International Journal of Pure and Applied Mathematics
Volume 78 No. 5 2012, 751-772
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

GOODNESS-OF-FIT TESTS FOR EXPONENTIALITY AND RAYLEIGH DISTRIBUTION

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Abstract: We give some goodness-of-fit tests for exponentiality and Rayleigh distribution derived from characterizations of continuous distributions via expected values of two functions of record values.

AMS Subject Classification: 62G10, 62G30, 62G32
Key Words: order statistics, record values, exponential, Rayleigh distribution, characterizations, goodness-of-fit tests, powers

1. Introduction

Goodness-of-fit tests for exponentiality have a large literature and they are constructed by different techniques (see survey papers: Ascher [1], Henze and Meintanis [6], Doksum and Yandell [4]).

Tests of fit for the Rayleigh distribution were proposed in Meintanis and Iliopoulos [9], Morris and Szynal [11], Nosalewicz et al. [12]. Among many techniques for constructing goodness-of-fit tests characterizations of distributions play an important role. Characterizations of continuous distributions via expected values of two functions of order statistics and record values given in Lin [7], Grudzień and Szynal [5], Malinowska et al. [8] were used to construct

Received: May 8, 2012

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families of goodness-of-fits by Morris and Szynal [10], Szynal [13]. We extend here some families of tests based on moments of record values, proposed in Morris and Szynal [11], and Szynal [14]. This extension can be done by choosing particular values of parameters appearing in the mentioned characterizations.

2. Characterization Conditions

Let $\{X_n, n \geq 1\}$ be a sequence of i.i.d. random variables with a common cumulative distribution function F, $\overline{F} = 1 - F$, and probability density f. For a fixed integer $k \geq 1$ we define the sequence $U_k(1), U_k(2), \ldots$ of k-th upper record times of $\{X_n, n \geq 1\}$ as follows:

$$U_k(1) = 1,\ U_k(n+1) = \min\{j > U_k(n) : X_{j:j+k-1} > X_{U_k(n):U_k(n)+k-1}\},$$

where $X_{i:l}$ denotes the order statistic. Write

$$Y_n^{(k)} = X_{U_k(n):U_k(n)+k-1},\ n \geq 1.$$

The sequence $\{Y_n^{(k)}, n \geq 1\}$ is called the sequence of kth (upper) record values of the above sequence. For convenience we also take $Y_0^{(k)} = 0$ and note that

$$Y_1^{(k)} = X_{1:k} = \min(X_1, \ldots, X_k).$$

Here we discuss and apply characterizations in terms of the expected values of two functions of upper record values. Such conditions were first given by Lin [7] and generalized by Malinowska et al. [8]. We recall them now.

Let $\{X_n, n \geq 1\}$ be a sequence of random variables distributed as a variate X. Assume that $n \geq 1,\ k \geq 1$ and $s \geq 0,\ s \neq n$, are given integers and $r \neq 0$ a given real number such that $n + r + 1 > 0$. Then under some conditions $X \sim F$ and F is continuous iff

$$\begin{cases} E\left[-\log\left(\overline{F}\left(Y_{n+1}^{(k)}\right)\right)\right]^r = \dfrac{\Gamma(n+r+1)}{n!k^r}, \\ E\left[-\log\left(\overline{F}\left(Y_{s+1}^{(k)}\right)\right)\right]^{r+n-s} = \dfrac{\Gamma(n+r+1)}{s!k^{r+n-s}}. \end{cases} \tag{1}$$

We see that choosing particular values of parameters n and s we can use different characterization conditions leading to different families (depending on r and k) of fit tests. The case $n = 1,\ s = 0$, was proposed in Morris and Szynal [11]. Then allowed $r > -1/2$. The case $n = 2,\ s = 1$, was considered in Szynal [14].

Then the class of tests depends on $r > -3/2$. Here we apply the condition (1) with $n = 6,\ s = 0$, looking for pairs (r, k) of parameters (here $r > -6,\ k \geq 1$) which allow to choose from family of tests, statistic tests with powers greater than powers of recommended tests. Note that in the tests in Szynal [14] the form $X_i/\overline{X}_n$ appears with a power > 1 and the term $X_{i:n}/\overline{X}_n$ with a power > 0. A similar property characterizes the tests in Morris and Szynal [11]. Here we present a family of tests where $X_i/\overline{X}_n$ and $X_{i:n}/\overline{X}_n$ appear with the same power.

For exponential distribution $F(x) = 1 - e^{-x},\ x > 0$, letting $n = 6,\ s = 0$, we have

$$X \sim F(x) = 1 - e^{-x} \Leftrightarrow \begin{cases} E\left[Y_7^{(k)}\right]^r = \dfrac{\Gamma(r+7)}{6!k^r}, \\ E\left[Y_1^{(k)}\right]^{r+6} = \dfrac{\Gamma(r+7)}{k^{r+6}}. \end{cases} \tag{2}$$

Using the formulae for moments of record values (cf. Bieniek and Szynal [2]) we lead to

$$X \sim F(x) = 1 - e^{-x} \Rightarrow \begin{cases} E\left[X^{r+6}e^{-(k-1)X}\right] = \dfrac{\Gamma(r+7)}{k^{r+7}}, \\ E\left[X_{1:k}^{r+6}\right] = \dfrac{\Gamma(r+7)}{k^{r+6}}. \end{cases} \tag{3}$$

For Rayleigh distribution $F(x) = 1 - e^{-x^2},\ x > 0$, letting $n = 6,\ s = 0$, we have

$$X \sim F(x) = 1 - e^{-x^2} \Leftrightarrow \begin{cases} E\left[\left(Y_7^{(k)}\right)^2\right]^r = \dfrac{\Gamma(r+7)}{6!k^r}, \\ E\left[\left(Y_1^{(k)}\right)^2\right]^{r+6} = \dfrac{\Gamma(r+7)}{k^{r+6}}. \end{cases} \tag{4}$$

Hence we get

$$X \sim F(x) = 1 - e^{-x^2} \Rightarrow \begin{cases} E\left[\left(X^2\right)^{(r+6)} e^{-(k-1)X^2}\right] = \dfrac{\Gamma(r+7)}{k^{r+7}}, \\ E\left[\left(X_{1:k}\right)^2\right]^{r+6} = \dfrac{\Gamma(r+7)}{k^{r+6}}. \end{cases} \tag{5}$$

It is a starting point to construct the goodness-of-fit tests for exponentiality and for Rayleigh distribution.

3. Tests for Exponentiality: $X \sim Exp(\alpha)$

We consider the hypothesis $H : X \sim F$ when $F(x) = 1 - e^{-\alpha x}$, $x > 0$; $\alpha > 0$, where α is unknown parameter. We use

$$\hat{\alpha}_n = \frac{1}{\overline{X}_n}, \quad \overline{X}_n = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

(i) Applying (3) to construct goodness-of-fit tests for exponentiality we introduce statistics

$$\widehat{V}_{n1}^{(r,k)} := \frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{\overline{X}_n}\right)^{r+6} e^{-(k-1)X_i / \overline{X}_n},$$

$$\widehat{V}_{n2}^{(r,k)} := \frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}}{\overline{X}_n}\right)^{r+6}.$$

Then we need the quantities (see Morris and Szynal [11], and Szynal [14])

$$a_{n1}^{(r,k)} = \frac{1}{n}\left[\frac{\Gamma(2r+13)}{(2k-1)^{2r+13}} - \frac{\Gamma^2(r+8)}{k^{2r+16}} + \frac{2\Gamma(r+7)\left(\Gamma(r+8) - k\Gamma(r+7)\right)}{k^{2r+15}}\right],$$

$$b_{n1}^{(r,k)} = \frac{k}{n}\left[\frac{\Gamma(2r+14)}{(k-1)^{r+6}k^{r+7}} B_{\frac{k-1}{2k-1}}(r+7, r+7) + \frac{\Gamma(2r+13)}{(2k-1)^{2r+13}} - \frac{\Gamma^2(r+7)}{k^{2r+13}} - \frac{\left(\Gamma(r+8) - k\Gamma(r+7)\right)\left(\Gamma(r+8) - \Gamma(r+7)\right)}{k^{2r+15}}\right],$$

$$c_{n1}^{(r,k)} = \frac{1}{\binom{n}{k}}\sum_{j=1}^{k-1}\binom{k}{j}\binom{n-k}{k-j}\left[2\frac{\Gamma(2r+14)}{k^{r+6}(k-j)^{r+6}} B_{\frac{k-j}{2k-j}}(r+7, r+7) + j\frac{\Gamma(2r+13)}{(2k-j)^{2r+13}} - \frac{\Gamma^2(r+7)}{k^{2r+12}}\right] + \frac{1}{\binom{n}{k}}\left[\frac{\Gamma(2r+13) - \Gamma^2(r+7)}{k^{2r+12}}\right] - \frac{\left(\Gamma(r+8) - \Gamma(r+7)\right)^2}{nk^{2r+12}},$$

where $B_x(a,b) = \int_0^x t^{a-1}(1-t)^{b-1}dt,\ 0 < t < 1,\ a, b > 0$. When $k = 1$, we have

$$a_{n1}^{(r,1)} = b_{n1}^{(r,1)} = c_{n1}^{(r,1)} = \frac{1}{n}\left[\Gamma(2r+13) - \Gamma^2(r+7) - \left(\Gamma(r+8) - \Gamma(r+7)\right)^2\right]$$

and

$$\widehat{V}_{n1}^{(r,1)} = \widehat{V}_{n2}^{(r,1)} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{\overline{X}_n}\right)^{r+6}.$$

Also we use

$$\Sigma_{n1}^{(r,k)} := \begin{bmatrix} a_{n1}^{(r,k)} & b_{n1}^{(r,k)} \\ b_{n1}^{(r,k)} & c_{n1}^{(r,k)} \end{bmatrix}, \quad \Delta_{n1}^{(r,k)} := det\left(\Sigma_{n1}^{(r,k)}\right),$$

$$\left(\Sigma_{n1}^{(r,k)}\right)^{-1} = \frac{1}{\Delta_{n1}^{(r,k)}}\begin{bmatrix} c_{n1}^{(r,k)} & -b_{n1}^{(r,k)} \\ -b_{n1}^{(r,k)} & a_{n1}^{(r,k)} \end{bmatrix}.$$

Then the test statistics are as follows

$$\begin{aligned}
\widehat{T}_{n}^{(r,k)} =& \left(\begin{bmatrix} \widehat{V}_{n1}^{(r,k)} \\ \widehat{V}_{n2}^{(r,k)} \end{bmatrix} - \begin{bmatrix} \frac{\Gamma(r+7)}{k^{r+7}} \\ \frac{\Gamma(r+7)}{k^{r+6}} \end{bmatrix}\right)\left(\Sigma_{n1}^{(r,k)}\right)^{-1}\left(\begin{bmatrix} \widehat{V}_{n1}^{(r,k)} \\ \widehat{V}_{n2}^{(r,k)} \end{bmatrix} - \begin{bmatrix} \frac{\Gamma(r+7)}{k^{r+7}} \\ \frac{\Gamma(r+7)}{k^{r+6}} \end{bmatrix}\right)^{\prime} \\
=& \frac{1}{\Delta_{n1}^{(r,k)}}\left[c_{n1}^{(r,k)}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{\overline{X}_n}\right)^{r+6} e^{-(k-1)X_i/\overline{X}_n} - \frac{\Gamma(r+7)}{k^{r+7}}\right)^2\right. \\
& -2b_{n1}^{(r,k)}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{\overline{X}_n}\right)^{r+6} e^{-(k-1)X_i/\overline{X}_n} - \frac{\Gamma(r+7)}{k^{r+7}}\right) \\
& \cdot\left(\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}}{\overline{X}_n}\right)^{r+6} - \frac{\Gamma(r+7)}{k^{r+6}}\right) \\
& \left.+a_{n1}^{(r,k)}\left(\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}}{\overline{X}_n}\right)^{r+6} - \frac{\Gamma(r+7)}{k^{r+6}}\right)^2\right] \\
=& \widehat{T}_{n;c_1}^{(r,k)} + \widehat{T}_{n;c_2}^{(r,k)} = \widehat{T}_{n;c_3}^{(r,k)} + \widehat{T}_{n;c_4}^{(r,k)},
\end{aligned}$$

where

$$\begin{aligned}
\widehat{T}_{n;c_1}^{(r,k)} =& \frac{1}{a_{n1}^{(r,k)}}\left(\widehat{V}_{n1}^{(r,k)} - \frac{\Gamma(r+7)}{k^{r+7}}\right)^2 \\
=& \frac{1}{a_{n1}^{(r,k)}}\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{\overline{X}_n}\right)^{r+6} e^{-(k-1)X_i/\overline{X}_n} - \frac{\Gamma(r+7)}{k^{r+7}}\right]^2, \\
\widehat{T}_{n;c_2}^{(r,k)} =& \frac{1}{\Delta_{n1}^{(r,k)} a_{n1}^{(r,k)}}\left[a_{n1}^{(r,k)}\widehat{V}_{n2}^{(r,k)} - b_{n1}^{(r,k)}\widehat{V}_{n1}^{(r,k)} - \frac{\Gamma(r+7)}{k^{r+7}}\left(ka_{n1}^{(r,k)} - b_{n1}^{(r,k)}\right)\right]^2 \\
=& \frac{1}{\Delta_{n1}^{(r,k)} a_{n1}^{(r,k)}}\left[a_{n1}^{(r,k)}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}}{\overline{X}_n}\right)^{r+6}\right.
\end{aligned}$$

$$-b_{n1}^{(r,k)}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{\overline{X}_n}\right)^{r+6}e^{-(k-1)X_i/\overline{X}_n}-\frac{\Gamma(r+7)}{k^{r+7}}\left(ka_{n1}^{(r,k)}-b_{n1}^{(r,k)}\right)\Bigg]^2,$$

$$\widehat{T}_{n;c_3}^{(r,k)}=\frac{1}{c_{n1}^{(r,k)}}\left(\widehat{V}_{n2}^{(r,k)}-\frac{\Gamma(r+7)}{k^{r+7}}\right)^2$$

$$=\frac{1}{c_{n1}^{(r,k)}}\left[\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}}{\overline{X}_n}\right)^{r+6}-\frac{\Gamma(r+7)}{k^{r+6}}\right]^2,$$

$$\widehat{T}_{n;c_4}^{(r,k)}=\frac{1}{\Delta_{n1}^{(r,k)}c_{n1}^{(r,k)}}\left[c_{n1}^{(r,k)}\widehat{V}_{n1}^{(r,k)}-b_{n1}^{(r,k)}\widehat{V}_{n2}^{(r,k)}-\frac{\Gamma(r+7)}{k^{r+7}}\left(c_{n1}^{(r,k)}-kb_{n1}^{(r,k)}\right)\right]^2$$

$$=\frac{1}{\Delta_{n1}^{(r,k)}c_{n1}^{(r,k)}}\Bigg[c_{n1}^{(r,k)}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{\overline{X}_n}\right)^{r+6}e^{-(k-1)X_i/\overline{X}_n}$$

$$-b_{n1}^{(r,k)}\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}}{\overline{X}_n}\right)^{r+6}-\frac{\Gamma(r+7)}{k^{r+7}}\left(c_{n1}^{(r,k)}-kb_{n1}^{(r,k)}\right)\Bigg]^2.$$

4. Tests for Rayleigh Distribution: $X \sim Ral(\alpha)$

Here $F(x) := F(x;\alpha) = 1 - e^{-\alpha x^2}$, $f(x) = 2\alpha x e^{-\alpha x^2}$, $x > 0$; $\alpha > 0$, and we write

$$\hat{\alpha}_n = 1\Big/\overline{X_n^2}\,,\quad \overline{X_n^2} = \frac{1}{n}\sum_{j=1}^{n} X_j^2.$$

The quantities $a_{n1}^{(r,k)}$, $b_{n1}^{(r,k)}$, $c_{n1}^{(r,k)}$, are as for the exponential distribution and the test statistics are given by the following formulae:

$$\widehat{T}_n^{(r,k)}=\left(\begin{bmatrix}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i^2}{\overline{X_n^2}}\right)^{r+6}e^{-(k-1)X_i^2/\overline{X_n^2}}\\ \frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}^2}{\overline{X_n^2}}\right)^{r+6}\end{bmatrix}-\begin{bmatrix}\frac{\Gamma(r+7)}{k^{r+7}}\\ \frac{\Gamma(r+7)}{k^{r+6}}\end{bmatrix}\right)\left(\Sigma_{n1}^{(r,k)}\right)^{-1}$$

$$\cdot\left(\begin{bmatrix}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i^2}{\overline{X_n^2}}\right)^{r+6}e^{-(k-1)X_i^2/\overline{X_n^2}}\\ \frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}^2}{\overline{X_n^2}}\right)^{r+6}\end{bmatrix}-\begin{bmatrix}\frac{\Gamma(r+7)}{k^{r+7}}\\ \frac{\Gamma(r+7)}{k^{r+6}}\end{bmatrix}\right)$$

$$=\frac{1}{\Delta_{n1}^{(r,k)}}\Bigg[c_{n1}^{(r,k)}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i^2}{\overline{X_n^2}}\right)^{r+6}e^{-(k-1)X_i^2/\overline{X_n^2}}-\frac{\Gamma(r+7)}{k^{r+7}}\right)^2$$

$$
\begin{aligned}
&-2b_{n1}^{(r,k)}\left(\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i^2}{\overline{X_n^2}}\right)^{r+6} e^{-(k-1)X_i^2/\overline{X_n^2}}-\frac{\Gamma(r+7)}{k^{r+7}}\right)\\
&\cdot\left(\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}^2}{\overline{X_n^2}}\right)^{r+6}-\frac{\Gamma(r+7)}{k^{r+7}}\right)\\
&+a_{n1}^{(r,k)}\left(\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}^2}{\overline{X_n^2}}\right)^{r+6}-\frac{\Gamma(r+7)}{k^{r+6}}\right)^2\Bigg]\\
&=\widehat{T}_{n;c_1}^{(r,k)}+\widehat{T}_{n;c_2}^{(r,k)}=\widehat{T}_{n;c_3}^{(r,k)}+\widehat{T}_{n;c_4}^{(r,k)},
\end{aligned}
$$

where

$$
\begin{aligned}
\widehat{T}_{n;c_1}^{(r,k)}=&\frac{1}{a_{n1}^{(r,k)}}\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i^2}{\overline{X_n^2}}\right)^{r+6} e^{-(k-1)X_i^2/\overline{X_n^2}}-\frac{\Gamma(r+7)}{k^{r+7}}\right]^2,\\
\widehat{T}_{n;c_2}^{(r,k)}=&\frac{1}{\Delta_{n1}^{(r,k)}a_{n1}^{(r,k)}}\Bigg[a_{n1}^{(r,k)}\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}^2}{\overline{X_n^2}}\right)^{r+6}\\
&-b_{n1}^{(r,k)}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i^2}{\overline{X_n^2}}\right)^{r+6} e^{-(k-1)X_i^2/\overline{X_n^2}}-\frac{\Gamma(r+7)}{k^{r+7}}\left(ka_{n1}^{(r,k)}-b_{n1}^{(r,k)}\right)\Bigg]^2,\\
\widehat{T}_{n;c_3}^{(r,k)}=&\frac{1}{c_{n1}^{(r,k)}}\left[\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}^2}{\overline{X_n^2}}\right)^{6}-\frac{\Gamma(r+7)}{k^{r+6}}\right]^2,\\
\widehat{T}_{n;c_4}^{(r,k)}=&\frac{1}{\Delta_{n1}^{(r,k)}c_{n1}^{(r,k)}}\Bigg[c_{n1}^{(r,k)}\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i^2}{\overline{X_n^2}}\right)^{r+6} e^{-(k-1)X_i^2/\overline{X_n^2}}\\
&-b_{n1}^{(r,k)}\frac{1}{\binom{n}{k}}\sum_{i=1}^{n-k+1}\binom{n-i}{k-1}\left(\frac{X_{i:n}^2}{\overline{X_n^2}}\right)^{r+6}-\frac{\Gamma(r+7)}{k^{r+7}}\left(c_{n1}^{(r,k)}-kb_{n1}^{(r,k)}\right)\Bigg]^2.
\end{aligned}
$$

5. Simulations

5.1. Exponential Distribution

We have selected tests and alternatives in Table 1 from Cabaña and Cabaña [3] as standards of comparisons with tests $\widehat{T}_n^{(r,k)}$ and their components $\widehat{T}_{n;c_1}^{(r,k)}$, $\widehat{T}_{n;c_2}^{(r,k)}$, $\widehat{T}_{n;c_3}^{(r,k)}$, $\widehat{T}_{n;c_4}^{(r,k)}$. When $n = 20$ and $n = 50$ the powers of tests were investigated for

$r = -5.95, -5.75, -5.5, -5.25, -5.0, -4.75, -4.5, -4.25, -4.0, -3.75, -3.5, -3.25, -3.0, -2.75, -2.5, -2.25, -2.0, -1.75, -1.5, -1.25, -1.0, -0.75, -0.5, -0.25, 0.25, 0.5, 0.75, 1.0$ with $k = 1, 2, 3, 4, 5$. Critical values were simulated using 100 000 samples and the associated powers were obtained using 100 000 samples, but only some results are presented here.

Moreover, we investigated simulation powers of tests derived from characterizing conditions (1) when there $n = 2$ and $s = 0$. These new tests have the same form as previous ones with r replaced by $r - 4$. For them we use the symbol $T^{\bullet}$.

Letting in the formulae for tests $r - 4$ instead of r we investigated their powers for $n = 20$ and $n = 50$ where $r = -1.95, -1.9, -1.7, -1.5, -1.3, -1.1, -1.0, -0.9, -0.7, -0.3, -0.1, 0.1, 0.3, 0.5, 0.7, 0.9, 1.0, 1.1, 1.3, 1.5, 2.0$ and $k = 1, 2, 3, 4, 5$.

For samples of size 20 we include simulations for some favorable omnibus tests with Av. powers ≥ 45.5 (Table 2). Note that the greatest Av. power 44.4 among the recommended tests in Table 1 belongs to the Cox and Oakes test. For samples of size 50 we include (Table 3) simulations for tests with Av. powers ≥ 75.5. Note that the greatest Av. power 75.2 among the recommended tests in Table 1 belongs to the Baringhause and Henze test based on the empirical characteristic function.

The meaning of the headings and the test-statistics can be found in Cabaña and Cabaña [3] and Henze and Meintanis [6] and their references:

TEEP: The transformed estimated empirical process (TEEP) test

EP: The Epps and Pulley test

BHKS: The Baringhaus and Henze test suggested by the Kolmogorov-Smirnov statistic

BHCM: The Baringhaus and Henze test suggested by the Cramér-von Mises statistic

S: The test based on spacing

CO: The Cox and Oakes test

BH: The Baringhaus and Henze test based on the empirical Laplace transform

T: The Henze and Meintanis test based on the empirical characteristic function (ECF).

The alternatives considered are:

$W(\theta)$- Weibull distribution with parameters $(1, \theta)$

$\Gamma(\theta)$- Gamma distribution with parameters $(1, \theta)$

	Alt.	$TEEP$	EP	$BHKS$	$BHCM$	S	CO	BH	T
	$W(0.8)$	26	24	17	22	24	28	24	1
	$W(1.4)$	35	36	28	35	35	37	37	45
	$\Gamma(0.4)$	85	76	71	75	76	91	80	11
	$\Gamma(2)$	54	48	40	47	46	54	51	56
	$LN(0.8)$	37	25	30	27	24	33	29	34
	$LN(1.5)$	64	67	58	66	67	60	66	2
	HN	17	21	18	22	21	19	21	31
	U	47	66	52	70	70	50	61	82
n=20	$CH(0.5)$	72	63	56	61	63	80	67	6
	$CH(1)$	12	15	13	16	15	13	15	23
	$CH(1.5)$	77	84	67	83	84	81	83	89
	$LF(2)$	24	28	24	30	29	25	25	39
	$LF(4)$	36	42	34	43	42	37	41	54
	$EV(0.5)$	11	13	18	16	15	13	15	23
	$EV(1.5)$	35	35	48	47	46	37	43	58
	$DL(1)$	25	20	20	21	19	25	23	28
	$DL(1.5)$	72	64	56	63	62	72	68	71
	Av.	42.9	42.8	38.2	43.8	43.4	44.4	44.1	38.4
	$W(0.8)$	53	48	35	46	48	56	50	17
	$W(1.4)$	82	80	71	77	79	82	81	81
	$\Gamma(0.4)$	100	99	97	99	98	100	99	90
	$\Gamma(2)$	94	91	86	90	90	96	93	92
	$LN(0.8)$	37	25	30	27	24	33	29	34
	$LN(1.5)$	95	95	92	95	95	92	95	54
	HN	45	54	50	53	54	45	52	60
	U	93	98	99	99	99	91	97	100
n=50	$CH(0.5)$	98	94	90	94	94	99	96	79
	$CH(1)$	31	38	36	37	38	30	35	44
	$CH(1.5)$	100	100	100	100	100	100	100	100
	$LF(2)$	61	69	65	69	69	60	68	74
	$LF(4)$	81	87	82	87	87	80	86	90
	$EV(0.5)$	30	38	36	37	38	30	35	44
	$EV(1.5)$	80	90	88	90	90	78	87	93
	$DL(1)$	57	39	43	44	39	55	47	54
	$DL(1.5)$	99	97	96	97	97	99	98	98
	Av.	74.8	74.2	72.2	74.9	74.2	74.1	75.1	72.7

Table 1: (Source: Cabaña and Cabaña [3]) Empirical comparison of the performances of the test TEEP and seven other tests, under several alternatives. The entries are simulated powers of 5% tests.

$LN(\theta)$- Lognormal distribution with parameters $(1, \theta)$

HN- Half-normal distribution: the law of $|Z|$, Z standard normal

U- Uniform distribution on $[0, 1]$

$EV(\theta)$- Modified extreme value distribution: the law of $\log(1 - \theta \log U)$, U uniform on $[0, 1]$

$LF(\theta)$- Linear increasing failure rate: the law of $\theta^{-1}(\sqrt{1 + 2Y\theta} - 1)$, $Y \sim$

k	2				
Tests	$\widehat{T}_{n;c_2}^{(r,k)}$	$\widehat{T}_{n;c_3}^{(r,k)}$	$\widehat{T}_{n;c_3}^{(r,k)}$	$\widehat{T}_{n;c_3}^{\bullet(r,k)}$	$\widehat{T}_{n;c_3}^{\bullet(r,k)}$
Alt \ r	-5.75	-5.5	-5.25	-1.5	-1.3
$W(0.8)$	27	26	25	26	25
$W(1.4)$	39	39	38	40	39
$G(0.4)$	90	88	83	88	84
$G(2)$	56	55	52	56	53
$LN(0.8)$	36	36	31	36	32
$LN(1.5)$	59	59	63	59	62
HN	20	20	21	20	21
U	56	57	66	58	64
$CH(0.5)$	79	77	70	77	71
$CH(1)$	14	14	15	14	15
$CH(1.5)$	82	82	84	82	84
$LF(2)$	28	28	30	28	29
$LF(4)$	40	41	44	42	43
$EV(0.5)$	14	14	15	14	15
$EV(1.5)$	40	41	45	41	44
$DL(1)$	26	26	23	27	24
$DL(1.5)$	73	73	70	74	70
Av.	45.8	45.7	45.6	46.1	45.7

$Exp(1)$

$DL(\theta)$- Dhillon's distribution: the law of $e^{(-\log U)^{1/(\theta+1)}} - 1$, U uniform on $[0,1]$

$CH(\theta)$- Chen distribution: the law of $(\log(1-\frac{1}{2}\log U))^{1/\theta}$, U uniform on $[0,1]$.

Comments. 1. For $k=1$ tests $\widehat{T}_n^{(r,1)}$ are not defined as $\triangle_{n1}^{(r,1)}=0$. But we can still use tests $\widehat{T}_{n;c_1}^{(r,1)} = \widehat{T}_{n;c_3}^{(r,1)}$ (or $\widehat{T}_{n;c_1}^{\bullet(r,1)} = \widehat{T}_{n;c_3}^{\bullet(r,1)}$). Then we can see that the simplest test

$$\widehat{T}_{n;c_1}^{\bullet(-1.5,1)} = \frac{1}{a_{n1}^{\bullet(-1.5,1)}}\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{\overline{\overline{X_n}}}\right)^{1/2} - \Gamma\left(\frac{3}{2}\right)\right]$$
$$= \frac{16n}{16-5\sqrt{\pi}}\left[\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{\overline{\overline{X_n}}}\right)^{1/2} - \frac{\sqrt{\pi}}{2}\right]^2 \qquad (E_1)$$

with Av. powers 45.0 and 74.5 for sample of sizes $n=20$ and 50, respectively, can be chosen instead of the Cox and Oakes test (with Av. powers 44.4 and 74.1) for the alternatives from Table 1. This test, among others things, was discussed in Szynal [13] and in Nosalewicz et al. [12].

2. In general, most powerful tests one can find in families tests $\widehat{T}_{n;c_2}^{(r,k)}$ ($\widehat{T}_{n;c_2}^{\bullet(r,k)}$) or $\widehat{T}_{n;c_3}^{(r,k)}$ ($\widehat{T}_{n;c_3}^{\bullet(r,k)}$). Tests with the index $n;c_3$ has the simpler form than with

k	3									
Tests	$\widehat{T}_{n;c_2}^{(r,k)}$	$\widehat{T}_{n;c_3}^{(r,k)}$	$\widehat{T}_{n;c_3}^{(r,k)}$	$\widehat{T}_{n;c_3}^{(r,k)}$	$\widehat{T}_{n;c_2}^{\bullet(r,k)}$	$\widehat{T}_{n;c_3}^{\bullet(r,k)}$	$\widehat{T}_{n;c_3}^{\bullet(r,k)}$	$\widehat{T}_{n;c_3}^{\bullet(r,k)}$	$\widehat{T}_{n;c_3}^{\bullet(r,k)}$	$\widehat{T}_{n;c_3}^{\bullet(r,k)}$
Alt $\backslash r$	-5.25	-5.25	-5.0	-4.75	-1.3	-1.3	-1.1	-1.0	-0.9	-0.7
$W(0.8)$	18	23	21	21	19	23	21	21	21	20
$W(1.4)$	43	42	43	41	43	42	42	42	42	41
$G(0.4)$	65	86	81	76	72	86	82	80	78	74
$G(2)$	57	60	58	55	58	60	59	58	57	54
$LN(0.8)$	33	43	39	33	36	43	40	38	36	32
$LN(1.5)$	61	53	57	60	60	52	55	57	58	61
HN	26	22	24	24	25	21	23	23	24	24
U	71	57	63	68	67	55	61	63	65	69
$CH(0.5)$	52	73	67	62	59	74	69	67	65	61
$CH(1)$	18	15	17	17	18	15	16	16	17	17
$CH(1.5)$	87	83	84	85	86	82	84	84	85	85
$LF(2)$	35	30	32	33	34	30	31	32	32	33
$LF(4)$	50	44	46	47	48	43	45	45	47	47
$EV(0.5)$	19	15	17	17	18	15	16	16	17	17
$EV(1.5)$	51	42	46	49	49	41	44	46	47	49
$DL(1)$	26	31	29	26	27	31	30	28	27	25
$DL(1.5)$	73	78	75	72	74	78	76	75	74	71
Av.	46.1	46.8	47.0	46.3	46.7	46.6	46.8	46.5	46.4	46.0

k	4									
Tests	$\widehat{T}_{n;c_3}^{(r,k)}$	$\widehat{T}_{n;c_2}^{(r,k)}$	$\widehat{T}_{n;c_3}^{(r,k)}$	$\widehat{T}_{n;c_3}^{(r,k)}$	$\widehat{T}_{n;c_3}^{(r,k)}$	$\widehat{T}_{n;c_3}^{(r,k)}$	$\widehat{T}_{n;c_3}^{\bullet(r,k)}$	$\widehat{T}_{n;c_2}^{\bullet(r,k)}$	$\widehat{T}_{n;c_3}^{\bullet(r,k)}$	$\widehat{T}_{n;c_2}^{\bullet(r,k)}$
Alt $\backslash r$	-5.25	-5.0	-5.0	-4.75	-4.5	-4.25	-1.3	-1.1	-1.1	-1.0
$W(0.8)$	19	15	17	16	16	15	20	18	19	16
$W(1.4)$	42	46	44	45	45	44	42	45	44	46
$G(0.4)$	85	72	81	76	72	67	86	81	83	73
$G(2)$	62	63	63	62	60	57	86	81	83	63
$LN(0.8)$	51	45	49	45	41	36	52	49	51	45
$LN(1.5)$	41	51	45	48	51	53	41	48	45	52
HN	21	26	23	25	26	26	21	24	23	26
U	50	63	56	61	66	69	48	57	54	63
$CH(0.5)$	72	58	67	61	58	52	73	67	70	58
$CH(1)$	15	18	17	18	19	19	15	17	16	19
$CH(1.5)$	79	85	82	84	85	85	79	84	82	86
$LF(2)$	29	35	32	34	35	35	28	33	31	35
$LF(4)$	41	49	45	47	49	49	41	46	44	49
$EV(0.5)$	15	19	17	18	19	19	15	17	16	19
$EV(1.5)$	38	47	43	46	49	50	37	44	42	48
$DL(1)$	35	33	34	33	31	28	35	35	36	33
$DL(1.5)$	80	80	80	79	77	74	80	81	81	80
Av.	45.7	47.3	46.7	47.0	47.0	45.9	45.6	47.6	47.0	47.8

the index $n;c_2$. For instance, we can recommend the simple test

$$\widehat{T}_{n;c_3}^{(-5,3)} = \frac{1}{c_{n1}^{(-5,3)}} \left[\frac{1}{\binom{n}{3}} \sum_{i=1}^{n-2} \binom{n-i}{2} \frac{X_{i:n}}{\overline{X_n}} - \frac{1}{3} \right]^2$$

k	4						5		
Tests	$\widehat{T}^{\bullet(r,k)}_{n;c_3}$	$\widehat{T}^{\bullet(r,k)}_{n;c_2}$	$\widehat{T}^{\bullet(r,k)}_{n;c_3}$	$\widehat{T}^{\bullet(r,k)}_{n;c_3}$	$\widehat{T}^{\bullet(r,k)}_{n;c_3}$	$\widehat{T}^{\bullet(r,k)}_{n;c_3}$	$\widehat{T}^{(r,k)}_{n;c_3}$	$\widehat{T}^{(r,k)}_{n;c_3}$	$\widehat{T}^{(r,k)}_{n;c_3}$
Alt \ r	-1.0	-0.9	-0.9	-0.7	-0.5	-0.3	-5.0	-4.5	-4.25
$W(0.8)$	17	13	16	15	16	15	13	10	8
$W(1.4)$	45	46	44	44	45	44	45	47	47
$G(0.4)$	81	83	73	82	59	79	80	68	60
$G(2)$	82	59	79	75	72	68	65	65	63
$LN(0.8)$	49	51	45	49	42	48	56	51	47
$LN(1.5)$	49	42	48	44	41	37	33	36	36
HN	24	23	26	23	27	24	23	26	27
U	23	27	24	25	26	26	50	60	63
$CH(0.5)$	67	70	58	68	45	64	64	52	44
$CH(1)$	68	45	64	60	57	53	16	19	20
$CH(1.5)$	84	82	86	83	86	83	81	84	85
$LF(2)$	33	31	35	32	36	32	31	35	36
$LF(4)$	32	36	32	34	35	35	43	48	50
$EV(0.5)$	17	16	19	17	19	17	16	19	19
$EV(1.5)$	17	19	17	18	19	19	40	47	49
$DL(1)$	35	31	34	32	31	28	38	37	34
$DL(1.5)$	81	78	80	79	77	75	83	82	80
Av.	47.3	46.1	46.7	46.6	46.9	46.0	45.7	46.2	45.2

k	5								
Tests	$\widehat{T}^{(r,k)}_{n;c_2}$	$\widehat{T}^{(r,k)}_{n;c_3}$	$\widehat{T}^{\bullet(r,k)}_{n;c_3}$	$\widehat{T}^{\bullet(r,k)}_{n;c_2}$	$\widehat{T}^{\bullet(r,k)}_{n;c_3}$	$\widehat{T}^{\bullet(r,k)}_{n;c_2}$	$\widehat{T}^{\bullet(r,k)}_{n;c_3}$	$\widehat{T}^{\bullet(r,k)}_{n;c_3}$	$\widehat{T}^{\bullet(r,k)}_{n;c_3}$
Alt \ r	-4.75	-4.75	-1.0	-0.9	-0.9	-0.7	-0.7	-0.5	-0.3
$W(0.8)$	10	11	13	13	13	9	11	9	9
$W(1.4)$	47	46	45	45	46	47	46	46	47
$G(0.4)$	69	74	80	79	78	63	72	67	63
$G(2)$	65	65	65	65	65	65	65	64	64
$LN(0.8)$	52	54	57	56	55	51	53	50	48
$LN(1.5)$	38	34	33	32	34	39	34	35	37
HN	25	24	23	23	24	26	24	26	27
U	58	55	50	51	53	60	56	60	62
$CH(0.5)$	52	57	64	63	61	47	55	50	46
$CH(1)$	18	17	16	17	17	19	18	18	19
$CH(1.5)$	84	83	81	81	81	84	83	84	85
$LF(2)$	34	33	31	31	32	35	33	35	36
$LF(4)$	48	46	43	44	45	49	46	48	50
$EV(0.5)$	19	18	16	16	17	19	18	19	19
$EV(1.5)$	46	44	40	41	41	46	44	46	48
$DL(1)$	37	38	38	39	39	37	37	36	35
$DL(1.5)$	83	83	83	83	83	82	82	82	81
Av.	46.1	45.9	45.8	45.8	46.1	45.8	45.8	45.7	45.6

Table 2: Powers of 5% test based on 100 000 simulations using empirical critical values with Av. powers ≥ 45.5; $n = 20$

where

$$c_{n1}^{(-5,3)} = \frac{4n^2 - 3n - 7}{45n(n-1)(n-2)}.$$

k	2			3							
Tests	$\widehat{T}^{(r,k)}_{n;c_3}$	$\widehat{T}^{\bullet(r,k)}_{n;c_2}$	$\widehat{T}^{\bullet(r,k)}_{n;c_3}$	$\widehat{T}^{(r,k)}_{n;c_2}$	$\widehat{T}^{(r,k)}_{n;c_3}$	$\widehat{T}^{(r,k)}_{n;c_3}$	$\widehat{T}^{(r,k)}_{n;c_3}$	$\widehat{T}^{\bullet(r,k)}_{n;c_2}$	$\widehat{T}^{\bullet(r,k)}_{n;c_3}$	$\widehat{T}^{\bullet(r,k)}_{n;c_3}$	$\widehat{T}^{\bullet(r,k)}_{n;c_2}$
Alt \ r	-5.25	-1.7	-1.3	-5.25	-5.25	-5.0	-4.75	-1.3	-1.1	-0.9	-0.7
$W(0.8)$	50	51	51	43	49	48	46	45	48	47	46
$W(1.4)$	82	82	82	82	82	82	82	83	83	82	81
$G(0.4)$	99	99	100	97	100	99	99	99	99	99	98
$G(2)$	94	93	94	93	96	95	93	94	95	94	93
$LN(0.8)$	60	56	63	56	79	70	61	62	74	67	58
$LN(1.5)$	94	94	93	94	90	92	94	94	91	93	94
HN	52	53	51	56	46	51	54	54	49	52	54
U	98	98	97	99	93	96	98	98	95	97	98
$CH(0.5)$	97	97	97	91	98	97	95	94	97	96	94
$CH(1)$	35	37	35	40	31	35	38	38	33	36	38
$CH(1.5)$	100	100	100	100	100	100	100	100	100	100	100
$LF(2)$	68	69	67	72	63	67	70	71	66	69	70
$LF(4)$	86	87	85	89	82	86	87	88	85	87	87
$EV(0.5)$	36	37	35	40	31	35	37	38	34	36	38
$EV(1.5)$	88	88	87	91	81	86	89	89	84	87	89
$DL(1)$	50	48	52	48	63	57	50	52	60	54	48
$DL(1.5)$	99	98	99	98	99	99	99	99	99	99	98
Av.	75.7	75.7	75.7	75.7	75.5	76.2	75.9	76.3	76.0	76.2	75.6

k	4										
Tests	$\widehat{T}^{(r,k)}_{n;c_2}$	$\widehat{T}^{(r,k)}_{n;c_3}$	$\widehat{T}^{(r,k)}_{n;c_2}$	$\widehat{T}^{(r,k)}_{n;c_3}$	$\widehat{T}^{(r,k)}_{n;c_3}$	$\widehat{T}^{\bullet(r,k)}_{n;c_2}$	$\widehat{T}^{\bullet(r,k)}_{n;c_2}$	$\widehat{T}^{\bullet(r,k)}_{n;c_3}$	$\widehat{T}^{\bullet(r,k)}_{n;c_2}$	$\widehat{T}^{\bullet(r,k)}_{n;c_3}$	$\widehat{T}^{\bullet(r,k)}_{n;c_3}$
Alt \ r	-5.0	-5.0	-4.75	-4.5	-4.25	-1.1	-1.0	-1.0	-0.9	-0.9	-0.7
$W(0.8)$	43	45	44	42	41	46	44	46	40	45	43
$W(1.4)$	84	83	83	82	81	83	84	83	83	83	82
$G(0.4)$	99	100	99	99	98	99	99	100	97	99	99
$G(2)$	96	96	96	94	93	96	96	96	95	96	95
$LN(0.8)$	75	84	77	70	62	83	76	84	68	81	75
$LN(1.5)$	91	87	90	91	93	89	91	88	92	88	90
HN	52	47	50	53	55	48	45	47	55	48	51
U	95	92	95	97	98	92	90	92	97	93	95
$CH(0.5)$	96	97	96	95	92	97	96	98	92	97	96
$CH(1)$	36	32	35	37	39	33	36	32	38	33	35
$CH(1.5)$	100	100	100	100	100	100	100	100	100	100	100
$LF(2)$	68	63	67	69	70	64	68	63	70	64	67
$LF(4)$	86	82	85	87	87	83	86	83	88	83	85
$EV(0.5)$	36	32	35	37	39	33	36	32	38	33	35
$EV(1.5)$	85	80	84	87	89	81	85	80	88	81	84
$DL(1)$	61	67	62	56	51	66	62	67	56	65	61
$DL(1.5)$	99	100	99	99	99	100	99	100	99	100	99
Av.	76.7	75.6	76.3	76.0	76.3	76.1	76.9	75.8	76.3	75.9	76.0

After some evaluations we get a simple test

$$\widehat{T}^{(-5,3)}_{n;c_3} = \frac{45n(n-1)(n-2)}{4n^2-3n-7}\left[\frac{1}{\binom{n}{3}}\sum_{i=1}^{n-2}\binom{n-i}{2}\frac{X_{i:n}}{\overline{X_n}} - \frac{1}{3}\right]^2$$

k	4		5								
Tests	$\widehat{T}^{\bullet(r,k)}_{n;c_3}$	$\widehat{T}^{\bullet(r,k)}_{n;c_3}$	$\widehat{T}^{(r,k)}_{n;c_2}$	$\widehat{T}^{(r,k)}_{n;c_3}$	$\widehat{T}^{(r,k)}_{n;c_3}$	$\widehat{T}^{(r,k)}_{n;c_3}$	$\widehat{T}^{\bullet(r,k)}_{n;c_2}$	$\widehat{T}^{\bullet(r,k)}_{n;c_3}$	$\widehat{T}^{\bullet(r,k)}_{n;c_2}$	$\widehat{T}^{\bullet(r,k)}_{n;c_3}$	$\widehat{T}^{\bullet(r,k)}_{n;c_3}$
Alt $\backslash r$	-0.5	-0.3	-4.5	-4.5	-4.25	-4.0	-0.7	-0.5	-0.5	-0.3	-0.1
$W(0.8)$	42	41	34	39	37	37	39	39	34	38	38
$W(1.4)$	82	82	83	83	83	83	84	83	84	83	83
$G(0.4)$	99	98	95	99	98	97	99	99	95	98	98
$G(2)$	94	93	95	96	95	94	96	96	95	95	95
$LN(0.8)$	70	63	72	82	76	70	82	82	72	78	73
$LN(1.5)$	91	93	91	87	89	90	88	87	91	88	90
HN	53	54	55	50	53	55	50	50	55	52	54
U	97	98	97	93	95	97	93	93	97	95	96
$CH(0.5)$	95	93	87	95	94	92	95	95	88	94	93
$CH(1)$	37	38	39	34	36	39	35	34	39	36	38
$CH(1.5)$	100	100	100	100	100	100	100	100	100	100	100
$LF(2)$	69	70	71	66	69	70	67	66	71	68	70
$LF(4)$	87	87	87	84	86	87	85	84	88	86	87
$EV(0.5)$	37	38	39	34	36	38	34	34	39	37	38
$EV(1.5)$	87	88	87	82	85	88	83	82	87	85	87
$DL(1)$	57	52	59	65	61	57	66	66	59	62	59
$DL(1.5)$	99	99	99	100	99	99	100	100	99	99	99
Av.	76.1	75.8	75.9	75.8	76.0	76.1	76.2	75.9	76.0	76.3	76.3

Table 3: Powers of 5% test based on 100 000 simulations using empirical critical values with Av. powers ≥ 75.5; $n = 50$

$$= \frac{5n(n-1)(n-2)}{4n^2-3n-7}\left[\frac{9}{n(n-1)(n-2)}\sum_{i=1}^{n-2}(n-i-1)(n-i)\frac{X_{i:n}}{\overline{X_n}}-1\right]^2 \qquad (E_2)$$

with Av. powers 47.0 and 76.2 for sample sizes $n = 20$ and 50, respectively.

Concluding Remark. To verify a test for exponentiality one can proceed as follows:

(i) First use the simple test (E_1) which behaviour similar as recomennded tests.

(ii) Next try to apply the test (E_2) which is more powerful than the discussed recommended tests.

(iii) At last our test $\widehat{T}^{\bullet(-1,4)}_{n;c_2}$ with the greatest Av. power can be proposed.

5.2. Rayleigh Distribution

We have selected tests and alternatives in Table 4 from Meintanis and Iliopoulos [9] as standards of comparisons with our tests. When $n = 20$ the test statistics $\widehat{T}^{(r,k)}_{n}$ and their components $\widehat{T}^{(r,k)}_{n;c_1}$, $\widehat{T}^{(r,k)}_{n;c_2}$, $\widehat{T}^{(r,k)}_{n;c_3}$, $\widehat{T}^{(r,k)}_{n;c_4}$ were investigated for $r = -5.95,\ -5.75,\ -5.5,\ -5.25,\ -5.0,\ -4.75,\ -4.5,\ -4.25,\ -4.0,\ -3.75,\ -3.5,$

-3.25, -3.0, -2.75, -2.5, -2.25, -2.0, -1.75, -1.5, -1.25, -1.0, -0.75, -0.5, -0.25, 0.25, 0.5, 0.75, 1.0 with $k = 1$, 2, 3, 4, 5. Critical values were simulated using 100 000 samples and the associated powers were obtained using 100 000 samples, but only some result are presented.

Moreover, we investigated simulation powers of tests derived from characterizing conditions (1) when there $n = 2$ and $s = 0$. These new tests have the same form as previous ones with r replaced by $r - 4$. For them we use the symbol $T^{\bullet}$.

Letting in the formulae for tests $r - 4$ instead of r we investigated their powers for $n = 20$ and $n = 50$ where $r = -1.95$, -1.9, -1.7, -1.5, -1.3, -1.1, -1.0, -0.9, -0.7, -0.3, -0.1, 0.1, 0.3, 0.5, 0.7, 0.9, 1.0, 1.1, 1.3, 1.5, 2.0 and $k = 1$, 2, 3, 4, 5.

For samples of size 20 we include simulations for some favorable omnibus tests with Av. powers ≥ 65.5 (Table 5) and for samples of size 50 with Av. powers ≥ 88.5 (Table 6). Note that Meintanis and Iliopoulos [9] did not present simulations of powers for samples of size 50.

The meaning of the headings and the test-statistics can be found in Meintanis and Iliopoulos [9]:

T: Meintanis and Iliopoulos;

BH: Baringhaus and Henze;

HE: Henze;

HM: Henze and Meintanis.

The alternatives considered are:

— the Weibull distribution with density

$$\theta x^{\theta-1}\exp(-x^{\theta}),\ \text{denoted by } W(\theta);$$

— the gamma distribution with density $\Gamma(\theta)^{-1}x^{\theta-1}\exp(-x)$, denoted by $\Gamma(\theta)$;

— the inverse Gaussian law $IG(\theta)$ with density

$$(\theta/2\pi)^{1/2}x^{3/2}\exp\left(-\theta(x-1)^2/2x\right);$$

— the lognormal law $LN(\theta)$ with density

$$(\theta x)^{-1}(2\pi)^{-1/2}\exp\left(-(logx)^2/(2\theta^2)\right);$$

a	Alt.	$T^L_{n,a}$	BH^L	HE^L	HM^L_1	HM^L_2	$T^{MO}_{n,a}$	BH^{MO}	HE^{MO}	HM^{MO}_1	HM^{MO}_2
	$W(1.0)$	97	93	93	86	83	96	94	94	80	83
	$W(3.0)$	40	52	53	45	41	35	47	47	41	16
	$\Gamma(1.5)$	76	71	71	56	57	72	73	72	47	58
	$\Gamma(2.0)$	43	43	43	30	36	37	44	45	23	38
	$IG(0.5)$	98	98	98	96	96	96	98	98	93	96
	$IG(1.5)$	48	63	64	51	61	33	62	64	38	62
	$LN(0.8)$	66	75	75	62	70	55	74	75	50	69
1.0	$LN(1.5)$	100	100	100	99	99	100	100	100	99	99
	$GO(0.5)$	84	70	69	55	47	84	75	72	52	53
	$GO(1.5)$	57	32	30	24	15	60	40	37	25	22
	$PW(1.0)$	43	14	12	23	9	50	24	19	28	22
	$PW(2.0)$	99	91	90	86	64	99	96	94	87	82
	$LF(2.0)$	70	52	51	38	33	70	57	55	35	37
	$LF(4.0)$	57	38	36	26	23	58	43	41	24	27
	$EP(1.0)$	70	47	46	36	26	71	55	52	35	34
	$EP(2.0)$	16	28	29	26	35	13	24	25	25	20
	Av.	66.5	60.4	60.0	52.4	49.7	64.3	62.9	61.9	48.9	51.1
	$W(1.0)$	96	92	91	85	83	96	95	94	84	86
	$W(3.0)$	47	51	50	47	27	43	44	41	35	0
	$\Gamma(1.5)$	75	69	68	59	61	75	75	73	57	63
	$\Gamma(2.0)$	44	43	43	36	41	43	48	47	34	44
	$IG(0.5)$	98	98	98	95	96	98	99	98	96	97
	$IG(1.5)$	57	65	65	60	65	50	67	69	57	69
	$LN(0.8)$	71	75	75	70	74	66	77	78	66	76
2.0	$LN(1.5)$	100	100	100	99	99	100	100	100	99	99
	$GO(0.5)$	80	65	63	52	49	83	75	71	54	56
	$GO(1.5)$	46	25	24	18	15	54	38	34	24	23
	$PW(1.0)$	30	8	7	13	6	37	21	19	26	24
	$PW(2.0)$	98	84	82	75	54	99	95	93	85	84
	$LF(2.0)$	63	47	46	35	36	68	58	54	38	41
	$LF(4.0)$	49	34	32	24	26	54	43	39	27	30
	$EP(1.0)$	61	40	40	30	27	67	54	49	36	35
	$EP(2.0)$	22	30	29	35	24	18	24	24	29	24
	Av.	64.8	57.9	57.1	52.1	48.9	65.7	62.6	61.4	52.9	53.2
	$W(1.0)$	95	90	90	83	81	96	96	96	92	92
	$W(3.0)$	52	46	46	22	0	45	40	37	14	2
	$\Gamma(1.5)$	74	67	67	62	61	77	77	76	70	70
	$\Gamma(2.0)$	45	43	44	42	43	49	49	48	48	46
	$IG(0.5)$	98	98	97	96	95	99	99	98	98	97
	$IG(1.5)$	63	66	67	67	67	62	66	65	70	67
	$LN(0.8)$	75	76	76	74	74	75	77	76	78	76
5.0	$LN(1.5)$	100	100	99	99	99	100	100	100	100	100
	$GO(0.5)$	74	59	59	49	47	80	79	80	68	68

— the Gompertz law $GO(\theta)$ with distribution function

$$1 - \exp\left(\theta^{-1}(1 - e^x)\right);$$

— the power distribution $PW(\theta)$ with density

$$\theta^{-1} x^{(1-\theta)/\theta}, \quad 0 \le x \le 1;$$

	$GO(1.5)$	34	20	20	14	14	46	44	50	32	34
	$PW(1.0)$	13	5	5	4	1	25	25	41	30	33
	$PW(2.0)$	93	73	72	51	41	97	97	98	94	96
	$LF(2.0)$	56	43	43	36	36	65	63	64	50	52
	$LF(4.0)$	41	30	31	26	26	50	48	50	38	39
	$EP(1.0)$	51	34	34	26	25	62	60	63	46	48
	$EP(2.0)$	28	29	29	19	0	21	19	17	17	17
	Av.	62.0	54.9	54.9	48.1	44.4	65.6	64.9	66.2	59.1	58.6
	$W(1.0)$	93	88	82	79	78	96	96	96	96	95
	$W(3.0)$	51	41	41	0	0	44	40	41	37	32
	$\Gamma(1.5)$	72	66	69	59	58	77	77	78	77	75
	$\Gamma(2.0)$	45	43	57	41	41	47	48	54	49	47
	$IG(0.5)$	98	97	94	94	94	99	99	98	99	98
	$IG(1.5)$	65	67	74	65	64	60	62	68	65	64
	$LN(0.8)$	76	75	79	72	72	74	75	77	76	75
10.0	$LN(1.5)$	100	99	97	98	98	100	100	100	100	100
	$GO(0.5)$	69	56	58	44	43	81	80	81	80	78
	$GO(1.5)$	29	18	32	12	12	47	45	51	46	46
	$PW(1.0)$	9	3	5	1	0	26	25	33	29	33
	$PW(2.0)$	88	65	43	37	34	98	97	98	97	98
	$LF(2.0)$	52	41	51	34	33	65	64	67	64	62
	$LF(4.0)$	38	29	44	25	24	50	49	54	49	47
	$EP(1.0)$	45	31	42	23	22	62	61	65	61	60
	$EP(2.0)$	28	25	25	0	0	20	18	18	17	14
	Av.	59.9	52.8	55.8	42.8	42.1	65.4	64.8	67.4	65.1	64.0

Table 4: Source: Meintanis and Iliopoulos [9]. Percentage of rejection for 10 000 Monte Carlo samples of size 20 at significance level $\alpha = 0.05$

— the linear increasing failure rate law $LF(\theta)$ with density

$$(1+\theta x)\exp\left(-x-\theta x^2/2\right);$$

— the exponential-power $EP(\theta)$ law with distribution function

$$1-\exp\Big(1-\exp(x^\theta)\Big).$$

Comments. 1. Here the simplest test we recommend are:

$$\widehat{T}^{\bullet(-1.95,1)}_{n;c_1} = \frac{400}{1600\Gamma(\frac{1}{10}) - 401\Gamma^2\left(\frac{1}{20}\right)} \left[\frac{20}{n}\sum_{i=1}^{n}\left(\frac{X_i^2}{\overline{X_n^2}}\right)^{1/20} - \Gamma\left(\frac{1}{20}\right)\right]^2$$

with Av. powers 66.9 and 88.6 for sample of sizes $n = 20$ and

k	1			2		
Tests	$\widehat{T}_{n;c_1}^{(r,k)}$	$\widehat{T}_{n;c_1}^{\bullet(r,k)}$	$\widehat{T}_{n;c_1}^{\bullet(r,k)}$	$\widehat{T}_{n;c_3}^{(r,k)}$	$\widehat{T}_{n}^{(r,k)}$	$\widehat{T}_{n}^{(r,k)}$
Alt \ r	-5.95	-1.95	-1.9	-5.95	-5.95	-5.75
$W(1.0)$	97	97	97	97	96	96
$W(3.0)$	45	45	47	43	26	31
$\Gamma(1.5)$	77	77	77	74	73	73
$\Gamma(2.0)$	44	43	44	39	42	43
$IG(0.5)$	98	98	98	97	98	98
$IG(1.5)$	50	50	52	38	59	59
$LN(0.8)$	67	68	69	59	70	71
$LN(1.5)$	100	100	100	100	100	100
$GO(0.5)$	84	84	84	84	82	81
$GO(1.5)$	55	56	53	58	58	54
$PW(1.0)$	39	39	36	44	57	55
$PW(2.0)$	99	99	99	99	99	99
$LF(2.0)$	70	70	69	70	68	65
$LF(4.0)$	56	56	55	57	55	52
$EP(1.0)$	69	69	67	70	68	65
$EP(2.0)$	20	20	21	18	15	17
Av.	66.8	66.9	66.7	65.5	66.6	66.2

k	2				3	
Tests	$\widehat{T}_{n}^{\bullet(r,k)}$	$\widehat{T}_{n}^{\bullet(r,k)}$	$\widehat{T}_{n;c_3}^{\bullet(r,k)}$	$\widehat{T}_{n}^{\bullet(r,k)}$	$\widehat{T}_{n}^{(r,k)}$	$\widehat{T}_{n}^{\bullet(r,k)}$
Alt \ r	-1.95	-1.9	-1.9	-1.7	-5.95	-1.9
$W(1.0)$	96	96	97	96	96	96
$W(3.0)$	24	27	45	32	28	30
$\Gamma(1.5)$	73	74	75	73	71	71
$\Gamma(2.0)$	42	42	40	43	39	38
$IG(0.5)$	97	98	97	98	97	97
$IG(1.5)$	59	59	41	59	58	58
$LN(0.8)$	70	70	61	71	68	68
$LN(1.5)$	100	100	100	100	100	100
$GO(0.5)$	82	82	84	80	81	81
$GO(1.5)$	57	57	56	53	57	56
$PW(1.0)$	57	57	41	53	55	54
$PW(2.0)$	99	99	99	99	99	99
$LF(2.0)$	67	67	69	64	67	66
$LF(4.0)$	55	55	56	51	54	53
$EP(1.0)$	68	68	69	64	67	66
$EP(2.0)$	14	16	17	17	16	18
Av.	66.3	66.7	65.6	65.7	65.8	65.7

Table 5: Powers of 5% tests when $n = 20$ based on 100 000 simulations using critical values from 100 000 samples

$$\widehat{T}_{n;c_1}^{(-5.75,1)} = \frac{16n}{128\sqrt{\pi} - 17\Gamma^2\left(\frac{1}{4}\right)} \left[\frac{4}{n}\sum_{i=1}^{n} \frac{X_i^{1/2}}{\overline{X_n^2}} - \Gamma\left(\frac{1}{4}\right)\right]^2$$

k	1		2							
Tests	$\widehat{T}^{\bullet(r,k)}_{n;c_1}$	$\widehat{T}^{\bullet(r,k)}_{n;c_1}$	$\widehat{T}^{(r,k)}_n$	$\widehat{T}^{(r,k)}_n$	$\widehat{T}^{(r,k)}_n$	$\widehat{T}^{\bullet(r,k)}_n$	$\widehat{T}^{\bullet(r,k)}_n$	$\widehat{T}^{\bullet(r,k)}_{n;c_3}$	$\widehat{T}^{\bullet(r,k)}_n$	$\widehat{T}^{\bullet(r,k)}_n$
Alt $\backslash r$	-1.95	-1.9	-5.95	-5.75	-5.5	-1.95	-1.9	-1.9	-1.7	-1.5
$W(1.0)$	100	100	100	100	100	100	100	100	100	100
$W(3.0)$	91	92	84	86	86	84	85	91	86	86
$\Gamma(1.5)$	98	98	97	97	97	97	97	98	97	97
$\Gamma(2.0)$	77	78	76	76	76	76	76	74	75	76
$IG(0.5)$	100	100	100	100	100	100	100	100	100	100
$IG(1.5)$	84	86	93	93	92	93	93	76	92	92
$LN(0.8)$	95	96	97	97	97	97	97	93	97	97
$LN(1.5)$	100	100	100	100	100	100	100	100	100	100
$GO(0.5)$	99	100	99	99	98	99	99	99	99	98
$GO(1.5)$	85	84	86	84	78	87	86	87	83	78
$PW(1.0)$	62	57	88	89	86	88	89	66	89	86
$PW(2.0)$	100	100	100	100	100	100	100	100	100	100
$LF(2.0)$	99	95	94	93	90	94	94	95	93	91
$LF(4.0)$	87	86	85	82	77	85	84	87	81	77
$EP(1.0)$	95	94	94	93	90	94	94	95	92	90
$EP(2.0)$	49	51	58	61	60	57	59	47	61	60
Av.	88.6	88.6	90.6	90.7	89.2	90.7	90.8	88.0	90.3	89.3

k	3						
Tests	$\widehat{T}^{(r,k)}_n$	$\widehat{T}^{(r,k)}_n$	$\widehat{T}^{(r,k)}_n$	$\widehat{T}^{\bullet(r,k)}_n$	$\widehat{T}^{\bullet(r,k)}_n$	$\widehat{T}^{\bullet(r,k)}_n$	$\widehat{T}^{\bullet(r,k)}_n$
Alt $\backslash r$	-5.95	-5.75	-5.5	-1.95	-1.9	-1.7	-1.5
$W(1.0)$	100	100	100	100	100	100	100
$W(3.0)$	84	86	88	84	85	87	88
$\Gamma(1.5)$	97	97	96	97	97	97	96
$\Gamma(2.0)$	73	72	71	73	73	72	71
$IG(0.5)$	100	100	100	100	100	100	100
$IG(1.5)$	93	93	92	93	93	93	92
$LN(0.8)$	97	96	96	97	97	96	96
$LN(1.5)$	100	100	100	100	100	100	100
$GO(0.5)$	99	99	98	99	99	98	98
$GO(1.5)$	86	84	80	86	86	83	80
$PW(1.0)$	84	85	85	84	84	85	85
$PW(2.0)$	100	100	100	100	100	100	100
$LF(2.0)$	94	92	90	94	93	92	90
$LF(4.0)$	84	81	77	84	84	80	77
$EP(1.0)$	94	92	90	94	93	92	90
$EP(2.0)$	53	59	64	52	54	60	63
Av.	89.8	89.8	89.3	89.8	89.8	89.7	89.2

with Av. powers 65.2 and 87.6 for sample sizes $n = 20$ and $n = 50$, respectively.

2. A more powerful test is

$$\widehat{T}^{(-5.75,2)}_n = \frac{1}{\Delta^{(-5.75,2)}} \left[c^{(-5.75,2)}_{n1} \left(\frac{1}{n} \sum_{i=1}^{n} \frac{X_i^{1/2}}{\left(\overline{X_n^2}\right)^{1/4}} e^{-X_i^2/\overline{X_n^2}} - \frac{\Gamma^2\left(\frac{1}{4}\right)}{8\sqrt[4]{2}} \right)^2 \right.$$

k	4					
Tests	$\widehat{T}_n^{(r,k)}$	$\widehat{T}_n^{(r,k)}$	$\widehat{T}_n^{\bullet(r,k)}$	$\widehat{T}_n^{\bullet(r,k)}$	$\widehat{T}_n^{\bullet(r,k)}$	$\widehat{T}_n^{\bullet(r,k)}$
Alt $\backslash r$	-5.95	-5.75	-1.95	-1.9	-1.7	-1.5
$W(1.0)$	100	100	100	100	100	100
$W(3.0)$	80	85	81	82	86	88
$\Gamma(1.5)$	97	96	97	97	96	95
$\Gamma(2.0)$	71	70	71	71	70	68
$IG(0.5)$	100	100	100	100	100	100
$IG(1.5)$	93	93	93	93	93	92
$LN(0.8)$	96	96	96	96	96	96
$LN(1.5)$	100	100	100	100	100	100
$GO(0.5)$	99	99	99	99	98	98
$GO(1.5)$	86	84	86	85	83	80
$PW(1.0)$	81	81	81	81	81	81
$PW(2.0)$	100	100	100	100	100	100
$LF(2.0)$	93	92	93	93	92	90
$LF(4.0)$	84	81	84	83	80	76
$EP(1.0)$	93	92	93	93	92	90
$EP(2.0)$	43	51	43	46	53	58
Av.	88.5	88.8	88.5	88.7	88.9	88.2

Table 6: Powers of 5% tests when $n = 50$ based on 100 000 simulations using critical values from 100 000 samples

$$
\begin{aligned}
&- 2b_{n1}^{(-5.75,2)} \left(\frac{1}{n} \sum_{i=1}^{n} \frac{X_i^{1/2}}{\left(\overline{X_n^2}\right)^{1/4}} e^{-X_i^2 / \overline{X_n^2}} - \frac{\Gamma\left(\frac{1}{4}\right)}{8\sqrt[4]{2}} \right) \\
&\times \left(\frac{1}{\binom{n}{2}} \sum_{i=1}^{n-1} (n-i) \frac{X_{i:n}^{1/2}}{\left(\overline{X_n^2}\right)^{1/4}} - \frac{\Gamma(\frac{1}{4})}{4\sqrt[4]{2}} \right) \\
&+ a_{n1}^{(-5.75,2)} \left(\frac{1}{\binom{n}{2}} \sum_{i=1}^{n-1} (n-i) \frac{X_{i:n}^{1/2}}{\left(\overline{X_n^2}\right)^{1/4}} - \frac{\Gamma(\frac{1}{4})}{4\sqrt[4]{2}} \right)^2 \Bigg],
\end{aligned}
$$

where

$$
\begin{aligned}
a_{n1}^{(-5.75,2)} &= \frac{1}{n} \left[\sqrt{\pi} \frac{1}{6\sqrt{3}} - \frac{73}{\sqrt{2}} \left(\frac{\Gamma(1/4)}{64} \right)^2 \right], \\
b_{n1}^{(-5.75,2)} &= \frac{1}{n} \left[\sqrt{\pi} \left(2^{-9/4} B_{\frac{1}{3}} \left(\frac{5}{4}, \frac{5}{4} \right) + \frac{1}{3\sqrt{3}} \right) - \frac{61}{\sqrt{2}} \left(\frac{\Gamma(1/4)}{32} \right)^2 \right], \\
c_{n1}^{(-5.75,2)} &= \frac{1}{n} \left[\sqrt{\pi} \left(2^{3/4} B_{\frac{1}{3}} \left(\frac{5}{4}, \frac{5}{4} \right) + \frac{2}{3\sqrt{3}} \right) - \frac{65}{\sqrt{2}} \left(\frac{\Gamma(1/4)}{16} \right)^2 \right]
\end{aligned}
$$

$$+\frac{1}{n(n-1)}\left[\sqrt{\pi}\left(\sqrt{2}-2^{3/4}B_{\frac{1}{3}}\left(\frac{5}{4},\frac{5}{4}\right)-\frac{2}{3\sqrt{3}}\right)+\sqrt{2}\left(\frac{\Gamma(1/4)}{4}\right)^2\right].$$

$$\Delta^{(-5.75,2)}=det\begin{bmatrix} a_{n1}^{(-5.75,2)} & b_{n1}^{(-5.75,2)} \\ b_{n1}^{(-5.75,2)} & c_{n1}^{(-5.75,2)} \end{bmatrix}$$

with Av. powers 66.2 and 90.7 for sample sizes $n = 20$ and $n = 50$, respectively.

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