FUZZY SUBGROUPS COMPUTATION OF
FINITE GROUP BY USING THEIR LATTICES

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Abstract: This paper gives a method to determine the number of fuzzy subgroups of finite group $G$ using diagram of lattice subgroups of $G$. This method can be used for abelian and non-abelian groups. First, an equivalence relation on the set of all fuzzy subgroups of group $G$ is defined. Second, this paper derive some theorems for determination of the number of fuzzy subgroups associated with some chains on the lattice subgroups of $G$. Then the demonstration of the method is given by determining the number of fuzzy subgroups of some finite group $G$.

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1. Introduction

The concept of fuzzy sets was first introduced by Zadeh in 1965 (see [1]). The study of fuzzy algebraic structures was started with the introduction of the concept of fuzzy subgroups by Rosenfeld in 1971 (see [2]). Since the first paper by Rosenfeld, researchers have sought to characterize the fuzzy subgroups

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of various groups, particularly of finite abelian groups. Several papers have
treated the particular case of finite cyclic group. Laszlo (see [3]) studied the
construction of fuzzy subgroups of groups of the orders one to six. Zhang and
Zou (see [4]) have determined the number of fuzzy subgroups of cyclic groups
of the order $p^n$ where $p$ is a prime number. Murali and Makamba (see [5], [6]),
considering a similar problem, found the number of fuzzy subgroups of abelian
groups of the order $p^n q^m$ where $p$ and $q$ are different primes, while Tarnauceanu
and Bentea (see [7]) have found this number for finite abelian groups.

In this paper, we take a different approach to determine the number of fuzzy
subgroups of some finite groups $G$. We determine this number by observing the
diagram of lattice subgroups of $G$. First, we define the equivalence relation on
the set of all fuzzy subgroups of group $G$ (see Definition 3). This definition
differs from that of Murali and Makamba (see [5], [6], [8]), but is equivalent
to the definitions of Dixit (see [9]), Zhang (see [4]) and Tarnauceanu (see [7],
[10]). We prefer to use this definition because it generalizes the definition used
by Murali. We observed patterns of diagram of lattice subgroups of $G$ and
determined its influence on the formula of the number of fuzzy subgroups. This
method can be used not only for abelian group, but also for non-abelian as well.

We found some formulas in more general. Results of Zhang (see theorem
3.1.(1) ) in [4] and Murali (see proposition 3.3 and theorem 3) in [5] can be
viewed as special cases from our formula (see Theorem 18, Example 17 and
Corollary 23, respectively). Tarnauceanu (see [7]) asserted that computing the
number of fuzzy subgroups of $Z_3^2 = Z_2 \times Z_2 \times Z_2$ by using ”the method of direct
calculation” is too difficult. We will show, however, that it is easy to compute
using our method (see Example 15).

2. Preliminaries

In this paper, a group $G$ is assumed to be a finite group. First of all, we present
some basic notions and results that will be used later.

**Definition 1.** Let $X$ be a nonempty set. A fuzzy subset of $X$ is a function
from $X$ into $[0, 1]$.

**Definition 2.** (Rosenfeld, see [2]). Let $G$ be a group. A fuzzy subset $\mu$ of
$G$ is called a fuzzy subgroup of $G$ if:

1. $\mu(xy) \geq \min\{\mu(x), \mu(y)\}, \forall x, y \in G,$
2. $\mu(x^{-1}) \geq \mu(x), \forall x \in G.$

If $\mu$ is a fuzzy subgroup of group $G$ and $\mu(G) = \{\theta_1, \theta_2, ..., \theta_n\}$, then we
assume that $\theta_1 > \theta_2 > ... > \theta_n$.

**Theorem 3.** (Rosenfeld, see [2]). Let $e$ denote the identity element of $G$. If $\mu$ is a fuzzy subgroup of $G$, then:
1. $\mu(e) \geq \mu(x), \forall x \in G$,
2. $\mu(x^{-1}) = \mu(x), \forall x \in G$.

**Theorem 4.** (Sulaiman and Abdul Ghafur, see [11]). A fuzzy subset $\mu$ of $G$ is a fuzzy subgroup of $G$ if and only if there is a chain of subgroups $G = P_1(\mu) \leq P_2(\mu) \leq P_3(\mu) \leq ... \leq P_n(\mu) = G$ such that $\mu$ is in the form:

$$
\mu(x) = \begin{cases} 
\theta_1, & x \in P_1(\mu) \\
\theta_2, & x \in P_2(\mu) \\
\vdots \\
\theta_m, & x \in P_m(\mu) 
\end{cases} \quad (1)
$$

If there is no confusion, then $P_i(\mu)$ as in (1) can simply be written as $P_i$. We define length (or order) of fuzzy subgroup $\mu$ in (1) as be $m$.

**Example 5.** Consider the group $G = Z_{12}$. Define functions $\mu, \gamma, \alpha$ and $\beta$ as follows:

$$
\mu(x) = \begin{cases} 
1, & x \in \{0, 2, 4, 6, 8, 10\} \\
\frac{1}{2}, & x \in \{1, 3, 5, 7, 9, 11\} 
\end{cases}, \\
\gamma(x) = \begin{cases} 
1, & x \in \{0, 1, 2, 3, 4, 5\} \\
\frac{1}{2}, & x \in \{6, 7, 8, 9, 10, 11\} 
\end{cases}, \\
\alpha(x) = \begin{cases} 
1, & x \in \{0, 4, 8\} \\
\frac{1}{2}, & x \in \{2, 6, 10\} \\
\frac{1}{3}, & x \in \{1, 3, 5, 7, 9, 11\} 
\end{cases}, \\
\beta(x) = \begin{cases} 
\frac{1}{2}, & x \in \{0, 4, 8\} \\
\frac{1}{3}, & x \in \{2, 6, 10\} \\
0, & x \in \{1, 3, 5, 7, 9, 11\} 
\end{cases}.
$$

Note that

$$P_1(\mu) = \{0, 2, 4, 6, 8, 10\}, P_2(\mu) = \{0, 2, 4, 6, 8, 10\} \cup \{1, 3, 5, 7, 9, 11\} = Z_{12}.$$

Thus $P_1(\mu)$ and $P_2(\mu)$ both are subgroups of $Z_{12}$. By Theorem 2, $\mu$ is fuzzy subgroup of $Z_{12}$. Similarly, we can show that $\alpha$ and $\beta$ both are fuzzy subgroups of $Z_{12}$. Note that $P_1(\gamma) = \{0, 1, 2, 3, 4, 5\}$ is not a subgroup of $Z_{12}$, so $\gamma$ is not a fuzzy subgroup of $Z_{12}$.
3. Equivalence Relation on Fuzzy Subgroups

Without any equivalence relation on fuzzy subgroups of group \( G \), the number of fuzzy subgroups is infinite, even for the trivial group \( \{e\} \). So we define equivalence relation on the set of all fuzzy subgroups of a given group. Using this equivalent, we compute the number of distinct equivalence classes of fuzzy subgroups of group \( G \).

**Definition 6.** Let \( \mu, \gamma \) be fuzzy subgroups of \( G \) of the form

\[
\mu(x) = \begin{cases} 
\theta_1, & x \in P_1 \\
\theta_2, & x \in P_2 \setminus P_1 \\
\theta_3, & x \in P_3 \setminus P_2 \\
\vdots \\
\theta_n, & x \in P_n \setminus P_{n-1}
\end{cases}, \quad \gamma(x) = \begin{cases} 
\delta_1, & x \in P_1 \\
\delta_2, & x \in P_2 \setminus P_1 \\
\delta_3, & x \in P_3 \setminus P_2 \\
\vdots \\
\delta_m, & x \in P_m \setminus P_{m-1}
\end{cases}.
\]

Then we define that \( \mu \) and \( \gamma \) are equivalent if:

1. \( m = n \), and
2. \( P_i = M_i, \forall i \in \{1, 2, 3, ..., n\} \).

We write \( \mu \sim \gamma \). It is easily checked that this relation is indeed an equivalence relation. Two fuzzy subgroups of \( G \) are said to be different if they are not equivalent. In this paper, the number of fuzzy subgroups of group \( G \) means the number of distinct equivalence classes of fuzzy subgroups.

**Example 7.** Let \( \mu, \alpha \) and \( \beta \) be fuzzy subgroups as in Example 5. Since \( |\mu| \neq |\alpha| \) and \( |\mu| \neq |\beta| \), by Definition 6 we determine that \( \mu \) and \( \alpha \) are not equivalent. Fuzzy subgroups \( \mu \) and \( \beta \) are not equivalent either. Note that \( |\alpha| = |\beta| \) and it is easy to show that \( P_i(\alpha) = P_i(\beta), \forall i \in \{1, 2, 3\} \). Thus, \( \alpha \sim \beta \).

4. Method of Counting Using Lattices

In this section we discuss a method to determine the number of fuzzy subgroups of a group \( G \) using a diagram of lattice subgroups of \( G \). We observe at every chain and pattern of diagram of lattice subgroups of \( G \) to compute the number of fuzzy subgroup. This method can be used both for abelian and non-abelian groups. We denote the number of fuzzy subgroups of \( G \) as \( n(F_G) \), while the number of fuzzy subgroups \( \mu \) of \( G \) with \( P_1(\mu) = H \) is denoted by \( n(F_{P_1=H}) \). From Theorem 4, we have

\[
n(F_G) = \sum_{H \leq G} n(F_{P_1=H})
\] (2)
**Theorem 8.** Let $G_1 < G_2 < ... < G_{k-1} < G_k = G$ be the only maximal chain from $G_1$ to $G$ on the lattice subgroups of $G$. Then:

$$n(F_{P_1=G_1}) = \begin{cases} 1, & k=1,2 \\ 2^{k-2}, & k > 2 \end{cases}.$$

**Proof.** If $k = 1$, then $G_1 = G$. We have only one fuzzy subgroup of $G$ where $P_1 = G$. Its length is $1$, that is, $\mu(x) = \theta_1, \forall x \in G$. If $k = 2$, we have only one fuzzy subgroup of $G$ where $P_1 = G_1$. Its length is $2$, that is,

$$\mu(x) = \begin{cases} \theta_1, & x \in G_1 \\ \theta_2, & x \in G \setminus G_1 \end{cases}.$$

If $k > 2$, the part of the diagram of lattice subgroups of $G$ is $G_1 - G_2 - ... - G_k = G$. Let $\mu$ be the fuzzy subgroup of $G$ where $P_1(\mu) = G_1$. Since $G_1 \neq G$ and since the maximal chain $G_1 \subset G_2 \subset ... \subset G_k = G$ contains $k$ subgroups of $G$, then the possible length of $\mu$ is $2, 3, 4, ..., (k-1)$ or $k$. Let $n$ be an arbitrary element of $\{2, 3, 4, ..., k\}$. We will count the number of fuzzy subgroups $\mu$ of length $n$ where $P_1 = G_1$ and $P_n = G$. For $P_2, P_3, ..., P_n$, we can choose one of $(k-2)$ pieces of the existing subgroup $G_2, G_2, ..., G_{k-1}$. So the number of fuzzy subgroups of $G$ with $P_1 = G_1$ of length $n$ is $C_{n-2}^k$. Thus, the number of fuzzy subgroups of $G$ with $P_1 = G_1$ is equal to $C_0^k + C_1^k + C_2^k + ... + C_{k-2}^k + C_{k-2}^k = 2^{k-2}$. □

**Corollary 9.** Let $G_1 < G_2 < ... < G_{k-1} < G_k = G, k \geq 2$ be the only maximal chain from $G_1$ to $G$ on the lattice subgroups of $G$. For $m, 1 \leq m \leq m-1$ we have $n(F_{P_1=G_m}) = 2^{k-m-1}$.

**Proof.** Use Theorem 8. □

**Corollary 10.** Let $G_1 < G_2 < ... < G_{k-1} < G_k = G, k \geq 3$ be the only maximal chain from $G_1$ to $G$ on the lattice subgroups of $G$. For $m, 1 \leq m \leq m-2$ we have $n(F_{P_1=G_m}) = 2n(F_{P_1=G_{m+1}})$.

**Proof.** Using Theorem 3, we have $n(F_{P_1=G_m}) = C_0^{k-m-1} + C_1^{k-m-1} + C_2^{k-m-1} + ... + C_{k-m-1}^{k-m-1}$ and $n(F_{P_1=G_{m+1}}) = C_0^{k-m-2} + C_1^{k-m-2} + C_2^{k-m-2} + ... + C_{k-m-2}^{k-m-2}$. Then using Pascal’s recursion formula (see [12]), $C_k^m = C_{k-1}^{m-1} + C_{k-1}^{m-1}$, we obtain $n(F_{P_1=G_m}) = C_0^{k-m-1} + C_0^{k-m-2} + C_1^{k-m-2} + C_1^{k-m-2} + C_2^{k-m-2} + C_3^{k-m-2} + ... + C_{k-m-2}^{k-m-2} + C_{k-m-3}^{k-m-2} + C_{k-m-2}^{k-m-2} + C_{k-m-1}^{k-m-2} = 2[n(F_{P_1=G_{m+1}})]$. □

**Theorem 11.** Let $H$ be a subgroup of $G$, and let the set of all subgroups of $G$ which contain $H$ (but are not equal to $H$) be $\{H_1, H_2, H_3, ..., H_k\}$. Then $n(F_{P_1=H}) = \Sigma_{i=1}^{k} n(F_{P_1=H_i})$. 
Proof. We have \( n(F_{P_1=H}) = \sum_{i=1}^{k} n(F_{P_1=H;P_2=H_i}). \) Note that \( \forall i \in \{1, 2, ..., k\} \),
\[
n(F_{P_1=H;P_2=H_i}) = n(F_{P_1=H_i}).
\]
Hence, we obtain \( n(F_{P_1=H}) = \sum_{i=1}^{k} n(F_{P_1=H_i}) \).

**Theorem 12.** Let \( G_1 \) be a subgroup of \( G \). If there are \( m (m \geq 2) \) maximal chains from \( G_1 \) to \( G \) on the lattice subgroup \( G \) of length \( k_1, k_2, ..., k_m \), and every pair of those chains is disjoint except in \( G_1 \) and \( G \), then \( n(F_{P_1=G_1}) = \sum_{i=1}^{m} 2^{k_i-2} - (m-1) \).

Proof. Let the chains be
\[
G_1 \subset G_2^1 \subset G_3^1 \subset ... \subset G_{k_1-1}^1 \subset G,
G_1 \subset G_2^2 \subset G_3^2 \subset ... \subset G_{k_2-1}^2 \subset G,
G_1 \subset G_2^3 \subset G_3^3 \subset ... \subset G_{k_3-1}^3 \subset G,
\]
...,
\[
G_1 \subset G_2^m \subset G_3^m \subset ... \subset G_{k_m-1}^m \subset G.
\]
Then part of the diagram of lattice subgroup \( G \) is as shown in Figure 1.

According to Theorem 11, \( n(F_{P_1 = G}) = \sum_{i=2}^{k_1-1} n(F_{P_1 = G_i^1}) + n(F_{P_1 = G}) + \sum_{i=2}^{k_2-1} n(F_{P_1 = G_i^2}) + ... + \sum_{i=2}^{k_m-1} n(F_{P_1 = G_i^m}) \). Applying Theorem 12 and with a little algebraic manipulation, we have
\[
n(F_{P_1 = G_1}) = \sum_{i=1}^{m} 2^{k_i-2} - (m - 1).
\]

**Corollary 13.** Let \( G \) be a group. Then
\[
n(F_{P_1 = \{e\}}) = \sum_{\{e\} \neq H \leq G} n(F_{P_1 = H}).
\]

**Proof.** The proof follows immediately from Theorem 11. \( \square \)

**Corollary 14.** Let \( G \) be a group. Then \( n(F_G) = 2.n(F_{P_1 = \{e\}}) \).

**Proof.** From (2), we have \( n(F_G) = \sum_{H \leq G} n(P_1 = H) \). Therefore
\[
n(F_G) = n(F_{P_1 = \{e\}}) + \sum_{\{e\} \neq H \leq G} n(F_{P_1 = H}).
\]
Using Corollary 2, we have
\[
n(F_G) = n(F_{P_1 = \{e\}}) + n(F_{P_1 = \{e\}}) = 2.n(F_{P_1 = \{e\}}).
\]

**Example 15.** Consider a diagram of lattice subgroups of \( Z_2^3 \) (see Figure 1 in [7]). Using Theorem 8, we have
\[
n(F_{P_1 = H_1}) = n(F_{P_1 = H_8}) = n(F_{P_1 = H_9})
= n(F_{P_1 = H_{10}}) = n(F_{P_1 = H_{11}}) = n(F_{P_1 = H_{12}})
= n(F_{P_1 = H_{13}}) = n(F_{P_1 = H_{14}}) = 1.
\]

Applying Theorem 11 we have
\[
n(F_{P_1 = H_1}) = n(F_{P_1 = H_2}) = n(F_{P_1 = H_3})
= n(F_{P_1 = H_4}) = n(F_{P_1 = H_5}) = n(F_{P_1 = H_6})
= n(F_{P_1 = H_7}) = 4.
\]

Therefore, using Theorem 11 we obtain \( n(F_{P_1 = H_0}) = 8(1) + 7(4) = 36 \). Using Corollary 14 we determine the number of fuzzy subgroups of \( Z_2^3 \) to be 72.

**Corollary 16.** Let \( \{e\} \subset G_1 \subset G_2 \subset \ldots \subset G_{k-1} < G_k = G \), be the only maximal chain from \( \{e\} \) to \( G \) on the lattice subgroups of \( G \). Then \( n(F_G) = 2^k \).
Proof. The proof follows immediately from Corollary 14 and Theorem 8.

**Example 17.** Let $p$ be a prime number. Then $\{0\} - Z_p - Z_p^2 - Z_p^3 - ... - Z_p^k$ is the only maximal chain from $\{0\}$ to $Z_p^k$. Using Corollary 4 we obtain the number of fuzzy subgroups of $Z_p^k$ to be $2^k$.

Note that Theorem 3.1(1) in [4] is a special case of Corollary 16. It is asserted by the following theorem.

**Theorem 18.** (The number of fuzzy subgroups of $p$-group). Let $G$ be a cyclic group of order $p^k$, where $k$ is natural number and $p$ is prime. Then $n(F_G) = 2^k$.

**Proof.** Let $G = \langle a \rangle$. The only maximal chain from $\{e\}$ to $G$ is $\{e\} \subset \subset a^{p^k-1} \subset \subset a^{p^k-2} \subset \subset a^{p^{k-3}} \subset \subset ... \subset \subset a^2 \subset \subset a \subset \subset G$. Applying Corollary 16 directly, we have $n(F_G) = 2^k$.

**Corollary 19.** Let $\{H_1, H_2, ..., H_k\}$ be the set of all nontrivial subgroups of $G$. If $\forall i, j \in \{1, 2, ..., k\}$ with $i \neq j$ implies $H_i$ is not subset of $H_j$ and $H_j$ is not subset of $H_i$, then the number of fuzzy subgroups of $G$ is $2(k + 1)$.

**Proof.** The diagram of lattice subgroups of $G$ is given in Figure 2. According to Theorem 8 we obtain $n(F_{P_1} = H_i) = 1, \forall i = 1, 2, ..., k$. From Theorem 8 and Theorem 11, we obtain $n(F_{P_1} = \{e\}) = 2(k) - (k - 1) = k + 1$. Using Corollary 14, we conclude that the number of fuzzy subgroups of $G$ is $2(k + 1)$.

**Lemma 20.** (Murali and Makamba, see Lemma 3.5 in [5]). For a prime $p$, the number of nontrivial subgroups of $G = Z_p \times Z_p$ is $p + 1$.

**Theorem 21.** For a prime $p$, the number of fuzzy subgroups of $G = Z_p \times Z_p$ is $2p + 4$.  

![Figure 2: Diagram of lattice subgroups $G$.](image-url)
Proof. By Lemma 20, the number of nontrivial subgroups of $G$ is $p + 1$. All of them are cyclic of the order $p$. By Corollary 19, we obtain the number of fuzzy subgroups of $G = \mathbb{Z}_p \times \mathbb{Z}_p = 2^p + 4$. \hfill \square

**Theorem 22.** Let $\{H_1, H_2, ..., H_m, K_1, K_2, ..., K_m\}$ be the set of all nontrivial subgroups of $G$ where $H_i \supset H_{i+1}, K_i \supset K_{i+1}, \forall i = 1, 2, ..., m - 1$ and $H_i$ is not subset of $K_i$, $K_i$ is not subset of $H_i, \forall i = 1, 2, ..., m$. Then $n(F_G) = 2^m(m + 2)$.

Proof. We prove by induction on $m$. The statement is true for the first few $m = 1, 2$. Assume that the statement is true for $m = k$. This means that the number of fuzzy subgroups of $G$ whose diagram lattice is shown in Figure 3 is $2^k(k + 2)$. For $m = k + 1$, the diagram lattice subgroups $G$ is shown in Figure 4.

For the case where $m = k + 1$, we have

$$n(F_G) = n(F_{P_1 = \{e\}}) + n(F_{P_1 = K_{k+1}}) + \sum_{i=1}^{k+1} n(F_{P_1 = H_i}) + \sum_{j=1}^{k} n(F_{P_1 = K_j}) + n(F_{P_1 = G}).$$
Using Theorem 8, we have \( n(F_{P_1 = K_{k+1}}) = 2^k \). Thus, we have

\[
\sum_{i=1}^{k+1} n(F_{P_1 = H_i}) + \sum_{j=1}^{k} n(F_{P_1 = K_j}) + n(F_{P_1 = G}) = 2^k (k + 2).
\]

Hence, \( n(F_G) = n(F_{P_1 = \{e\}}) + 2^k + 2^k (k + 2) \). Finally, using Corollary 13, we obtain

\[
n(F_G) = 2[2^k + 2^k (k + 2)] = 2^{k+1} (k + 3).
\]

This complete the induction.  

**Corollary 23.** (The number of fuzzy subgroups of \( \mathbb{Z}_p^n \times \mathbb{Z}_q \)) If \( p, q \) are distinct primes, then the number of fuzzy subgroups of \( \mathbb{Z}_p^n \times \mathbb{Z}_q \) is \( 2^n (n + 2) \).

**Proof.** Use Theorem 22.

**References**


