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EXISTENCE OF OPTIMAL PARAMETERS FOR THE BLACK-SCHOLES OPTION PRICING MODEL

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Abstract: In this paper we study parameters associated with the Black-Scholes option pricing model. The existence, uniqueness, and continuous dependence of the weak solution of the Black-Scholes model are established. The existence of optimal parameters is established.

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1. Introduction

Options are financial instruments that convey the right, but not the obligation, to engage in a future transaction on some underlying asset [3]. There are two main type of options, namely European options and American options. While European option can only be exercised at the expiration date T, an American option may be exercised during the whole lifetime of the option. In the Black-Scholes framework, price of the underlying asset S_t is assumed to follow a geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t \tag{1}$$

with W_t being the increment of a standard Wiener process. The quantities

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$\mu \in \mathcal{R}$ and $\sigma > 0$ are the drift and volatility. For a European option with maturity date T, striking price K, and payoff function g, the price $V = V(S,t)$ satisfies the following backward parabolic PDE.

$$\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}S^2\frac{\partial^2 V(S,t)}{\partial S^2} + rS\frac{\partial V(S,t)}{\partial S} - rV(S,t) = 0, \qquad (S,t) \in (0,\infty)\times[0,T), \tag{2}$$
$$V(S,0) = h(S).$$

We set $S = e^x, u(x,t) = V(e^x,t)$, and $h(e^x) = g(x)$. Moreover, we set reverse flow in time to obtain the following forward parabolic PDE.

$$\frac{\partial u(x,t)}{\partial t} - \alpha\frac{\partial^2 u(x,t)}{\partial x^2} - (\delta-\alpha)\frac{\partial u(x,t)}{\partial x} + \delta u(x,t) = 0, \qquad (x,t) \in \mathbb{Q}, \tag{3}$$
$$u(x,0) = g(x), \quad \in \mathbb{R},$$

where $g(x)$ is the pay-off function, for $T > 0$, $\mathbb{Q} = \mathcal{R}\times(0,T)$, $\alpha = \frac{\sigma^2}{2}$, and $\delta = r$.

In this paper we estimate the governing parameters of the Black-Scholes option pricing model such that the solution of (3) exhibits the desired behavior. More precisely, let

$$\mathcal{P}_{ad} = \{q = (\alpha,\delta) \in [\alpha_{min},\alpha_{max}]\times[\delta_{min},\delta_{max}]\},$$

where $\alpha_{min} > 0$ and $\delta_{min} > 0$. Define a functional $J(q)$ by

$$J(q) = \|u(q;t) - z_d\|^2_{L^2(0,T;H)} \tag{4}$$

where the data z_d can be thought of as the desired value of $u(q;t)$. The parameter identification problem for (3) with the objective function (4) is to find $q^* = (\alpha^*,\delta^*) \in \mathcal{P}_{ad}$ satisfying

$$J(q^*) = \inf_{q\in\mathcal{P}_{ad}} J(q). \tag{5}$$

Let $q \to u(q)$ from, $\mathcal{P}$ into $C([0,T];H)$ be the solution map. The existence and uniqueness of the weak solution of (3) is established in Section 2. The continuity of the solution with respect to data is established in Section 3. We show the existence of optimal parameters in (54) by restricting the cost functional $J(q)$ to $\mathcal{P}_{ad}$, a compact subset of $\mathcal{R}^2$. We conclude the paper with some discussions in Section 4.

2. Existence and Uniqueness of Weak Solution

Since typical pay-off functions like in (3) do not belong to $L^2(\mathcal{R})$, we introduce weighted Lebesgue and Sobolev spaces $L^2_\beta(\mathcal{R})$ and $H^1_\beta(\mathcal{R})$ for $\beta > 0$ as follows.

$$L^2_\beta(\mathcal{R}) = \left\{ u \in L^1_{loc}(\mathcal{R}) : ue^{-\beta|x|} \in L^2(\mathcal{R}) \right\}, \tag{6}$$

$$H^1_\beta(\mathcal{R}) = \left\{ u \in L^1_{loc}(\mathcal{R}) : ue^{-\beta|x|} \in L^2(\mathcal{R}), u'e^{-\beta|x|} \in L^2(\mathcal{R}) \right\}. \tag{7}$$

The respective inner products and norms are defined by

$$(u,v)_{L^2_\beta(\mathcal{R})} = \int_{\mathcal{R}} uve^{-2\beta|x|}dx, \tag{8}$$

$$(u,v)_{H^1_\beta(\mathcal{R})} = \int_{\mathcal{R}} uve^{-2\beta|x|}dx + \int_{\mathcal{R}} u'v'e^{-2\beta|x|}dx, \tag{9}$$

$$\|u\|_{L^2_\beta(\mathcal{R})} = \left(\int_{\mathcal{R}} |u|^2 e^{-2\beta|x|}dx \right)^{\frac{1}{2}}, \tag{10}$$

$$\|u\|_{H^1_\beta(\mathcal{R})} = \left(\int_{\mathcal{R}} |u|^2 e^{-2\beta|x|}dx + \int_{\mathcal{R}} |u'|^2 e^{-2\beta|x|}dx \right)^{\frac{1}{2}}. \tag{11}$$

We define the dual space of $H^1_\beta(\mathcal{R})$ as

$$(H^1_\beta(\mathcal{R}))^* = \left\{ u | u : H^1_\beta(\mathcal{R}) \to \mathcal{R} \textit{ is linear and continuous } \right\}. \tag{12}$$

The duality pairing between $H^1_\beta(\mathcal{R})$ and $(H^1_\beta(\mathcal{R}))^*$ is given by

$$\langle u, v \rangle = \int_{\mathcal{R}} uve^{-2\beta|x|}dx. \tag{13}$$

The following lemma is of critical importance to prove lemma 2.

Lemma 1. *Let $f \in L^2_\beta(\mathcal{R})$. For $\phi \in C_0^\infty$, supp $\phi = (-1,1)$, $\int_{\mathcal{R}} \phi(x)dx = 1$, and $\phi_\epsilon = \frac{1}{\epsilon}\phi(\frac{x}{\epsilon})$, then*

$$\phi_\epsilon * f \to f \quad in \quad L^2_\beta(\mathcal{R}). \tag{14}$$

Proof. Suppose $q = e^{-\beta|x|}$. Then we have

$$(\phi_\epsilon * f).q = \phi_\epsilon * (f.q) + ((\phi_\epsilon * f).q - \phi_\epsilon * (f.q)). \tag{15}$$

Since $f.q \in L^2$ and $\phi_\epsilon * (f.q) \to f.q$ in L^2, it suffices to show that

$$\|g_\epsilon(x)\|_{L^2} = (\|\phi_\epsilon * f).q - \phi_\epsilon * (f.q)\|_{L^2} \to 0 \;\; for \;\; \epsilon \to 0. \tag{16}$$

the fundamental theorem of calculus for q gives

$$g_\epsilon(x) = \int_{\mathcal{R}} \phi_\epsilon(x-y) f(y)(q(x) - q(y))dy \tag{17}$$

Using *supp* $\phi_\epsilon = (-\epsilon, \epsilon)$, we get

$$\begin{aligned} |g_\epsilon(x)| &\leq \int_{\mathcal{R}} \phi_\epsilon(x-y)|f(y)|(2\epsilon \sup_{t\in(x,y)} |q'(t)|)dy \\ &= 2\epsilon \int_{\mathcal{R}} \phi_\epsilon(x-y)|f(y)|(\sup_{|s|\leq\epsilon} |q'(y+s)|)dy = \bar{g_\epsilon}(x). \end{aligned} \tag{18}$$

Since $\bar{g_\epsilon}(x) \in L^2$ uniformly and $|\bar{g_\epsilon}(x)| \leq 2\epsilon|\bar{g_\epsilon}(x)|$, thus $\|g_\epsilon\|_{L^2} \to 0$ as $\epsilon \to 0$. □

Lemma 2. *$\mathcal{D}(\mathcal{R})$, the space of test functions in $\mathcal{R}$, is dense in $H^1_\beta(\mathcal{R})$.*

Proof. Let $f \in H^1_\beta(\mathcal{R})$ and $\Phi \in C^\infty$ such that

$$\Phi(x) = \begin{cases} 1, & \text{if } |x| \leq 1, \\ 0, & \text{if } |x| \geq 2. \end{cases}$$

Now we show that $f_\epsilon = (f.\Phi(\epsilon(.))) * \phi_\epsilon \in C_0^\infty$ where $\phi_\epsilon = \frac{1}{\epsilon}\phi(\frac{x}{\epsilon})$, $f_\epsilon \to f$ in $L^2_\beta(\mathcal{R})$. i.e.

$$f_\epsilon \to f \quad \text{and} \quad \nabla f_\epsilon \to \nabla f \;\; \textit{in } L^2_\beta(\mathcal{R}). \tag{19}$$

Since

$$\nabla f_\epsilon = (\nabla f.\Phi(\epsilon(.))) * \phi_\epsilon + \epsilon(f.(\nabla\Phi)(\epsilon(.))) * \phi_\epsilon, \tag{20}$$

it suffices to show

$$(f.\Phi(\epsilon(.))) * \phi_\epsilon \to f \quad \text{in} \quad L^2_\beta(\mathcal{R}). \tag{21}$$

By the Lebesgue Dominated Convergence Theorem, we get

$$f.\Phi(\epsilon(.)) \to f \quad \text{in} \quad L^2_\beta(\mathcal{R}). \tag{22}$$

Hence Lemma 1 concludes the proof. □

Since $\mathcal{D}(\mathcal{R})$ is dense in $H^1_\beta(\mathcal{R})$ and $L^2_\beta(\mathcal{R})$, the following lemma follows immediately.

Lemma 3.

$$H^1_\beta(\mathcal{R}) \subset L^2_\beta(\mathcal{R}) \subset (H^1_\beta(\mathcal{R}))^* \quad \textit{form a Gelfand triple.}$$

Note. Since $\mathcal{D}(\mathcal{R})$ is dense in H^1_β, the definition of $\langle .,. \rangle$ allows us to interpret the operator $\mathcal{A}$ as a mapping from $H^1_\beta \to (H^1_\beta)^*$.

For our simplicity, we use $V = H^1_\beta(\mathcal{R})$, $V^* = (H^1_\beta(\mathcal{R}))^*$, and $H = L^2_\beta(\mathcal{R})$. To use the variational formulation let us define the following bilinear form on $V \times V$

$$a_{(\alpha,\delta)}(u,v) = \alpha \int_{\mathcal{R}} u'v'e^{-2\beta|x|}dx + \delta \int_{\mathcal{R}} uve^{-2\beta|x|}dx - (\delta - \alpha)\int_{\mathcal{R}} u've^{-2\beta|x|}dx. \quad (23)$$

For $\alpha > 0$ and $\delta > 0$, one can show $a_{(\alpha,\delta)}(u,v)$ is bounded and coercive in V. Define a linear operator $A_{(\alpha,\delta)} : D(A_{(\alpha,\delta)}) = \{u : u \in V, A_{(\alpha,\delta)}u \in V^*\}$ into V^* by $a_{(\alpha,\delta)}(u,v) = (A_{(\alpha,\delta)}u, v)$ for all $u \in D(A_{(\alpha,\delta)})$ and for all $v \in V$.

Definition 4. Let X ba a Banach space and $a, b \in \mathcal{R}$ with $a < b, 1 \le p < \infty$. Then $L^2(0,T;X)$ and $L^\infty(0,T;X)$ denote the space of measurable functions u defined on (a,b) with values in V such that the function $t \to \|u(.,t)\|_X$ is square integrable and essentially bounded. The respective norms are defined by

$$\|u\|_{L^2(0,T;X)} = \left(\int_a^b \|u(.,t)\|_X^2 dt \right)^{\frac{1}{2}}, \quad (24)$$

$$\|u\|_{L^\infty(0,T;X)} = ess.sup_{a \le t \le b}\|u(.,t)\|_X. \quad (25)$$

For details on these function spaces, see [10].

Definition 5. A function $u : [0,T] \to V$ is a weak solution of (3) if:

(i) $u \in L^2(0,T;V)$ and $u_t \in L^2(0,T;V^*)$;

(ii) For every $v \in V, \langle u_t(t), v\rangle + a_{(\alpha,\delta)}(u(t),v) = 0$, for t pointwise a.e. in $[0,T]$;

(iii) $u(0) = g$.

Note. The time derivative u_t is understood in the distributional sense.

The following two Lemmas are of critical importance for the existence and uniqueness of weak solutions.

Lemma 6. *Let $V \hookrightarrow H \hookrightarrow V^*$ If $u \in L^2(0,T;V)$, $u' \in L^2(0,T;V^*)$, then $u \in C([0,T];H)$. Moreover, for any $v \in V$, the real-valued function $t \to \|u(t)\|_H^2$ is weakly differentiable in $(0,T)$ and satisfies*

$$\frac{1}{2}\frac{d}{dt}\{\|u\|^2\} = \langle u', u\rangle \quad (26)$$

For proof see [4].

Lemma 7. *(Gronwall's Lemma) Let $\xi(t)$ be a nonnegative, summable function on [0,T] which satisfies the integral inequality*

$$\xi(t) \leq C_1 \int_0^t \xi(s)ds + C_2 \quad \text{for constants } C_1 \,, C_2 \geq 0, \tag{27}$$

almost everywhere $t \in$ [0,T]. Then

$$\xi(t) \leq C_2(1 + C_1 t e^{C_1 t}) \quad \text{a.e. on } 0 \leq t \leq T. \tag{28}$$

In particular, if

$$\xi(t) \leq C_1 \int_0^t \xi(s)ds \text{ a.e. on } 0 \leq t \leq T, \text{ then } \xi(t) = 0 \text{ a.e. on } [0,\ T]. \tag{29}$$

For proof see [11].

Lemma 8. *The weak solution of (3) is unique if it exists.*

Proof. Let u_1 and u_2 be two weak solutions of (3). Let $u = u_1 - u_2$. To prove Lemma 8, it suffices to show that $u = 0$ pointwise a.e. on $[0, T]$. Since $\langle u_t(t), v\rangle + a_{(\alpha,\delta)}(u(t), v) = 0$ for any $v \in V$, we take $v = u \in V$ to get

$$\langle u_t(t), u\rangle + a_{(\alpha,\delta)}(u(t), u) = 0 \tag{30}$$

(30) is true pointwise a.e. on $[0, T]$. Using (6) and the coercivity estimate, we have

$$\frac{1}{2}\frac{d}{dt}\|u\|_H^2 \leq \gamma\|u\|_H^2 \,, u(0) = 0 \tag{31}$$

for some $\gamma > 0$. By Lemma 7, $\|u\|_H = 0$ for all $t \in [0, T]$. Thus $u = 0$ pointwise a.e. in $[0, T]$.

□

To show existence of the weak solution of (3), we first show existence and uniqueness of approximate soultion. Now we define the approximate solution of (3).

Definition 9. A function $u_M : [0, T] \to V_M$ is an approximate solutions of (3) if

(i) $u_M \in L^2(0, T; V_M)$ and $u_{M_t} \in L^2(0, T; V_M)$;

(ii) for every $v \in V_M$ and $(u_{M_t}(t), v)_H + a_{(\alpha,\delta)}(u_M(t), v) = 0$ pointwise a.e. in $[0, T]$

(iii) $u_M(0) = P_M g$.

To prove the existence of approximate solutions, we take $v = u_M$, in $\langle u_t(t), v\rangle + a_{(\alpha,\delta)}(u(t), v) = 0$ to get following system of ODEs.

$$C^j_{M_t} + \sum_{k=1}^{M} a^{jk} C^k_M = 0, \;\; C^j_M(0) = g^j \tag{32}$$

where $C^k_M \in H, C^k_{M_t} \in H \;\, for \;\, 1 \leq k \leq N, \;\, a^{jk}(t) = a(w_j, w_k), \;\, and \;\, g^j = (g, w_j)_H$ For $C : [0,T] \to \mathcal{R}^N$, equation (32) can be written as

$$\vec{C}_{M_t} + A(t)\vec{C}_M = 0, \;\; \vec{C}_M(0) = \vec{g} \tag{33}$$

Since $A \in L^\infty(0,T;\mathcal{R}^{M\times M})$, For $\vec{C}_M = \Psi(\vec{C}_M)$ (33) can be written as

$$\Psi(\vec{C}_M(t)) = \vec{g} - \int_0^t A(s)\vec{C}_M(s)ds \tag{34}$$

The following Lemma is immediate from contraction mapping theorem and (34).

Lemma 10. *For any $M \in \mathcal{N}$, there exists a unique approximate solution $u_M : [0,T] \to V_M$ of (3).*

The following theorem provides the energy estimates for approximate solutions.

Theorem 11. *There exists a constant C depending only on T and Ω such that the approximate solutions u_M satisfies*

$$\|u_M\|_{L^\infty(0,T;H)} + \|u_M\|_{L^2(0,T;V)} + \|u_{M_t}\|_{L^2(0,T;H)} \leq C\|g\|_H \tag{35}$$

Proof. For every $v \in V_M$ we have $(u_{M_t}(t), v)_H + a_{(\alpha,\delta)}(u_M(t), v) = 0$. Take $v = u_M(t)$ then we have

$$(u_{M_t}(t), u_M(t))_H + a_{(\alpha,\delta)}(u_M(t), u_M(t)) = 0, \text{ pointwise a.e. in } (0,T). \tag{36}$$

Using (36) and the coercivity estimate, we find that there exists constants $\rho > 0, \gamma > 0$ such that

$$\frac{1}{2}\frac{d}{dt}\left(e^{-2\gamma t}\|u_M\|^2_H\right) + \rho e^{-2\gamma t}\|u_M\|^2_V \leq 0, \tag{37}$$

Integrating (37) with respect to t, using the initial condition $u_M(0) = P_M(g)$, and $\|P_M(g)\|_H \leq \|g\|_H$, we get

$$\frac{1}{2}e^{-2\gamma t}\|u_M(t)\|^2_H + \rho \int_0^t e^{-2\gamma s}\|u_M\|^2_V \leq \frac{1}{2}\|g\|^2_H \tag{38}$$

Taking the supriemum over (38) with respect to t over $[0,T]$, we get

$$\|u_M\|^2_{L^2(0,T;H)} + \|u_M\|^2_{L^2(0,T;V)} \leq C\|g\|^2_H. \tag{39}$$

Since $u_{M_t}(t) \in V_M^*$, we have

$$\|u_{M_t}(t)\|_{V^*} = \sup_{v \in V_M} \frac{(u_{M_t}(t), v)_H}{\|v\|_V},\ v \neq\ 0 \tag{40}$$

Using the notion of approximate solution and boundedness of A we have

$$\|u_{M_t}(t)\|^2_{V^*} \leq C\|u_M(t)\|^2_V \tag{41}$$

Integrating (39) with respect to t and using estimate (39), we have

$$\|u_M\|_{L^\infty(0,T;H)} + \|u_M\|_{L^2(0,T;V)} + \|u_{M_t}\|_{L^2(0,T;H)} \leq C\|g\|_H. \quad \square \tag{42}$$

To complete the proof of existence of weak solution, we now show the convergence of approximate solutions by using weak compactness argument.

Definition 12. Let $L^2(0,T;V^*)$ be the dual space of $L^2(0,T;V)$. Let $f \in L^2(0,T;V^*)$ and $u \in L^2(0,T;V)$, then we say $u_M \to u$ in $L^2(0,T;V)$ weakly if

$$\int_0^T \langle f(t), u_M(t)\rangle dt \to \int_0^T \langle f(t), u(t)\rangle dt \quad \forall\ f \in L^2(0,T;V^*). \tag{43}$$

Lemma 13. *A subsequence $\{u_M\}$ of approximate solutions u_M converges weakly in $L^2(0,T;V^*)$ to a weak solution $u \in C([0,T];H) \cap L^2(0,T;V)$ of (3) with $u_t \in L^2(0,T;V^*)$. Moreover, it satisfies*

$$\|u\|_{L^\infty(0,T;H)} + \|u\|_{L^2(0,T;V)} + \|u_t\|_{L^2(0,T;H)} \leq C\|g\|_H. \tag{44}$$

Proof. Theorem 11 implies that the approximate solutions $\{u_M\}$ are bounded in $L^2(0,T;V)$ and their derivatives $\{u_{M_t}\}$ are bounded in $L^2(0,T;V^*)$. By the Banach-Alaoglu theorem, we can extract a subsequence $\{u_M\}$ such that

$$u_M \to u\ \ in\ L^2(0,T;V),\ u_{M_t} \to u_t\ in\ L^2(0,T;V^*) \quad \text{weakly}. \tag{45}$$

Let $\phi \in C_0^\infty(0,T)$ be a real-valued test function and $w \in V_N$ for some $N \in \mathcal{N}$. Replacing v by $\phi(t)w$ in $(u_{M_t}(t), v)_H + a_{(\alpha,\delta)}(u_M(t), v) = 0$ and integrating from 0 to T, we get

$$\int_0^T (u_{M_t}(t), \phi(t)w)_H dt + \int_0^T a_{(\alpha,\delta)}(u_M(t), \phi(t)w) dt = 0 \text{ for } M \geq N. \tag{46}$$

Taking the limit as $M \to \infty$, we get

$$\int_0^T (u_{M_t}(t), \phi w)_H dt = \int_0^T \langle u_t, \phi w \rangle dt. \tag{47}$$

By using boundedness of $a_{(\alpha,\delta)}$, we get

$$\int_0^T a_{(\alpha,\delta)}(u_M(t), \phi(t)w)dt = \int_0^T a_{(\alpha,\delta)}(u(t), \phi(t)w)dt \tag{48}$$

Using (46), (47), and (48), we get

$$\langle u_t(t), w \rangle + a_{(\alpha,\delta)}(u, w) = 0 \text{ pointwise a.e. in } (0, T). \tag{49}$$

Since (48) is true for all $w \in V_M$

$$\cup_{M \in \mathcal{N}} V_M \quad \text{and} \tag{50}$$

is dense in V, so (48) holds for all $w \in V$. Now it remains to show that $u(0) = g$. Using (48), integration by parts, and the Galerkin approximation we have

$$\langle u(0), w \rangle = \langle g, w \rangle \text{ as } M \to \infty, \tag{51}$$

for every $w \in V_M$. Thus $u(0) = g$. □

3. Existence of Optimal Parameters

Lemma 14. *Let $v \in V$. Then the mapping $(\alpha, \delta) \to A_{\alpha,\delta}\, v$ from*

$$\mathcal{P}_{ad} = \{q = (\alpha, \delta) \in [\alpha_{min}, \alpha_{max}] \times [\delta_{min}, \delta_{max}]\},$$

into V' is continuous.

Proof. Suppose that $q_n \to q$ in $\mathcal{R}^2$ as $n \to \infty$. We denote $A = A_{\alpha,\delta}$ and $A_n = A_{\alpha_n,\delta_n}$. We claim that $\|(A_n - A)v\|_{V'} \to 0$ as $n \to \infty$. Let $w \in V$ with $\|w\| \le 1$. Then

$$\begin{aligned} |\langle (A_n - A)v, w \rangle|^2 &\le \left(\int_{\mathcal{R}} |\alpha_n - \alpha||u'||w'|dx \right)^2 + \left(\int_{\mathcal{R}} |\delta_n - \delta||u||w|dx \right)^2 \\ &+ \left(\int_{\mathcal{R}} |\delta_n - \delta||u'||w|dx \right)^2 + \left(\int_{\mathcal{R}} |\alpha_n - \alpha||u'||w|dx \right)^2 \\ &\le 2|\alpha_n - \alpha|^2 \int_{\mathcal{R}} |u'(x)|^2 dx + |\delta_n - \delta|^2 \int_{\mathcal{R}} |u'(x)|^2 dx + |\delta_n - \delta|^2 \int_{\mathcal{R}} |u(x)|^2 dx \\ &\to 0 \quad \text{as} \quad n \to \infty. \end{aligned}$$

□

Lemma 15. *Suppose that $(\alpha_n, \delta_n) \to (\alpha, \delta)$ in $\mathcal{R}^2$, and $v_n \to v$ weakly in V, as $n \to \infty$. Then $A_n v_n \to Av$ weakly in V'.*

Proof. Let $w \in V$, then

$$|\langle A_n v_n, w\rangle - \langle Av, w\rangle| = |\langle A_n w, v_n\rangle - \langle Aw, v\rangle| \\ \leq |\langle (A_n - A)w, v_n\rangle| + |\langle Aw, v_n - v\rangle|. \quad (52)$$

Since a weakly convergent sequence is bounded, we have

$$|\langle (A_n - A)w, v_n\rangle| \leq \|A_n w - Aw\|_{V'} \|v_n\| \leq c\|A_n w - Aw\|_{V'} \to 0$$

as $n \to \infty$ by Lemma 14. The second term $|\langle Aw, v_n - v\rangle| \to 0$ since $v_n \to v$ weakly. □

Lemma 16. *Let $q \in \mathcal{P}_{ad}$. Then the solution map $q \to u(q)$ from $\mathcal{P}_{ad}$ into $C([0,T]; H)$ is continuous.*

Proof. Let $q_n \to q$ in $\mathcal{P}_{ad}$ as $n \to \infty$. Since $u(t;q)$ is the weak solution of (3) for any $q \in \mathcal{P}_{ad}$, we have the following estimate

$$\|u_M(t;q_n)\|_{L^\infty(0,T;H)} + \|u_M(t;q_n)\|_{L^2(0,T;V)} + \|u_{M_t}(t;q_n)\|_{L^2(0,T;H)} \\ \leq C\|g\|_H, \quad (53)$$

where C is a constant independent of $q \in \mathcal{P}_{ad}$. Estimate (53) shows that $u(t;q_n)$ is bounded in $W(0,T)$. Since $W(0,T)$ is reflexive, we can choose a subsequence $u(t;q_{n_k})$ weakly convergent to a function z in $W(0,T)$. The fact that $u(t;q_n)$ is bounded in $W(0,T)$ implies that $u(t;q_n)$ is bounded in $L^2(0,T;V)$, so $u(t;q_{n_k})$ weakly convergent to a function z in $L^2(0,T;V)$. Since V is compactly imbedded in H, then by the classical compactness theorem [4] $u(t;q_n) \to z$ in $L^2(0,T;H)$. By (53) the derivatives $u'(t;q_{n_k})$ and z' are uniformly bounded in $L^\infty(0,T;H)$. Therefore functions $\{u(t;q_{n_k}), z\}_{k=1}^\infty$ are equicontinuous in $C([0,T];H)$.Thus $u(t;q_{n_k}) \to z$ in $C([0,T];H)$. In particular, $u(t;q_{n_k}) \to z(t)$ in H and $u(t;q_{n_k}) \to z(t)$ weakly in V for any $t \in [0,T]$. By Lemma 15, $A_{n_k} u(t;q_{n_k}) \to Az(t)$ weakly in V'. Now we see that z satisfies the equation given in definition 5, i.e. it is the weak solution $u(q)$. The uniqueness of the weak solutions implies that $u(q_n) \to u(q)$ as $n \to \infty$ in $C([0,T];H)$ for the entire sequence $u(q_n)$ and not just for its subsequence. Thus $u(t;q_n) \to u(q)$ in $C([0,T];H)$ as $q_n \to q$ in P as claimed. □

The following theorem will follow immediately from Lemma 16.

Theorem 17. *If $\mathcal{P}_{ad}$ is compact, then there exists at least one optimal set of parameters that satisfies* (4). *Moreover, for optimal parameter q^* we have*

$$J(q^*) = \inf_{q \in \mathcal{P}_{ad}} J(q). \tag{54}$$

4. Conclusion

We have studied parameters associated with Black-Scholes option pricing model. The existence and uniqueness of weak solution of the Black-Scholes option pricing model is established. The continuity of weak solution with respect to parameters is shown. The existence of optimal model parameters is established.

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