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NONOSCILLATORY BOUNDED SOLUTIONS FOR A SYSTEM OF HIGHER ORDER NONLINEAR NEUTRAL DELAY DIFFERENTIAL EQUATIONS

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Abstract: A system of higher order nonlinear neutral delay differential equations is studied in this paper, and some sufficient conditions for existence of nonoscillatory bounded solutions for this system are established by Krasnoselkii fixed point theorem and Schauder fixed point theorem, and expressed through several theorems according to the range of the value of the functions $P_1(t), P_2(t)$ and their combinations.

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1. Introduction and Preliminaries

We consider the following nonlinear differential system

$$\begin{aligned}&\Big\{r_{in}(t)\cdots\Big[r_{i2}(t)\big[r_{i1}(t)[x_i(t)-P_i(t)x_i(t-\tau_i)]'\big]'\Big]'\cdots\Big\}'\\&\quad=F_i\big(t,x_{3-i}(t-\sigma_1),x_{3-i}(t-\sigma_2),\cdots,x_{3-i}(t-\sigma_m)\big),\quad t\geq t_0,\end{aligned}\tag{1.1}$$

where $\tau_i > 0, \sigma_1, \sigma_2, \cdots, \sigma_m \geq 0, r_{il} \in C([t_0,+\infty), \mathbb{R}^+), 1 \leq l \leq n, P_i(t) \in C([t_0,+\infty),\mathbb{R}), F_i \in C([t_0,+\infty) \times \mathbb{R}^m, \mathbb{R})$ and $i \in \{1,2\}$.

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Applying Krasnoselkii fixed point theorem and Schauder fixed point theorem, we obtained a few sufficient conditions for the existence of a nonoscillatory bounded solution of the system (1.1).

Lemma 1.1. (Krasnoselskii Fixed Point Theorem, see [4]) *Let Ω be a bounded closed convex subset of a Banach space X and $Q, S : \Omega \to X$ satisfy $Qx + Sy \in \Omega$ for each $x, y \in \Omega$. If Q is a contraction mapping and S is a completely continuous mapping, then the equation $Qx + Sx = x$ has at least one solution in Ω.*

Lemma 1.2. (Schauder Fixed Point Theorem, see [4]) *Let Ω be a closed, convex and nonempty subset of a Banach space X and $S : \Omega \to \Omega$ be a continuous mapping such that $S\Omega$ is a relatively compact subset of X. Then S has at least one fixed point in Ω. That is, there exists an $x \in \Omega$ such that $Sx = x$.*

2. Existence of Nonoscillatory Bounded Solutions

In this section, a few sufficient conditions of existence of nonoscillatory bounded solutions for system (1.1) will be given.

Theorem 2.1. *Let functions $h_i, q_i, r_{il} \in C([t_0, +\infty), \mathbb{R}^+)$ and $P_i(t) \in C([t_0, +\infty), \mathbb{R})$ satisfy that*

$$P_i(t) \equiv 1, \tag{2.1}$$

$$\begin{aligned} &\big|F_i(t, u_1, u_2, \cdots, u_m) - F_i(t, v_1, v_2, \cdots, v_m)\big| \\ &\qquad \le h_i(t) \max\big\{|u_j - v_j| : \ 1 \le j \le m\big\}, \end{aligned} \tag{2.2}$$

$$\big|F_i(t, u_1, u_2, \cdots, u_m)\big| \le q_i(t), \tag{2.3}$$

$$\int_{t_0}^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty} \frac{s_1 \max\{q_i(s), h_i(s)\}}{\big|\prod_{l=1}^{n} r_{il}(s_l)\big|} ds ds_n \cdots ds_1 < +\infty, \tag{2.4}$$

where $i \in \{1, 2\}, l \in \{1, 2, \cdots, n\}$. Then the system (1.1) has a nonoscillatory bounded solution.

Proof. According to a known result (Theorem 3.2.6 in [4]), (2.4) is equivalent to the condition

$$\sum_{k=0}^{\infty}\int_{t_0+k\tau_i}^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty} \frac{\max\{q_i(s), h_i(s)\}}{\big|\prod_{l=1}^{n} r_{il}(s_l)\big|} ds ds_n \cdots ds_1 < +\infty. \tag{2.5}$$

By (2.5), a sufficiently large $T > t_0$ can be chosen such that

$$\sum_{k=1}^{\infty}\int_{T+k\tau_i}^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty} \frac{\max\{q_i(s), h_i(s)\}}{\big|\prod_{l=1}^{n} r_{il}(s_l)\big|} ds ds_n \cdots ds_1 < 1. \tag{2.6}$$

Let $C([t_0,+\infty),\mathbb{R}^2)$ be the set of all continuous vector functions $x(t)=(x_1(t), x_2(t))$ with the norm $||x||=\sup_{t\geq t_0}\{|x_1(t)|,|x_2(t)|\}<+\infty$. Obviously, $C([t_0, +\infty),\mathbb{R}^2)$ is a Banach space. Now, define a bounded, closed and convex subset Ω of $C([t_0,+\infty),\mathbb{R}^2)$ as following:

$$\Omega=\Big\{x=(x_1,x_2)\in C([t_0,+\infty),\mathbb{R}^2):1\leq x_i(t)\leq 3, i\in\{1,2\}, t\geq t_0\Big\}.$$

Let mapping $S=(S_1,S_2):\Omega\to C([t_0,+\infty),\mathbb{R}^2)$ be defined as

$$(S_ix)(t)=\begin{cases}2+(-1)^n\sum_{k=1}^{\infty}\int_{t+k\tau_i}^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty}\\ \qquad\frac{F_i(s,x_{3-i}(s-\sigma_1),\cdots,x_{3-i}(s-\sigma_m))}{\prod_{l=1}^{n}r_{il}(s_l)}dsds_n\cdots ds_1, & t\geq T\\ (S_ix)(T), & t_0\leq t<T\end{cases}\tag{2.7}$$

for all $x\in\Omega$, where $i\in\{1,2\}$.

It is claimed that S is a self mapping on Ω. For all $x=(x_1,x_2)\in\Omega, i\in 1,2$ and $t\geq T$, by (2.3) and (2.6), we have

$$(S_ix)(t)\geq 2-\sum_{k=1}^{\infty}\int_{T+k\tau_i}^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty}\frac{q_i(s)}{|\prod_{l=1}^{n}r_{il}(s_l)|}dsds_n\cdots ds_1\geq 1,$$

and

$$(S_ix)(t)\leq 2+\sum_{k=1}^{\infty}\int_{T+k\tau_i}^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty}\frac{q_i(s)}{|\prod_{l=1}^{n}r_{il}(s_l)|}dsds_n\cdots ds_1\leq 3.$$

Therefore, $S\Omega\subset\Omega$.

Now we show that S is continuous. Let $x_k=(x_{1k}(t),x_{2k}(t))\in\Omega$ and $x_{ik}(t)\to x_i(t)$ as $k\to+\infty$. Since Ω is closed, $x=(x_1(t),x_2(t))\in\Omega$. For $t\geq T$, (2.2) guarantees that

$$\begin{aligned}\big|(S_ix_k)(t)-(S_ix)(t)\big|\leq&\sum_{k=1}^{\infty}\int_{t+k\tau_i}^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty}\frac{1}{|\prod_{l=1}^{n}r_{il}(s_l)|}\\ &\big|F_i(s,x_{3-i\ k}(s-\sigma_1),\cdots,x_{3-i\ k}(s-\sigma_m))\\ &-F_i(s,x_{3-i}(s-\sigma_1),\cdots,x_{3-i}(s-\sigma_m))\big|dsds_n\cdots ds_1\\ \leq&\sum_{k=1}^{\infty}\int_{t+k\tau_i}^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty}\\ &\frac{h_i(s)\max\left\{|x_{3-i\ k}(s-\sigma_j)-x_{3-i}(s-\sigma_j)|:1\leq j\leq m\right\}}{|\prod_{l=1}^{n}r_{il}(s_l)|}dsds_n\cdots ds_1\\ \leq&\|x_k-x\|\sum_{k=1}^{\infty}\int_{T+k\tau_i}^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty}\frac{h_i(s)}{|\prod_{l=1}^{n}r_{il}(s_l)|}dsds_n\cdots ds_1.\end{aligned}$$

This above inequality together with (2.6) implies that S is continuous.

Next, we prove $S\Omega$ is relatively compact. It is sufficient to show that the family of functions $\{Sx : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, +\infty)$. $S\Omega \subset \Omega$ ensures the uniform boundedness. For the equicontinuity, according to Levitan's result [10], it is only need to prove that, for any given $\varepsilon > 0$, $[t_0, +\infty)$ can be decomposed into finite subintervals in such a way that on each subinterval all functions of the family have change of amplitude less than ε. By (2.6), for any $\varepsilon > 0$, take $T' \geq T$ large enough so that

$$\sum_{k=1}^{\infty} \int_{T'+k\tau_i}^{+\infty} \int_{s_1}^{+\infty} \cdots \int_{s_n}^{+\infty} \frac{q_i(s)}{\left|\prod_{l=1}^{n} r_{il}(s_l)\right|} ds ds_n \cdots ds_1 < \frac{\varepsilon}{2}. \tag{2.8}$$

Then, for any $x \in \Omega$ and $t_2 > t_1 \geq T'$, (2.8) ensures that

$$\begin{aligned}\big|(S_i x)(t_2) - (S_i x)(t_1)\big| \leq & \sum_{k=1}^{\infty} \int_{t_2+k\tau_i}^{+\infty} \int_{s_1}^{+\infty} \cdots \int_{s_n}^{+\infty} \\ & \frac{\big|F_i(s, x_{3-i}(s-\sigma_1), \cdots, x_{3-i}(s-\sigma_m))\big|}{\left|\prod_{l=1}^{n} r_{il}(s_l)\right|} ds ds_n \cdots ds_1 \\ & + \sum_{k=1}^{\infty} \int_{t_1+k\tau_i}^{+\infty} \int_{s_1}^{+\infty} \cdots \int_{s_n}^{+\infty} \\ & \frac{\big|F_i(s, x_{3-i}(s-\sigma_1), \cdots, x_{3-i}(s-\sigma_m))\big|}{\left|\prod_{l=1}^{n} r_{il}(s_l)\right|} ds ds_n \cdots ds_1 \\ \leq & \sum_{k=1}^{\infty} \int_{T'+k\tau_i}^{+\infty} \int_{s_1}^{+\infty} \cdots \int_{s_n}^{+\infty} \frac{q_i(s)}{\left|\prod_{l=1}^{n} r_{il}(s_l)\right|} ds ds_n \cdots ds_1 \\ & + \sum_{k=1}^{\infty} \int_{T'+k\tau_i}^{+\infty} \int_{s_1}^{+\infty} \cdots \int_{s_n}^{+\infty} \frac{q_i(s)}{\left|\prod_{l=1}^{n} r_{il}(s_l)\right|} ds ds_n \cdots ds_1 \\ < & \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.\end{aligned}$$

For any $x \in \Omega$ and $T \leq t_1 < t_2 \leq T'$, there exists $\delta > 0$ such that if $0 < t_2 - t_1 < \delta$, then

$$\begin{aligned}\big|(S_i x)(t_2) - (S_i x)(t_1)\big| \leq & \sum_{k=1}^{\infty} \int_{t_1+k\tau_i}^{t_2+k\tau_i} \int_{s_1}^{+\infty} \cdots \int_{s_n}^{+\infty} \\ & \frac{\big|F_i(s, x_{3-i}(s-\sigma_1), \cdots, x_{3-i}(s-\sigma_m))\big|}{\left|\prod_{l=1}^{n} r_{il}(s_l)\right|} ds ds_n \cdots ds_1\end{aligned}$$

$$\leq \sum_{k=1}^{\infty} \int_{t_1+k\tau_i}^{t_2+k\tau_i} \int_{s_1}^{+\infty} \cdots \int_{s_n}^{+\infty} \frac{q_i(s)}{\left|\prod_{l=1}^{n} r_{il}(s_l)\right|} ds ds_n \cdots ds_1 < \varepsilon.$$

For any $x \in S$ and $t_0 \leq t_1 < t_2 \leq T$, it is easy to get that

$$\left|(S_i x)(t_2) - (S_i x)(t_1)\right| = 0 < \varepsilon.$$

Consequently, $\{S_i x : x \in \Omega\}$ is uniformly bounded and equicontinuous on $[t_0, +\infty)$. Therefore $S\Omega$ is relatively compact. Applying Lemma 1.2, we could find a $x_0 = (x_{01}, x_{02}) \in \Omega$ such that $Sx_0 = x_0$. That is

$$x_{0i}(t) = \begin{cases} 2 + (-1)^n \sum_{k=1}^{\infty} \int_t^{+\infty} \int_{s_1}^{+\infty} \cdots \int_{s_n}^{+\infty} \\ \quad \frac{F_i(s, x_{0\ 3-i}(s-\sigma_1), \cdots, x_{0\ 3-i}(s-\sigma_m))}{\prod_{l=1}^{n} r_{il}(s_l)} ds ds_n \cdots ds_1, & t \geq T \\ x_{0i}(T), & t_0 \leq t < T \end{cases} \tag{2.9}$$

where $i \in \{1, 2\}$. For $t \geq T$,

$$x_{0i}(t) - x_{0i}(t-\tau_i) = (-1)^{n+1} \int_t^{+\infty} \int_{s_1}^{+\infty} \cdots \int_{s_n}^{+\infty} \frac{F_i(s, x_{0\ 3-i}(s-\sigma_1), \cdots, x_{0\ 3-i}(s-\sigma_m))}{\prod_{l=1}^{n} r_{il}(s_l)} ds ds_n \cdots ds_1.$$

Then,

$$[x_{0i}(t) - x_{0i}(t-\tau_i)]' = (-1)^n \int_t^{+\infty} \int_{s_2}^{+\infty} \cdots \int_{s_n}^{+\infty} \frac{F_i(s, x_{0\ 3-i}(s-\sigma_1), \cdots, x_{0\ 3-i}(s-\sigma_m))}{r_{i1}(t) \prod_{l=2}^{n} r_{il}(s_l)} ds ds_n \cdots ds_2,$$

which we can rewrite it as

$$r_{i1}(t)[x_{0i}(t) - x_{0i}(t-\tau_i)]' = (-1)^n \int_t^{+\infty} \int_{s_2}^{+\infty} \cdots \int_{s_n}^{+\infty} \frac{F_i(s, x_{0\ 3-i}(s-\sigma_1), \cdots, x_{0\ 3-i}(s-\sigma_m))}{\prod_{l=2}^{n} r_{il}(s_l)} ds ds_n \cdots ds_2.$$

Finding the derivative,

$$\left[r_{i1}(t)[x_{0i}(t) - x_{0i}(t-\tau_i)]'\right]' = (-1)^{n-1} \int_t^{+\infty} \int_{s_3}^{+\infty} \cdots \int_{s_n}^{+\infty} \frac{F_i(s, x_{0\ 3-i}(s-\sigma_1), \cdots, x_{0\ 3-i}(s-\sigma_m))}{r_{i2}(t) \prod_{l=3}^{n} r_{il}(s_l)} ds ds_n \cdots ds_3.$$

Proceeding as before, we get

$$\Big\{r_{in}(t)\cdots\Big[r_{i2}(t)\big[r_{i1}(t)[x_{0i}(t)-x_{0i}(t-\tau_i)]'\big]'\Big]'\cdots\Big\}'$$
$$=F_i\big(t,x_{0\ 3-i}(t-\sigma_1),\cdots,x_{0\ 3-i}(t-\sigma_m)\big).$$

Therefore, $x_0(t)$ is a bounded nonoscillatory solution of the system (1.1). This completes the proof.

Theorem 2.2. *Let functions* $h_i, q_i, r_{il} \in C([t_0,+\infty),\mathbb{R}^+)$ *and* $P_i(t) \in C([t_0,+\infty),\mathbb{R})$ *satisfy that (2.2), (2.3) and*

$$|P_i(t)| \le \overline{P_i} < \frac{1}{2}, \tag{2.10}$$

$$\int_{t_0}^{+\infty} \max\Big\{\frac{1}{|r_{il}(t)|}, h_i(t), q_i(t)\Big\}dt < +\infty, \tag{2.11}$$

where $i \in \{1,2\}, l \in \{1,2,\cdots,n\}$. *Then the system (1.1) has a nonoscillatory bounded solution.*

Proof. In virtue of (2.11), a sufficiently large $T > t_0$ can be chosen such that

$$\int_T^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty}\frac{q_i(s)}{|\prod_{l=1}^n r_{il}(s_l)|}dsds_n\cdots ds_1 \le \frac{1}{2}-\overline{P_i}, \tag{2.12}$$

where $i \in \{1,2\}$.

Let $C([t_0,+\infty),\mathbb{R}^2)$ be the set like that in the proof of Theorem 2.1 and define a bounded, closed and convex subset Ω of $C([t_0,+\infty),\mathbb{R}^2)$ as following:

$$\Omega = \Big\{x=(x_1,x_2)\in C([t_0,+\infty),\mathbb{R}^2) : 0 \le x_i(t) \le 1, i\in\{1,2\}, t \ge t_0\Big\}.$$

Let mappings $Q=(Q_1,Q_2)$ and $S=(S_1,S_2):\Omega\to C([t_0,+\infty),\mathbb{R}^2)$ be defined by

$$(Q_ix)(t)=\begin{cases}\frac{1}{2}+P_i(t)x_i(t-\tau_i), & t\ge T\\ (Q_ix)(T), & t_0\le t<T\end{cases} \tag{2.13}$$

$$(S_ix)(t)=\begin{cases}(-1)^{n+1}\int_t^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty} \\ \qquad \frac{F_i(s,x_{3-i}(s-\sigma_1),\cdots,x_{3-i}(s-\sigma_m))}{\prod_{l=1}^n r_{il}(s_l)}dsds_n\cdots ds_1, & t\ge T\\ (S_ix)(T), & t_0\le t<T\end{cases} \tag{2.14}$$

for all $x\in\Omega$, where $i\in\{1,2\}$.

(i) It is claimed that $Qx + Sy \in \Omega$ for all $x, y \in \Omega$, i.e. $Q\Omega \cup S\Omega \subset \Omega$.

In fact, for each $x, y \in \Omega$ and $t \geq T$, it follows from (2.3), (2.10) and (2.12) that

$$\begin{aligned}(Q_ix)(t)+(S_iy)(t) \geq & \frac{1}{2} - \overline{P_i}x_i(t-\tau_i) - \int_t^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty} \\ & \Big|\frac{F_i(s, y_{3-i}(s-\sigma_1), \cdots, y_{3-i}(s-\sigma_m))}{\prod_{l=1}^{n} r_{il}(s_l)}\Big| ds ds_n \cdots ds_1 \\ \geq & \frac{1}{2} - \overline{P_i} - \int_T^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty} \frac{q_i(s)}{|\prod_{l=1}^{n} r_{il}(s_l)|} ds ds_n \cdots ds_1 \geq 0,\end{aligned}$$

and

$$\begin{aligned}&(Q_ix)(t) + (S_iy)(t) \\ &\leq \frac{1}{2} + \overline{P_i} + \int_T^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty} \frac{q_i(s)}{|\prod_{l=1}^{n} r_{il}(s_l)|} ds ds_n \cdots ds_1 \leq 1.\end{aligned}$$

Thus, $0 \leq (Q_ix)(t) + (S_iy)(t) \leq 1, i \in \{1, 2\}$ for $t \geq t_0$.

(ii) It is declared that Q is a contraction mapping on Ω.

In reality, for any $x, y \in \Omega$ and $t \geq T$, it is easy to derive that

$$\begin{aligned}\big|(Q_ix)(t) - (Q_iy)(t)\big| \leq & |P_i(t)|\big|x_i(t-\tau_i) - y_i(t-\tau_i)\big| \\ \leq & \overline{P_i}\big|x_i(t-\tau_i) - y_i(t-\tau_i)\big| \leq \frac{1}{2}||x-y||,\end{aligned}$$

which implies that

$$||Q_ix - Q_iy|| \leq \frac{1}{2}||x-y||.$$

That is, Q is a contraction mapping on Ω.

(iii) It can be asserted that S is completely continuous, just like what we did in Theorem 2.1. Hence, we omit it.

It follows from Lemma 1.1 that there is $x_0 \in \Omega$ such that $Qx_0 + Sx_0 = x_0$. Obviously, $x_0(t)$ is a nonoscillatory bounded solution of the system (1.1). This completes the proof.

Theorem 2.3. *Let functions $h_i, q_i, r_{il} \in C([t_0, +\infty), \mathbb{R}^+)$ and $P_i(t) \in C([t_0, +\infty), \mathbb{R})$ satisfy that (2.2), (2.3) and*

$$P_1(t) \equiv 1, \tag{2.15}$$

$$|P_2(t)| \leq \overline{P_2} < \frac{1}{2}, \tag{2.16}$$

$$\int_{t_0}^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty}\frac{s_1\max\{q_1(s),h_1(s)\}}{\left|\prod_{l=1}^{n}r_{1l}(s_l)\right|}dsds_n\cdots ds_1<+\infty, \tag{2.17}$$

$$\int_{t_0}^{+\infty}\max\Big\{\frac{1}{|r_{2l}(t)|},h_2(t),q_2(t)\Big\}dt<+\infty, \tag{2.18}$$

where $i\in\{1,2\},l\in\{1,2,\cdots,n\}$. Then the system (1.1) has a nonoscillatory bounded solution.

Proof. By (2.17) and (2.18), a sufficiently large $T>t_0$ can be chosen such that

$$\sum_{k=1}^{\infty}\int_{T+k\tau_1}^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty}\frac{\max\{q_1(s),h_1(s)\}}{\left|\prod_{l=1}^{n}r_{1l}(s_l)\right|}dsds_n\cdots ds_1<1, \tag{2.19}$$

$$\int_{T}^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty}\frac{\max\{q_2(s),h_2(s)\}}{\left|\prod_{l=1}^{n}r_{2l}(s_l)\right|}dsds_n\cdots ds_1\le\frac{1}{2}-\overline{P_2}, \tag{2.20}$$

Let $C([t_0,+\infty),\mathbb{R}^2)$ be the set as in the proof of Theorem 2.1 and define a bounded, closed and convex subset Ω of $C([t_0,+\infty),\mathbb{R}^2)$ as following:

$$\Omega=\Big\{x=(x_1,x_2)\in C([t_0,+\infty),\mathbb{R}^2):1\le x_1(t)\le 3,0\le x_2(t)\le 1\Big\}.$$

Let mappings S_1,Q_2 and $S_2:\Omega\to C([t_0,+\infty),\mathbb{R}^2)$ be defined as

$$(S_1x)(t)=\begin{cases}2+(-1)^n\sum_{k=1}^{\infty}\int_{t+k\tau_1}^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty} \\ \qquad\frac{F_1(s,x_2(s-\sigma_1),\cdots,x_2(s-\sigma_m))}{\prod_{l=1}^{n}r_{1l}(s_l)}dsds_n\cdots ds_1, & t\ge T\\ (S_1x)(T), & t_0\le t<T\end{cases} \tag{2.21}$$

$$(Q_2x)(t)=\begin{cases}\frac{1}{2}+P_2(t)x_2(t-\tau_2), & t\ge T\\ (Q_2x)(T), & t_0\le t<T\end{cases} \tag{2.22}$$

$$(S_2x)(t)=\begin{cases}(-1)^n\sum_{k=1}^{\infty}\int_{t+k\tau_2}^{+\infty}\int_{s_1}^{+\infty}\cdots\int_{s_n}^{+\infty} \\ \qquad\frac{F_2(s,x_1(s-\sigma_1),\cdots,x_1(s-\sigma_m))}{\prod_{l=1}^{n}r_{2l}(s_l)}dsds_n\cdots ds_1, & t\ge T\\ (S_2x)(T), & t_0\le t<T\end{cases} \tag{2.23}$$

for all $x\in\Omega$.

Proceeding similarly as in the proof of Theorem 2.1 and 2.2, we get that there are $x_{01},x_{02}\in\Omega$ such that $S_1x_{01}=x_{01}$ and $Q_2x_{02}+S_2x_{02}=x_{02}$. Then $x_0(t)=(x_{01}(t),x_{02}(t))$ is a nonoscillatory bounded solution of the system (1.1). This finishes the proof.

Remark 2.4. Proceeding as before, we can prove that no matter $P_i(t)$ belongs to which cases:

(1) $P_i(t) \equiv 1$,

(2) $P_i(t) \equiv -1$,

(3) $|P_i(t)| \leq \overline{P_i} < \frac{1}{2}$,

(4) $0 < P_i(t) \leq \overline{P_i} < 1$,

(5) $1 < \underline{P_i} \leq P_i(t) \leq \overline{P_i} < +\infty$,

(6) $-1 < \underline{P_i} \leq P_i(t) < 0$,

(7) $-\infty < \underline{P_i} \leq P_i(t) \leq \overline{P_i} < -1$,

(8) any combination of the above, one for $i = 1$, the other for $i = 2$,

the system (1.1) has a bounded nonoscillatory solution.

References

[1] R.P. Agarwal, D. O'Regan, S.H. Saker, Oscillation criteria for second-order nonlinear neutral delay dynamic equations, *J. Math. Anal. Appl.*, **300** (2004), 203-217.

[2] D.D. Bainov, D.P. Mishev, *Oscillation Theory for Neutral Differential Equation with Delay*, Hilger, Bristol (1991).

[3] J.F. Cheng, Z. Annie, Existence of nonoscillatory solution to second order linear neutral delay equation, *J. Sys. Sci. Math. Scis.*, **24**, No. 3 (2004), 389-397.

[4] L.H. Erbe, Q.K. Kong, B.G. Zhang, *Oscillation Theory for Functional Differential Equations*, Marcel Dekker, New York (1995).

[5] H. El-Metwally, M.R.S. Kulenovic, S. Hadziomerspahic, Nonoscillatory solutions for system of neutral delay equation, *Nonlinear Anal.*, **54** (2003), 63-81.

[6] I. Gyori, G. Ladas, *Oscillation Theory for Delay Differential Equations with Applications*, Oxford Univ. Press, London (1991)

[7] J.K. Hale, *Theory of Functional Differential Equatin*, Springer-Verlag, New York (1977).

[8] Y. Liu, W. Ge, Asymptopic behavior of certain delay differential equations with forcing term, *J. Math. Anal. Appl.*, **280** (2003), 350-363.

[9] J. Shen, X. Tang, New nonoscillation criteria for delay differential equations, *J. Math. Anal. Appl.*, **290** (2004), 1-9.

[10] X. Tang, Y. Liu, Bounded oscillation for second-order delay differential equations with unstable type in a critical case, *Appl. Math. Letters*, **16** (2003), 263-268.

[11] Y. Yu, H. Wang, Nonoscillatory solutions of second-order nonlinear neutral delay equations, *J. Math. Anal. Appl.*, **311** (2005), 445-456.