3,3)$:32400	≥ 36
NRD$(16,6,2,2)$:18	4	NRD$(31,6,1,1)$:32400	$\geq 50\ 000$
NRD$(21,5,1,1)$:21500	$\geq 200\ 000$	NRD$(37,9,2,2)$	0
	$\mu=\lambda+1$		
NRD$(7,2,0,1)$:0	1	NRD$(23,10,4,5)$	0
NRD$(11,4,1,2)$	0	NRD$(25,8,2,3)$:9000	$\geq 4\ 070$
NRD$(13,3,0,1)$:0	5	NRD$(27,12,5,6)$	?
NRD$(15,6,2,3)$:25	0	NRD$(31,5,0,1)$:25200	$\geq 65\ 000$
NRD$(16,5,1,2)$	16	NRD$(31,9,2,3)$	?
NRD$(19,8,3,4)$	0	NRD$(37,8,1,2)$	?
NRD$(21,4,0,1)$:1800	187		

Table 3: Normally regular digraphs with $\mu \in \{\lambda, \lambda+1\}$

We remark that the presented approach in this paper is not so efficient for the class of k-regular digraphs on large vertex numbers. Brinkmann [2] recently described a somehow more efficient generation procedure of regular digraphs where he construct regular digraphs from regular bipartite (undirected) graphs. However, it is always good to have different ways to do the same job. Moreover, our goal is to describe a more general algorithm to generate other classes of digraphs as well.

The results of the classification of normally regular digraphs (with special value of μ equals to λ or $\lambda+1$) are presented in Table 3. Note that "*isot.*" indicates the number of non-isomorphic NRDs found in the search.

The results of classification of tournament regular digraphs (i.e. doubly regular tournaments) are presented in Table 4. We remark that, all of the doubly regular tournaments with $n \leq 27$ were classified previously by Spence

NRD(n, k, λ, μ):time	isot.
NRD$(3, 1, 0, \cdot)$:0	1
NRD$(7, 3, 1, \cdot)$:0	1
NRD$(11, 5, 2, \cdot)$:0	1
NRD$(15, 7, 3, \cdot)$:7	2
NRD$(19, 9, 4, \cdot)$:290	2
NRD$(23, 11, 5, \cdot)$:5400	37
NRD$(27, 13, 6, \cdot)$:18000	722
NRD$(31, 15, 7, \cdot)$:36000	$\geq$ 13 330

Table 4: Doubly regular tournaments with at most 31 vertices

[14]. The case of NRD$(31, 15, 7, \cdot)$ was previously considered by McKay where he constructed five NRD$(31, 15, 7, \cdot)$. In our search, we have confirmed the results of Spence [14], and succeeded to construct $13, 330$ NRDs$(31, 15, 7, \cdot)$.

In Table 5, we present the results of the search for NRDs with parameters (n, k, λ, μ) so that $\mu \notin \{0, k, \lambda, \lambda + 1\}$ and that the parameters satisfy the conditions of Theorem 3.6. The results presented in Table 3, Table 4, and Table 5 were in full agreement with those in Jørgensen [7]. Moreover, some of these entries were extended by our search.

Finally, we present the results of the search for SRDs in Table 6. The case of SRD$(16, 7, 3, 3, 4)$ was not found in [7], but its existence was proven in [5]. All cases that were stated with a 'question mark' in the presented results seems to be more difficult to completed. For such cases, a specialized program would be useful.

NRD(n, k, λ, μ):time	isot.	NRD(n, k, λ, μ):time	isot.
NRD$(19, 6, 1, 3)$:1	1	NRD$(31, 10, 2, 5)$	0
NRD$(21, 8, 3, 2)$:1035	1	NRD$(31, 10, 4, 1)$	0
NRD$(23, 8, 2, 4)$	0	NRD$(31, 12, 4, 6)$	≥ 1
NRD$(25, 8, 3, 1)$	0	NRD$(31, 12, 5, 2)$	?
NRD$(27, 8, 1, 4)$	0	NRD$(35, 10, 1, 5)$	0
NRD$(27, 10, 3, 5)$	≥ 1	NRD$(35, 12, 3, 6)$	?
NRD$(28, 9, 2, 4)$	?	NRD$(35, 14, 5, 7)$	?
NRD$(28, 12, 5, 4)$	?	NRD$(36, 7, 0, 2)$:180	2
NRD$(29, 7, 2, 1)$	4	NRD$(36, 10, 3, 2)$	?
NRD$(29, 12, 5, 3)$	?		

Table 5: Normally regular digraphs with parameters satisfying the conditions of Theorem 3.6

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NRD(n, k, λ, μ, t)	isot.
SRD$(6, 2, 0, 1, 1)$	1
SRD$(8, 3, 1, 1, 2)$	1
SRD$(10, 4, 1, 2, 2)$	16
SRD$(12, 3, 0, 1, 1)$	1
SRD$(12, 4, 0, 2, 2)$	1
SRD$(12, 5, 2, 2, 3)$	20
SRD$(14, 5, 1, 2, 4)$	0
SRD$(14, 6, 2, 3, 3)$	16 495
SRD$(15, 4, 1, 1, 2)$	5
SRD$(15, 5, 1, 2, 2)$	1 292
SRD$(16, 6, 1, 3, 3)$	0
SRD$(16, 7, 4, 2, 5)$	1
SRD$(16, 7, 3, 3, 4)$	$\geq$ 3 328
SRD$(18, 4, 0, 1, 3)$	1
SRD$(18, 5, 2, 1, 3)$	2
SRD$(18, 6, 0, 3, 3)$	1

Table 6: Strongly regular digraphs with at most 18 vertices

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