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# FUZZY SUBGROUPS COMPUTATION OF FINITE GROUP BY USING THEIR LATTICES

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**Abstract:** This paper gives a method to determine the number of fuzzy subgroups of finite group $G$ using diagram of lattice subgroups of $G$. This method can be used for abelian and non-abelian groups. First, an equivalence relation on the set of all fuzzy subgroups of group $G$ is defined. Second, this paper derive some theorems for determination of the number of fuzzy subgroups associated with some chains on the lattice subgroups of $G$. Then the demonstration of the method is given by determining the number of fuzzy subgroups of some finite group $G$.

**AMS Subject Classification:** 06B05, 20N25, 01-08
**Key Words:** equivalence, lattice, chain, fuzzy subgroup

## 1. Introduction

The concept of fuzzy sets was first introduced by Zadeh in 1965 (see [1]). The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups by Rosenfeld in 1971 (see [2]). Since the first paper by Rosenfeld, researchers have sought to characterize the fuzzy subgroups

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of various groups, particularly of finite abelian groups. Several papers have treated the particular case of finite cyclic group. Laszlo (see [3]) studied the construction of fuzzy subgroups of groups of the orders one to six. Zhang and Zou (see [4]) have determined the number of fuzzy subgroups of cyclic groups of the order $p^n$ where $p$ is a prime number. Murali and Makamba (see [5], [6]), considering a similar problem, found the number of fuzzy subgroups of abelian groups of the order $p^n q^m$ where $p$ and $q$ are different primes, while Tarnauceanu and Bentea (see [7]) have found this number for finite abelian groups.

In this paper, we take a different approach to determine the number of fuzzy subgroups of some finite groups $G$. We determine this number by observing the diagram of lattice subgroups of $G$. First, we define the equivalence relation on the set of all fuzzy subgroups of group $G$ (see Definition 3). This definition differs from that of Murali and Makamba (see [5], [6], [8]), but is equivalent to the definitions of Dixit (see [9]), Zhang (see [4]) and Tarnauceanu (see [7], [10]). We prefer to use this definition because it generalizes the definition used by Murali. We observed patterns of diagram of lattice subgroups of $G$ and determined its influence on the formula of the number of fuzzy subgroups. This method can be used not only for abelian group, but also for non-abelian as well.

We found some formulas in more general. Results of Zhang (see theorem 3.1.(1) ) in [4] and Murali (see proposition 3.3 and theorem 3) in [5] can be viewed as special cases from our formula (see Theorem 18, Example 17 and Corollary 23, respectively). Tarnauceanu (see [7]) asserted that computing the number of fuzzy subgroups of $Z_3^2 = Z_2 \times Z_2 \times Z_2$ by using "the method of direct calculation" is too difficult. We will show, however, that it is easy to compute using our method (see Example 15).

## 2. Preliminaries

In this paper, a group $G$ is assumed to be a finite group. First of all, we present some basic notions and results that will be used later.

**Definition 1.** Let $X$ be a nonempty set. A fuzzy subset of $X$ is a function from $X$ into [0, 1].

**Definition 2.** (Rosenfeld, see [2]). Let $G$ be a group. A fuzzy subset $\mu$ of $G$ is called a fuzzy subgroup of $G$ if:

(1) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}, \forall x, y \in G$,

(2) $\mu(x^{-1}) \geq \mu(x), \forall x \in G$.

If $\mu$ is a fuzzy subgroup of group $G$ and $\mu(G) = \{\theta_1, \theta_2, ..., \theta_n\}$, then we assume that $\theta_1 > \theta_2 > ... > \theta_n$.

**Theorem 3.** *(Rosenfeld, see [2]). Let $e$ denote the identity element of $G$. If $\mu$ is a fuzzy subgroup of $G$, then:*

*(1) $\mu(e) \geq \mu(x), \forall x \in G$,*

*(2) $\mu(x^{-1}) = \mu(x), \forall x \in G$.*

**Theorem 4.** *(Sulaiman and Abdul Ghafur, see [11]). A fuzzy subset $\mu$ of $G$ is a fuzzy subgroup of $G$ if and only if there is a chain of subgroups $G$, $P_1(\mu) \leq P_2(\mu) \leq P_3(\mu) \leq ... \leq P_n(\mu) = G$ such that $\mu$ is in the form:*

$$\mu(x) = \begin{cases} \theta_1, & x \in P_1(\mu) \\ \theta_2, & x \in P_2(\mu) \\ \vdots & \\ \theta_m, & x \in P_m(\mu) \end{cases} \tag{1}$$

If there is no confusion, then $P_i(\mu)$ as in (1) can simply be written as $P_i$. We define *length* (or *order*) of fuzzy subgroup $\mu$ in (1) as be $m$.

**Example 5.** Consider the group $G = Z_{12}$. Define functions $\mu, \gamma, \alpha$ and $\beta$ as follows:

$$\mu(x) = \begin{cases} 1, & x \in \{0, 2, 4, 6, 8, 10\} \\ \frac{1}{2}, & x \in \{1, 3, 5, 7, 9, 11\} \end{cases},$$
$$\gamma(x) = \begin{cases} 1, & x \in \{0, 1, 2, 3, 4, 5\} \\ \frac{1}{2}, & x \in \{6, 7, 8, 9, 10, 11\} \end{cases},$$
$$\alpha(x) = \begin{cases} 1, & x \in \{0, 4, 8\} \\ \frac{1}{2}, & x \in \{2, 6, 10\} \\ \frac{1}{3}, & x \in \{1, 3, 5, 7, 9, 11\} \end{cases},$$
$$\beta(x) = \begin{cases} \frac{1}{2}, & x \in \{0, 4, 8\} \\ \frac{1}{4}, & x \in \{2, 6, 10\} \\ 0, & x \in \{1, 3, 5, 7, 9, 11\} \end{cases}.$$

Note that

$$P_1(\mu) = \{0, 2, 4, 6, 8, 10\}, P_2(\mu) = \{0, 2, 4, 6, 8, 10\} \cup \{1, 3, 5, 7, 9, 11\} = Z_{12}.$$

Thus $P_1(\mu)$ and $P_2(\mu)$ both are subgroups of $Z_{12}$. By Theorem 2, $\mu$ is fuzzy subgroup of $Z_{12}$. Similarly, we can show that $\alpha$ and $\beta$ both are fuzzy subgroups of $Z_{12}$. Note that $P_1(\gamma) = \{0, 1, 2, 3, 4, 5\}$ is not a subgroup of $Z_{12}$, so $\gamma$ is not a fuzzy subgroup of $Z_{12}$.

## 3. Equivalence Relation on Fuzzy Subgroups

Without any equivalence relation on fuzzy subgroups of group $G$, the number of fuzzy subgroups is infinite, even for the trivial group $\{e\}$. So we define equivalence relation on the set of all fuzzy subgroups of a given group. Using this equivalent, we compute the number of distinct equivalence classes of fuzzy subgroups of group $G$.

**Definition 6.** Let $\mu, \gamma$ be fuzzy subgroups of $G$ of the form

$$\mu(x) = \begin{cases} \theta_1, & x \in P_1 \\ \theta_2, & x \in P_2 \setminus P_1 \\ \theta_3, & x \in P_3 \setminus P_2 \\ ... & \\ \theta_n, & x \in P_n \setminus P_{n-1} \end{cases}, \gamma(x) = \begin{cases} \delta_1, & x \in P_1 \\ \delta_2, & x \in P_2 \setminus P_1 \\ \delta_3, & x \in P_3 \setminus P_2 \\ ... & \\ \delta_m, & x \in P_m \setminus P_{m-1} \end{cases}.$$

Then we define that $\mu$ and $\gamma$ are *equivalent* if:

(1) $m = n$, and

(2) $P_i = M_i, \forall i \in \{1, 2, 3, ..., n\}$.

We write $\mu \sim \gamma$. It is easily checked that this relation is indeed an equivalence relation. Two fuzzy subgroups of $G$ are said to be different if they are not equivalent. In this paper, the number of fuzzy subgroups of group $G$ means the number of distinct equivalence classes of fuzzy subgroups.

**Example 7.** Let $\mu, \alpha$ and $\beta$ be fuzzy subgroups as in Example 5. Since $|\mu| \neq |\alpha|$ and $|\mu| \neq |\beta|$, by Definition 6 we determine that $\mu$ and $\alpha$ are not equivalent. Fuzzy subgroups $\mu$ and $\beta$ are not equivalent either. Note that $|\alpha| = |\beta|$ and it is easy to show that $P_i(\alpha) = P_i(\beta), \forall i \in \{1, 2, 3\}$. Thus, $\alpha \sim \beta$.

## 4. Method of Counting Using Lattices

In this section we discuss a method to determine the number of fuzzy subgroups of a group $G$ using a diagram of lattice subgroups of $G$. We observe at every chain and pattern of diagram of lattice subgroups of $G$ to compute the number of fuzzy subgroup. This method can be used both for abelian and non-abelian groups. We denote the number of fuzzy subgroups of $G$ as $n(F_G)$, while the number of fuzzy subgroups $\mu$ of $G$ with $P_1(\mu) = H$ is denoted by $n(F_{P_1=H})$. From Theorem 4, we have

$$n(F_G) = \sum_{H \leq G} n(F_{P_1=H}) \tag{2}$$

**Theorem 8.** *Let $G_1 < G_2 < ... < G_{k-1} < G_k = G$ be the only maximal chain from $G_1$ to $G$ on the lattice subgroups of $G$. Then:*

$$n(F_{P_1=G_1}) = \begin{cases} 1, & \text{k=1,2} \\ 2^{k-2}, & k > 2 \end{cases} .$$

*Proof.* If $k = 1$, then $G_1 = G$. We have only one fuzzy subgroup of $G$ where $P_1 = G$. Its length is 1, that is, $\mu(x) = \theta_1, \forall x \in G$. If $k = 2$, we have only one fuzzy subgroup of $G$ where $P_1 = G_1$. Its length is 2, that is,

$$\mu(x) = \begin{cases} \theta_1, & x \in G_1 \\ \theta_2, & x \in G \setminus G_1 \end{cases} .$$

If $k > 2$, the part of diagram of lattice subgroups of $G$ is $G_1 - G_2 - ... - G_k = G$. Let $\mu$ be the fuzzy subgroup of $G$ where $P_1(\mu) = G_1$. Since $G_1 \neq G$ and since the maximal chain $G_1 \subset G_2 \subset ... \subset G_k = G$ contains $k$ subgroups of $G$, then the possible length of $\mu$ is $2, 3, 4, ..., (k-1)$ or $k$. Let $n$ be an arbitrary element of {2,3,4,...,k}. We will count the number of fuzzy subgroups $\mu$ of length $n$ where $P_1 = G_1$ and $P_n = G$. For $P_2, P_3, ..., P_n$, we can choose one of $(k-2)$ pieces of the existing subgroup $G_2, G_2, ..., G_{k-1}$. So the number of fuzzy subgroups of $G$ with $P_1 = G_1$ of length $n$ is $C^{k-2}_{n-2}$. Thus, the number of fuzzy subgroups of $G$ with $P_1 = G_1$ is equal to $C^{k-2}_0 + C^{k-2}_1 + C^{k-2}_2 + ... + C^{k-2}_{k-3} + C^{k-2}_{k-2} = 2^{k-2}$. □

**Corollary 9.** *Let $G_1 < G_2 < ... < G_{k-1} < G_k = G, k \geq 2$ be the only maximal chain from $G_1$ to $G$ on the lattice subgroups of $G$. For $m$, $1 \leq m \leq m - 1$ we have $n(F_{P_1=G_m}) = 2^{k-m-1}$.*

*Proof.* Use Theorem 8. □

**Corollary 10.** *Let $G_1 < G_2 < ... < G_{k-1} < G_k = G, k \geq 3$ be the only maximal chain from $G_1$ to $G$ on the lattice subgroups of $G$. For $m$, $1 \leq m \leq m - 2$ we have $n(F_{P_1=G_m}) = 2.n(F_{P_1=G_{m+1}})$.*

*Proof.* Using Theorem 3, we have $n(F_{P_1=G_m}) = C^{k-m-1}_0 + C^{k-m-1}_1 + C^{k-m-1}_2 + ... + C^{k-m-1}_{k-m-1}$ and $n(F_{P_1=G_{m+1}}) = C^{k-m-2}_0 + C^{k-m-2}_1 + C^{k-m-2}_2 + ... + C^{k-m-2}_{k-m-2}$. Then using Pascal's recursion formula (see [12]), $C^n_k = C^{n-1}_{k-1} + C^{n-1}_k$, we obtain $n(F_{P_1=G_m}) = C^{k-m-1}_0 + C^{k-m-2}_0 + C^{k-m-21}_1 + C^{k-m-2}_1 + C^{k-m-2}_2 + C^{k-m-2}_2 + C^{k-m-2}_3 + ... + + C^{k-m-2}_{k-m-3} + C^{k-m-2}_{k-m-2} + C^{k-m-1}_{k-m-1} = 2[n(F_{P_1=G_{m+1}})]$. □

**Theorem 11.** *Let $H$ be a subgroup of $G$, and let the set of all subgroups of $G$ which contain $H$ (but are not equal to $H$ ) be $\{H_1, H_2, H_3, ..., H_k\}$. Then $n(F_{P_1=H}) = \Sigma^k_{i=1} n(F_{P_1=H_i})$.*

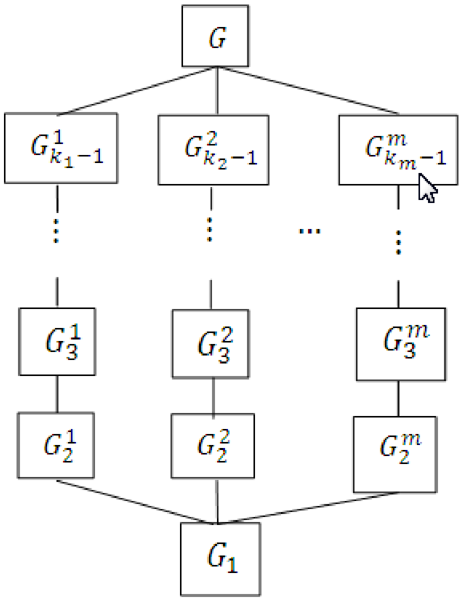


Figure 1: Part of the diagram of lattice subgroups $G$.

*Proof.* We have $n(F_{P_1=H}) = \sum_{i=1}^{k} n(F_{P_1=H;P_2=H_i})$. Note that $\forall i \in \{1, 2, ..., k\}$,

$$n(F_{P_1=H;P_2=H_i}) = n(F_{P_1=H_i}).$$

Hence, we obtain $n(F_{P_1=H}) = \Sigma_{i=1}^{k} n(F_{P_1=H_i})$. $\square$

**Theorem 12.** *Let $G_1$ be a subgroup of $G$. If there are $m (m \geq 2)$ maximal chains from $G_1$ to $G$ on the lattice subgroup $G$ of length $k_1, k_2, ..., k_m$, and every pair of those chains is disjoint except in $G_1$ and $G$, then $n(F_{P_1=G_1}) = \sum_{i=1}^{m} 2^{k_i-2} - (m-1)$.*

*Proof.* Let the chains be

$$\begin{gathered}
G_1 \subset G_2^1 \subset G_3^1 \subset ... \subset G_{k_1-1}^1 \subset G, \\
G_1 \subset G_2^2 \subset G_3^2 \subset ... \subset G_{k_2-1}^2 \subset G, \\
G_1 \subset G_2^3 \subset G_3^3 \subset ... \subset G_{k_3-1}^3 \subset G, \\
\vdots \\
G_1 \subset G_2^m \subset G_3^m \subset ... \subset G_{k_m-1}^m \subset G.
\end{gathered}$$

Then part of the diagram of lattice subgroup $G$ is as shown in Figure 1.

According to Theorem 11, $n(F_{P_1=G_1}) = \sum_{i=2}^{k_1-1} n(F_{P_1=G_i^1}) + n(F_{P_1=G}) + \sum_{i=2}^{k_2-1} n(F_{P_1=G_i^2}) + ... + \sum_{i=2}^{k_m-1} n(F_{P_1=G_i^m})$. Applying Theorem 12 and with a little algebraic manipulation, we have $n(F_{P_1=G_1}) = \sum_{i=1}^{m} 2^{k_i-2} - (m-1)$. □

**Corollary 13.** *Let $G$ be a group. Then*

$$n(F_{P_1=\{e\}}) = \Sigma_{\{e\}\neq H\leq G} n(F_{P_1=H}).$$

*Proof.* The proof follows immediately from Theorem 11. □

**Corollary 14.** *Let $G$ be a group. Then $n(F_G) = 2.n(F_{P_1=\{e\}})$.*

*Proof.* From (2), we have $n(F_G) = \sum_{H\leq G} n(P_1 = H)$. Therefore

$$n(F_G) = n(F_{P_1=\{e\}}) + \sum_{\{e\}\neq H\leq G} n(F_{P_1=H}).$$

Using Corollary 2, we have

$$n(F_G) = n(F_{P_1\{e\}}) + n(F_{P_1\{e\}}) = 2.n(F_{P_1\{e\}}).$$

□

**Example 15.** Consider a diagram of lattice subgroups of $Z_2^3$ (see Figure 1 in [7]). Using Theorem 8, we have

$$\begin{aligned} n(F_{P_1=H_{15}}) =& n(F_{P_1=H_8}) = n(F_{P_1=H_9}) \\ =& n(F_{P_1=H_{10}}) = n(F_{P_1=H_{11}}) = n(F_{P_1=H_{12}}) \\ =& n(F_{P_1=H_{13}}) = n(F_{P_1=H_{14}}) = 1. \end{aligned}$$

Applying Theorem 11 we have

$$\begin{aligned} n(F_{P_1=H_1}) =& n(F_{P_1=H_2}) = n(F_{P_1=H_3}) \\ =& n(F_{P_1=H_4}) = n(F_{P_1=H_5}) = n(F_{P_1=H_6}) \\ =& n(F_{P_1=H_7}) = 4. \end{aligned}$$

Therefore, using Theorem 11 we obtain $n(F_{P_1=H_0}) = 8(1) + 7(4) = 36$. Using Corollary 14 we determine the number of fuzzy subgroups of $Z_2^3$ to be 72.

**Corollary 16.** *Let $\{e\} \subset G_1 \subset G_2 \subset ... \subset G_{k-1} < G_k = G$, be the only maximal chain from $\{e\}$ to $G$ on the lattice subgroups of $G$. Then $n(F_G) = 2^k$.*

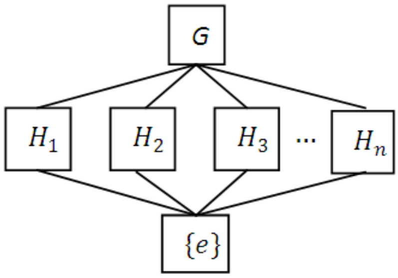


Figure 2: Diagram of lattice subgroups $G$.

*Proof.* The proof follows immediately from Corollary 14 and Theorem 8. □

**Example 17.** Let $p$ be a prime number. Then $\{0\}-Z_p-Z_{p^2}-Z_{p^3}-...-Z_{p^k}$ is the only maximal chain from $\{0\}$ to $Z_{p^k}$. Using Corollary 4 we obtain the number of fuzzy subgroups of $Z_{p^k}$ to be $2^k$.

Note that Theorem 3.1(1) in [4] is a special case of Corollary 16. It is asserted by the following theorem.

**Theorem 18.** *(The number of fuzzy subgroups of p-group). Let $G$ be a cyclic group of order $p^k$, where $k$ is natural number and $p$ is prime. Then $n(F_G) = 2^k$.*

*Proof.* Let $G =< a >$. The only maximal chain from $\{e\}$ to $G$ is $\{e\} \subset< a^{p^{k-1}} >\subset< a^{p^{k-2}} >\subset< a^{p^{k-3}} >\subset ... \subset< a^{p^2} >\subset< a^p >\subset G$. Applying Corollary 16 directly, we have $n(F_G) = 2^k$. □

**Corollary 19.** *Let $\{H_1, H_2, ..., H_k\}$ be the set of all nontrivial subgroups of $G$. If $\forall i, j \in \{1, 2, ..., k\}$ with $i \neq j$ implies $H_i$ is not subset of $H_j$ and $H_j$ is not subset of $H_i$ , then the number of fuzzy subgroups of $G$ is $2(k + 1)$.*

*Proof.* The diagram of lattice subgroups of $G$ is given in Figure 2. According to Theorem 8 we obtain $n(F_{P_1} = H_i) = 1, \forall i = 1, 2, ..., k$. From Theorem 8 and Theorem 11, we obtain $n(F_{P_1} = \{e\}) = 2(k) - (k - 1) = k + 1$. Using Corollary 14, we conclude that the number of fuzzy subgroups of $G$ is $2(k + 1)$. □

**Lemma 20.** *(Murali and Makamba, see Lemma 3.5 in [5]). For a prime $p$, the number of nontrivial subgroups of $G = Z_p \times Z_p$ is $p + 1$.*

**Theorem 21.** *For a prime $p$, the number of fuzzy subgroups of $G = Z_p \times Z_p$ is $2p + 4$.*

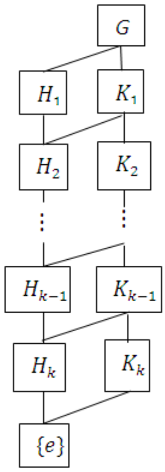


Figure 3: Diagram of lattice subgroups $G$ for m=k.

*Proof.* By Lemma 20, the number of nontrivial subgroups of $G$ is $p+1$. All of them are cyclic of the order $p$. By Corollary 19, we obtain the number of fuzzy subgroups of $G = Z_p \times Z_p = 2p + 4$. □

**Theorem 22.** *Let $\{H_1, H_2, ..., H_m, K_1, K_2, ..., K_m\}$ be the set of all nontrivial subgroups of $G$ where $H_i \supset H_{i+1}, K_i \supset K_{i+1}$, $\forall i = 1, 2, ..., m-1$ and $H_i$ is not subset of $K_i$, $K_i$ is not subset of $H_i, \forall i = 1, 2, ..., m$. Then $n(F_G) = 2^m(m + 2)$.*

*Proof.* We prove by induction on $m$. The statement is true for the first few $m = 1, 2$. Assume that the statement is true for $m = k$. This means that the number of fuzzy subgroups of $G$ whose diagram lattice is shown in Figure 3 is $2^k(k + 2)$. For $m = k + 1$, the diagram lattice subgroups $G$ is shown in Figure 4.

For the case where $m = k + 1$, we have

$$n(F_G) = n(F_{P_1=\{e\}}) + n(F_{P_1=K_{k+1}}) + \sum_{i=1}^{k+1} n(F_{P_1=H_i}) + \sum_{j=1}^{k} n(F_{P_1=K_j}) + n(F_{P_1=G}).$$

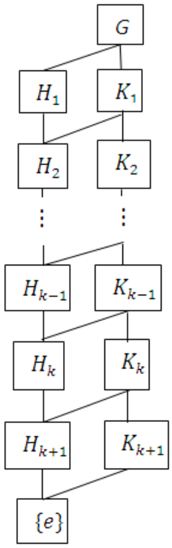


Figure 4: Diagram of lattice subgroups $G$ for m=k+1.

Using Theorem 8, we have $n(F_{P_1=K_{k+1}}) = 2^k$. Thus, we have

$$\sum_{i=1}^{k+1} n(F_{P_1=H_i}) + \sum_{j=1}^{k} n(F_{P_1=K_j}) + n(F_{P_1=G}) = 2^k(k+2).$$

Hence,$n(F_G) = n(F_{P_1=\{e\}})+2^k+2^k(k+2)$. Finally, using Corollary 13, we obtain $n(F_G) = 2[2^k + 2^k(k+2)] = 2^{k+1}(k+3)$. This complete the induction. □

**Corollary 23.** (The number of fuzzy subgroups of $Z_p^n \times Z_q$) *If $p, q$ are distinct primes, then the number of fuzzy subgroups of $Z_p^n \times Z_q$) is $2^n(n+2)$.*

*Proof.* Use Theorem 22. □

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