International Journal of Pure and Applied Mathematics
Volume 78 No. 5 2012, 721-733
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

A FOURIER SERIES-BASED ANALYTICAL SOLUTION FOR THE OSCILLATING AIRFLOW IN A HUMAN RESPIRATORY TRACT

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Abstract: Understanding of the airflow behavior in a human respiratory tract is very important to the drug delivery treatment. In this work, we propose a mathematical modelling of the airflow in a human upper respiratory tract. The governing equations are composed of the Navier-Stokes equations and the continuity equation. The oscillating flow is described by setting one side of boundaries be a periodic pressure function. Due to the complexity and requirement of high performance computing resources of numerical methods, we therefore, present an efficient alternative method that is the method of analytical expression. The obtained solution is in a Fourier series-based form. The simulation of an airflow field is done on a two-dimensional geometry of a human upper respiratory tract model. The obtained results show a good agreement to the fact of the airflow in a human respiratory tract and other related publications.

AMS Subject Classification: 92B05, 97M60
Key Words: airflow in a human respiratory tract, Navier-Stokes equations, analytical solution, Fourier series, mathematical modelling, oscillating flow

Received: April 24, 2012

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1. Introduction

The aerosolized medicine is the popular treatment in respiratory disease because droplet particles are directly diffused in pathologically respiratory position and they have slight side effects. The transportation of delivered droplets is most affected by the oscillating air flow driven by the pressure gradient. The airflow is an important factor that defines particle trajectories and final particle locations [1], [2], [3] and [4]. Since the air is one of fluids, most of researches tend to express the airflow velocity by the Navier-Stokes equations. To find a solution of the Navier-Stokes equations however it depends on a numerical analysis which is too complex as well as it needs a high-performance computing. However, in some specific conditions, the solution of the Navier-Stokes equations can be obtained by an analytical method. In order to save time and computing resources, some researchers then try to find the solution of the Navier-Stokes equations in an analytical expression instead.

In 1995, Andersson [5] presented an exact solution of the Navier-Stokes equations for magnetohydrodynamic flow. In 2003, moreover, Tsangaris and Vlachakis [6] studied an analytical solution of the fully developed laminar flow. The solution is given in a Fourier series form for the case of isosceles, right angled triangle. In 2006, S. Otarod, D. Otarod [7] showed an analytical solution for the Navier-Stokes equations in two-dimensions for laminar incompressible flow. In 2008, furthermore, Mohyuddin et al. [8] presented an analytical solution of the two-dimensional Navier-Stokes equations governing the unsteady incompressible flow. In 2009, Emin and Erdem [9] also showed an analytical solution of the Navier-Stokes equations for flow over a moving plate bounded by two side walls. Nevertheless, the analytical expression for the oscillating air flow in a human upper airway has never been given.

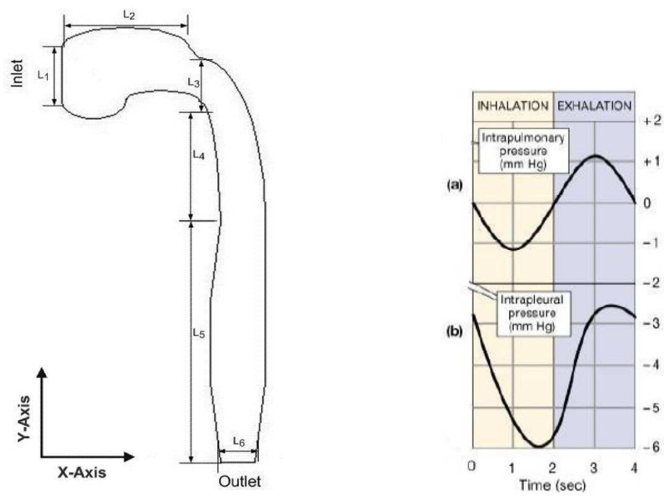
In this research, we propose the model that the governing equations are described by the Navier-Stokes equations with one side of boundaries, pressure is described by the oscillating function sine of time. Then an analytical expression of the model has been solved for a two-dimensional domain of a human upper airway including oral cavity , nasopharynx region and trachea. The solution is easily derived by consulting the research of Tsangaris and Vlachakis.

2. Construction of the Model

Due to the complexity of the geometry of a human respiratory tract and for a convenient way to derive an analytical expression for the solution of the airflow velocity, we simulated an airflow field based on a two-dimensional respiratory tract model. The domain is started from the beginning of the oral cavity to the end of the trachea in a human upper airway. Their dimensions are summarized as in Table 1 [10]. The two-dimensional model generated from the parameters in Table 1 is shown in Figure 1.

Parameter	Length (cm)
The diameter of inlet (L_1)	3.125
The length of oral cavity (L_2)	7.0
The diameter of upper trachea (L_3)	2.5
The length of upper trachea (L_4)	7.0
The length of lower trachea (L_5)	9.0
The diameter of outlet (L_6)	2.0

Table 1: Parameters of a human upper respiratory tract



(A) (B)

Figure 1: (A) 2-D respiratory tract model; (B) graphs of (a) intrapulmonary and (b) intrapleural pressure(modified from [17])

2.1. Governing Equations and Boundary Conditions

The air is assumed to be an incompressible Newtonian fluid which has constant density and viscosity. The flow is also assumed that there is no effect from any external force, it is driven by the oscillating pressure gradient. The governing equations for the oscillating two-dimensional airflow are the Navier-Stokes equations and the continuity equation. In the Cartesian coordinates, these equations can be expressed in the following form.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}) = -\frac{\partial p}{\partial x} + \mu(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}), \tag{2}$$

$$\rho(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}) = -\frac{\partial p}{\partial y} + \mu(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}) \tag{3}$$

where u and v are the components of the velocity in the Cartesian coordinates and p stands for the pressure. ρ and μ are the density and dynamic viscosity of the air, respectively.

The non-slip boundary condition, $\boldsymbol{u} = [u, v] = [0, 0]$, is assigned to the inner walls[16]. The pressure at the inlet is zero when compare to the outside, while the pressure at outlet is an oscillating function sine of time, $p(t) = Psin(\omega t)$, where P is the amplitude of the oscillating pressure, taken from the graph of the intrapulmonary pressure as shown in Figure 1(B(a)).

Now, we have a boundary value problem which states as follows.

BVP. Find u, v and p such that all equations (1)-(3) are satisfied and all boundary conditions are satisfied on the relevant boundary segments.

3. Method of Analytical Solution

Due to the complexity and requirement of high performance computing resources of numerical methods. We therefore, try to give a solution of the **BVP.** in an analytical expression which is based upon the Fourier series expansion.

To simplify our problem and for the convenience to derive an analytical expression of the airflow velocity, we divide our 2-D respiratory tract domain into subregions as shown in Figure 2(A). These subregions are looked like two types of regions that are horizontal rectangular channels, areas 1-3, say horizontal regions; and vertical rectangular channels, areas 4-7, say vertical regions. We

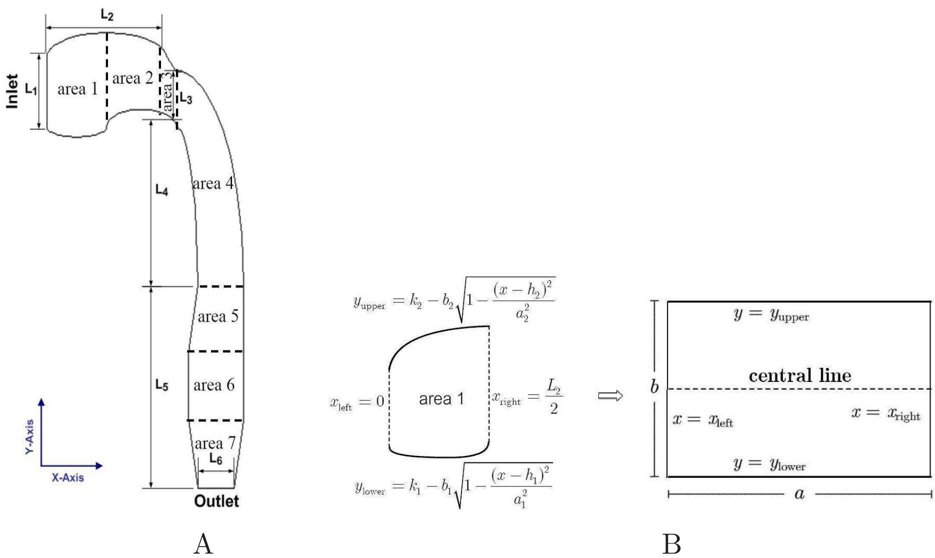


Figure 2: (A) Division of the domain; (B) transformation of a region

therefore divide the method of solution into 2 parts that are solution to the horizontal regions and solution to the vertical regions.

As finding a solution of each subregion, we need to transform the original shape of the subregion into a rectangular channel as an example shown in Figure 2(B). The transformation of each region can be made by approximating each boundary with a curve in mathematics such as a straight line, one part of an ellipse. Each curve of a region is then transformed into an horizontal straight line or a vertical straight line to obtain a rectangular channel. Therefore, the airflow is then transformed to be the flow in straight channels. The airflow is now reasonable to obtain an analytical expression[6].

3.1. Solution for the Horizontal Regions

We firstly find an analytical solution of the airflow in the horizontal areas. For equations (1) - (3), we see that the three partial differential equations have three unknowns u, v and p as functions of three independent variables x, y and t. By assuming fully developed flow on these regions, $v = 0$ and $u = u(t, y)$, the continuity equation is satisfied and when developed flow conditions across the x-axis; therefore, we can omit equation (3). Then the following partial differential equations should be satisfied because we are interested in the cases of the flow due to an oscillating pressure gradient:

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu(\frac{\partial^2 u}{\partial y^2}), \qquad \frac{\partial p}{\partial x} = \frac{P}{a}\sin(\omega t), \tag{4}$$

in which $\frac{P}{a}$ is the amplitude of the imposed pressure gradient, a is the length measured on x-axis of the considered region, ν is its kinematic viscous coefficient such that $\nu = \frac{\mu}{\rho}$ and ω is the cyclic frequency of the oscillating pressure gradient.

To define the oscillatory solution, we assume that u is periodic so that:

$$u(y,t) = u_s(y)\sin(\omega t) + u_c(y)\cos(\omega t). \tag{5}$$

By introducing dimensionless variables $\tilde{x}$, $\tilde{y}$, $\tilde{u}$ and α such that

$$\tilde{x} = \frac{x}{a}, \quad \tilde{y} = \frac{y}{b}, \quad \tilde{u} = \frac{u}{Pb^2}\mu a, \quad \alpha = b\sqrt{\frac{\omega\rho}{\mu}}, \tag{6}$$

where b is the length measured on y-axis of the considered region. After the introducing dimensionless variables, each rectangular region is then transformed to be a one-unit rectangular region as shown in Figure 3.

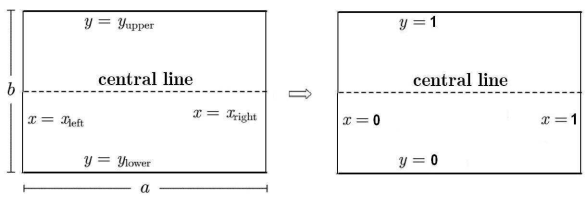


Figure 3: A diminution of a horizontal rectangular region

Equation (4) together with equation (5) are reduced to a system of non-homogeneous Helmholtz equations in one dimension[12]:

$$\alpha^2\tilde{u_s} = \frac{d^2\tilde{u_c}}{d\tilde{y}^2}, \quad -\alpha^2\tilde{u_c} = -1 + \frac{d^2\tilde{u_s}}{d\tilde{y}^2}, \tag{7}$$

where α is the reduced frequency. The boundary conditions for $\tilde{u_s}$ and $\tilde{u_c}$ result from using the boundary conditions for the velocity and the equation for oscillating flow assumption (5):

$$\tilde{u_s}(0) = 0, \quad \tilde{u_c}(0) = 0, \quad \tilde{u_s}(1) = 0, \quad \tilde{u_c}(1) = 0. \tag{8}$$

For equations (7), the analytical solution which satisfies the boundary conditions (8), can be determined by using a Fourier series analysis of $\tilde{u_s}$, $\tilde{u_c}$ for $\tilde{y}$. Hence, $\tilde{u_s}$, $\tilde{u_c}$ and 1 are expressed as the Fourier expansions [13]:

$$\begin{aligned} \tilde{u_s} &= \sum_{k=1}^{\infty} A_k \sin(k\pi\tilde{y}), \quad \tilde{u_c} = \sum_{k=1}^{\infty} B_k \sin(k\pi\tilde{y}), \\ 1 &= \sum_{k=1}^{\infty} C_k \sin(k\pi\tilde{y}), \end{aligned} \tag{9}$$

where C_k can be calculated by

$$C_k = \frac{\int_0^1 \sin(k\pi\tilde{y})d\tilde{y}}{\int_0^1 \sin^2(k\pi\tilde{y})d\tilde{y}} = \frac{2}{k\pi}(1-(-1)^k); \quad k = 1, 2, 3, \cdots . \tag{10}$$

Substituting the above series expansions (9)-(10) in the system of differential equations (7), we get a system of algebraic equations, which has the following solution for the unknown coefficients A_k and B_k:

$$A_k = C_k(\frac{-\pi^2 k^2}{\alpha^4 + k^4\pi^4}), \ B_k = C_k(\frac{\alpha^2}{\alpha^4 + k^4\pi^4}); \ k = 1, 2, 3, \cdots . \tag{11}$$

For the imposed oscillating pressure gradient the resulting periodic velocity can be written as:

$$\tilde{u} = \tilde{u}_a \cos(\omega t), \qquad \tilde{u}_a = \sqrt{\tilde{u}_s^2 + \tilde{u}_c^2},$$

where $\tilde{u}_a$ is the amplitude resulting from the expression of $\tilde{u}$. We can obtain the airflow velocity u by substituting $\tilde{u}$ back into (6), we yield

$$u = \frac{Pb^2}{\mu a}\tilde{u}.$$

3.2. Solution for the Vertical Regions

Now, we find the solution to the vertical subregions, areas 4-7. We do the same fashion as we do in the vertical part. By assuming fully developed flow, $u = 0$,$v = v(t, x)$, the continuity equation is satisfied and when developed flow conditions across the y-axis; therefore, we can omit equation (2). The following partial differential equation should be satisfied :

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu(\frac{\partial^2 v}{\partial x^2}), \quad \frac{\partial p}{\partial y} = -\frac{P}{a}\sin(\omega t), \tag{12}$$

where a is the length measured on y-axis of the considered region.

We also assume that v is periodic

$$v(x,t) = v_s(x)\sin(\omega t) + v_c(x)\cos(\omega t). \tag{13}$$

By introducing dimensionless variables $\tilde{x}$, $\tilde{y}$, $\tilde{v}$ and α such that

$$\tilde{x} = \frac{x}{b}, \quad \tilde{y} = \frac{y}{a}, \quad \tilde{v} = \frac{v}{Pb^2}\mu a, \quad \alpha = b\sqrt{\frac{\omega\rho}{\mu}}, \tag{14}$$

where b is the length measured on x-axis of the considered region. Each rectangular region is then transformed to be a one-unit rectangular region as shown in Figure 4.

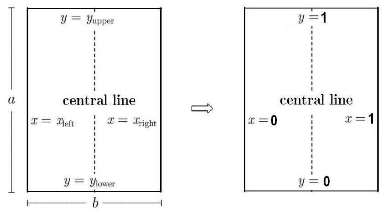


Figure 4: A diminution of a vertical rectangular region

Equation (12) together with equation (13) are reduced to a system of non-homogeneous Helmholtz equations in one-dimension:

$$\alpha^2\tilde{v_s} = \frac{d^2\tilde{v_c}}{d\tilde{x}^2}, \quad -\alpha^2\tilde{v_c} = 1 + \frac{d^2\tilde{v_s}}{d\tilde{x}^2}, \tag{15}$$

The boundary conditions for $\tilde{v_s}$ and $\tilde{v_c}$:

$$\tilde{v_s}(0) = 0, \quad \tilde{v_c}(0) = 0, \quad \tilde{v_s}(1) = 0, \quad \tilde{v_c}(1) = 0. \tag{16}$$

For equations (15), the analytical solution, which satisfies the boundary conditions (16), can be determined by using a Fourier series analysis of $\tilde{v_s}$, $\tilde{v_c}$ for $\tilde{x}$. Hence, $\tilde{v_s}$, $\tilde{v_c}$ and 1 are expressed as Fourier expansions:

$$\begin{aligned} \tilde{v_s} &= \sum_{m=1}^{\infty} A_m \sin(m\pi\tilde{x}), \quad \tilde{v_c} = \sum_{m=1}^{\infty} B_m \sin(m\pi\tilde{x}), \qquad (17) \\ 1 &= \sum_{m=1}^{\infty} C_m \sin(m\pi\tilde{x}), \end{aligned}$$

where C_m can be calculated by:

$$C_m = \frac{\int_0^1 \sin(m\pi\tilde{x})d\tilde{x}}{\int_0^1 \sin^2(m\pi\tilde{x})d\tilde{x}} = \frac{2}{m\pi}(1-(-1)^m);\ m = 1,2,3,\cdots. \tag{18}$$

Substituting the above series expansions (17)-(18) in the system of differential equations (15), we get a system of algebraic equations, which have the following solution for the unknown coefficients A_m and B_m:

$$A_m = C_m(\frac{\pi^2 m^2}{\alpha^4 + m^4\pi^4}),\ B_m = C_m(\frac{-\alpha^2}{\alpha^4 + m^4\pi^4});\ m = 1,2,3,\cdots. \tag{19}$$

For the imposed oscillating pressure gradient the resulting periodic velocity can be written as:

$$\tilde{v} = \tilde{v}_a \cos(\omega t + \pi), \qquad \tilde{v}_a = \sqrt{\tilde{v}_s^2 + \tilde{v}_c^2},$$

where $\tilde{v}_a$ is the amplitude resulting from the expression of $\tilde{v}$. We then obtain the airflow velocity

$$v = \frac{Pb^2}{\mu a}\tilde{v}.$$

4. Results and Discussion

Based on medical literatures, the breathing period is about $4s$ [11]. Therefore, in this paper, we use $\omega = \frac{\pi}{2}$. The analysis is carried out with $\rho = 1.148\ kg/m^3$, $\mu = 1.82 \times 10^{-5}\ Pa \cdot s$ and the amplitude of the oscillating intrapulmonary pressure $P = -133.32Pa$.

The arrow plots of the velocity field at $t = 1.4s$, $t = 2.05s$ and $t = 2.5s$ of each breathing period, respectively, are shown in Figures 5(a)-5(c). When $t = 1.4s$ air flows into the respiratory tract cavity which corresponds to the pulmonary relaxation. In contrast, when $t = 2.05s$ and $t = 2.5s$, air flows out the respiratory tract cavity which corresponds to pulmonary contraction. The high speed velocity occurs in the central area and reduces to the zero for the area which is more close to the walls.

When we consider the contour plots of the velocity field as shown in Figure 6, it is found that the velocity profiles become flat in the central area of the respiratory cavity except of the area which is close to the walls. This shows a boundary layer behavior with a high velocity gradient close to the boundaries. The amplitude of velocity always shows the maximum value in the central area

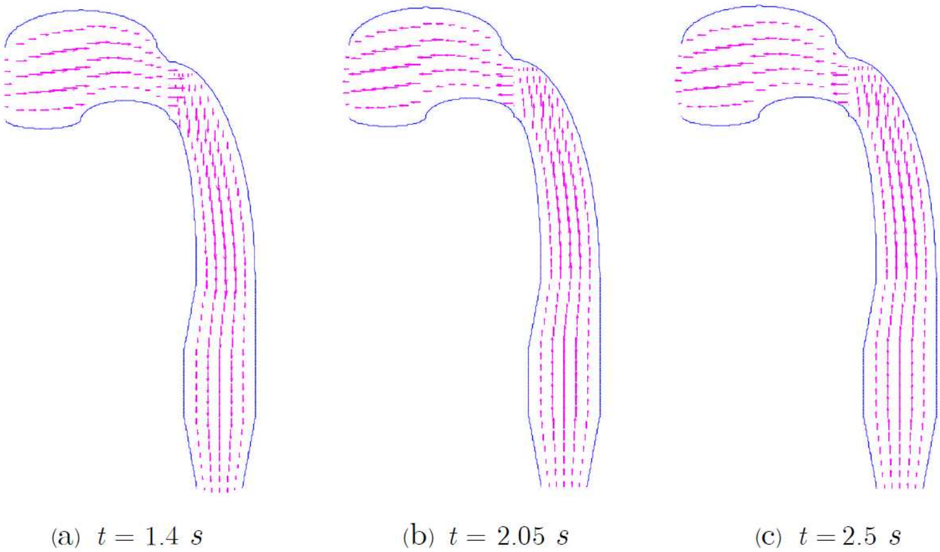
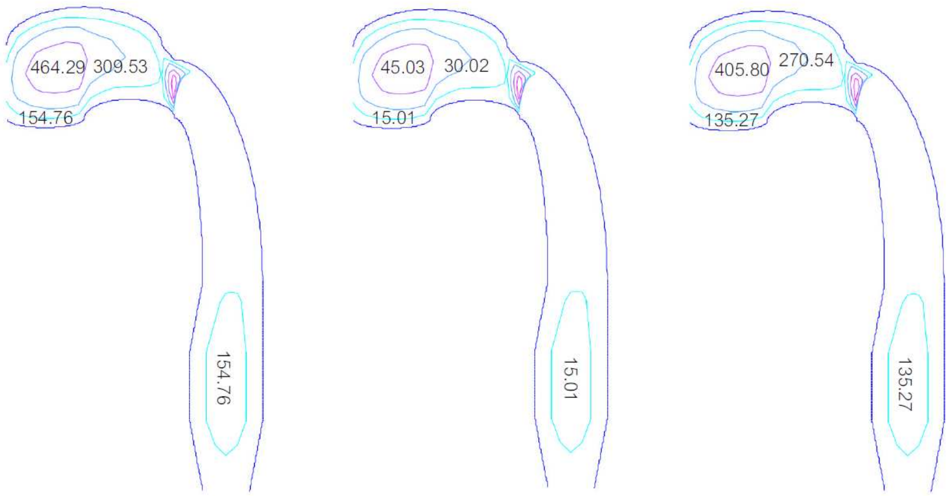


Figure 5: Arrow plots of the velocity field(cm/s) in the respiratory tract model

and reduces to the zero value for the area which is more close to the walls which corresponds to the arrow plots. When compare the velocities at different times, they are different and when $t = 2.05s$ the amplitude of the velocity is close to zero. The obtained velocities are in the range of $[0, 900]cm/s$ which agree to other publications [2] and [10].



(a) $t = 1.4\ s$ (b) $t = 2.05\ s$ (c) $t = 2.5\ s$

Figure 6: Contour plots of the airflow field(cm/s) in the respiratory tract model

5. Conclusions

An effective method as analytical expression for the solution of airflow in a human respiratory tract has been carried out as the objective of this research. The analytical solution of the oscillating flow is given in a Fourier series-based form. The obtained airflow profiles are reasonable and feasible. Because they show a good agreement with the fact that the laminar flow has the maximum velocity in the central area and reduces to the zero value close to the walls [15] and agree with other related publications [2], [10] and [11].

Acknowledgments

This research is supported by the Centre of Excellence in Mathematics, the Commission on Higher Education, Thailand. The authors are also very grateful to Department of Mathematics and Statistics, Thammasat University for giving the facilities support.

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