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A UNIQUE COMMON FIXED POINT THEOREM FOR GENERALIZED $\varphi-$TRIPLE ON CONE METRIC SPACES

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Abstract: In the present work, existence of coincidence points and common fixed point theorems for generalized φ-triple on cone metric spaces are proved. These results extends and generalize the results of F. Sabetghadam and H.P. Masiha, see [8] and others.

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1. Introduction and Preliminaries

In 2007 Huang and Zhang [6] have introduced the concept of cone metric spaces and established some fixed point theorems for contractive mappings in these spaces. Subsequently Abbas and Jungck [1] and Abbas and Rhoades [2] have studied common fixed point theorems in cone metric spaces (see also [6], [7] and the references mentioned therein). In [5] Di Bari and Vetro have introduced the concept of φ-map and proved some fixed point theorems generalizing some known results. We define the concept of generalized φ -triple and prove some results about common fixed points for such mappings. Our results generalize some results of Huang and Zhang [6], Di Bari and Vetro[5], Abbas and Jugck [1] and F. Sabetghadam and H.P. Masiha [8].

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In all that follows, E is a real Banach space and θ is the zero element of E. For the mappings $f, g : X \longrightarrow X$,let $C(f, g)$ denotes set of coincidence points of f, g that is $C(f, g) := \{z \in X : fz = gz\}$.

We recall some definitions of cone metric spaces and some of their properties (see [6]).

Definition 1.1. Let E be a real Banach space and P be a subset of E. The set P is called a cone if and only if:

(a) P is closed, nonempty and $P \neq \{0\}$;

(b) $a, b \in R$,$a, b \geq 0$,$x, y \in P \Longrightarrow ax + by \in P$;

(c) $x \in P$ and $-x \in P \Longrightarrow x = 0$.

Definition 1.2. Let P be a cone in a Banach space E define partial ordering $\leq$ with respect to p by $x \leq y$ if and only if $y - x \in P$. We shall write $x < y$ to indicate $x \leq y$ but $x \neq y$ while $x \ll y$ will stand for $y - x \in intP$, where $intP$ denotes the interior of the set P. This Cone P is called an order cone.

Definition 1.3. Let E be a Banach Space and $P \subset E$ be an order cone. The order cone P is called normal if there exists $K > 0$ such that for all $x, y \in E$, $0 \leq x \leq y$ implies $\| x \| \leq K \| y \|$. The least positive number K satisfying the above inequality is called the normal constant of P.

Definition 1.4. Let X be a nonempty set of E. Suppose that the map $d : X \times X \longrightarrow E$ satisfies:

$(d1)$ $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

$(d2)$ $d(x, y) = d(y, x)$ for all $x, y \in X$;

$(d3)$ $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

It is obvious that the cone metric spaces generalize metric spaces.

Definition 1.5. Let (X, d) be a cone metric space.We say that $\{x_n\}$ is:

(i) a Cauchy sequence if for every c in E with $0 \ll c$, there is N such that for all $n, m > N$, $d(x_n, x_m) \ll c$;

(ii) a convergent sequence if for any $0 \ll c$, there is an N such that for all $n > N$, $d(x_n, x) \ll c$, for some fixed x in X. We denote this $x_n \longrightarrow x$ $(n \longrightarrow \infty)$. A cone metric space X is said to be complete if every Cauchy sequence in X is convergent in X.

Definition 1.6. Let $f, g : X \longrightarrow X$. Then the pair (f, g) is said to be(IT)-Commuting at $z \in X$ if $f(g(z)) = g(f(z))$ with $f(z) = g(z)$.

Definition 1.7. Let P be an order cone. A non-decreasing function $\varphi : P \longrightarrow P$ is called a φ-map if:

(φ_1) $\varphi(\theta) = \theta$ and $\theta < \varphi(\omega)$ for, $\omega \in P \setminus \theta$;

(φ_2) $\omega \in IntP$ implies $\omega - \varphi(\omega) \in IntP$;

(φ_3) $\lim_{n\longrightarrow\infty} \varphi^n(\omega) = \theta$ for every $\omega \in P \setminus \theta$.

Definition 1.8. Let P be a cone and let $\{\omega_n\}$ be a sequence in P. One say that $\omega_n \xrightarrow{\ll} \theta$ if for every $\epsilon \in P$ with $\theta \ll \epsilon$, there exists a natural number N such that $\omega_n \ll \epsilon$ for all $n \geq N$.

2. Common Fixed Point Theorem

In this section, we introduce the notation of generalized $\varphi-$ mapping and contractive condition called generalizes $\varphi-$triple. We prove some results on common fixed points of these mappings on cone metric spaces.

We define the following definitions.

Definition 2.1. Let P be a cone. Let $F : P \longrightarrow P$ be a non-decreasing mapping satisfies the following conditions:

(F1) $F(\omega) \geq \theta$ for all $\omega \in p$ and $F(\omega) = \theta$ if and only if $\omega = \theta$,

(F2) for every $\omega_n \in P$, $\omega_n \xrightarrow{\ll} \theta$, if and only if $F(\omega_n) \xrightarrow{\ll} \theta$,

(F3) for every ω_1, $\omega_2 \in P$ for every, $F(\omega_1 + \omega_2) \leq F(\omega_1) + F(\omega_2)$.

Definition 2.2. The self-mappings $f, g, h : X \longrightarrow X$ are called generalized $\varphi-$*triple*, if there exists a φ-mapping and a mapping F satisfying the conditions $(F1), (F2)$ and $(F3)$ such that

$$F(d(fx, gy)) \leq \varphi(F(d(fx, hx)) + F(d(gy, hy))) \quad \text{for every} \quad x, y \in X.$$

Remark 2.3. Replace the condition φ_1 with the following condition: $\varphi_1^{'}$ there exists $k \in [0, 1/2)$ such that $\varphi(\omega) \leq k\omega$ for $\omega \backslash \{\theta\}$ and $\varphi(\theta)$=θ. Then we have the following Theorem for φ-triple.

Theorem 2.4. *Let (X,d) be a cone metric space and let $f, g, h : X \longrightarrow X$ be self -mappings such that*

$$F(d(fx, gy)) \leq \varphi(F(d(fx, hx)) + F(d(gy, hy))) \quad \textit{for every} \quad x, y \in X, \tag{1}$$

where φ is non decreasing mapping from P into P satisfying the conditions $\varphi_1^{'}$, φ_2, φ_3 and $F : P \longrightarrow P$ is non decreasing mapping satisfying the conditions

$(F1)$-$(F3)$. If $f(X) \cup g(X) \subset h(X)$ and $h(X)$ is a complete subspace of X. Then the maps f, g and h have a coincidence point p in X. Moreover if (f,h) and (g,h) are $(IT)-$commuting at p, then f, g and h have a unique common fixed point.

Proof. Suppose x_0 is an arbitrary point of X, and define the sequence $\{y_n\}$ in X such that

$$y_{2n} = fx_{2n+1} = hx_{2n+1} \quad \text{and} \quad y_{2n+1} = gx_{2n+1} = hx_{2n+1},$$

for all $n = 0, 1, 2, \ldots$. By (1), we have

$$\begin{aligned} F(d(y_{2n}, y_{2n+1})) &= F(d(fx_{2n}, gx_{2n+1})) \\ &\leq \varphi(F(d(fx_{2n}, hx_{2n})) + F(d(gx_{2n+1}, hx_{2n+1})) \\ &\leq k(F(d(y_{2n}, y_{2n-1})) + F(d(y_{2n+1}, y_{2n})), \\ (1-k)F(d(y_{2n}, y_{2n+1})) &\leq k(F(d(y_{2n}, y_{2n-1})), \\ F(d(y_{2n}, y_{2n+1})) &\leq \delta(F(d(y_{2n}, y_{2n-1})), \text{ where } \delta = \frac{k}{1-k}. \end{aligned}$$

Similarly, it can be shown that

$$F(d(y_{2n+1}, y_{2n+2})) \leq \delta(F(d(y_{2n}, y_{2n+1})).$$

Therefore for all n:

$$F(d(y_{2n+1}, y_{2n+2})) \leq \delta(F(d(y_n, y_{n+1})) \leq \cdots \leq \delta^{n+1}(F(d(y_0, y_1)).$$

Now for any $m > n$

$$\begin{aligned} F(d(y_n, y_m)) \leq& F(d(y_n, y_{n+1})) + F(d(y_{n+1}, y_{n+2})) + \cdots + F(d(y_{m-1}, y_m)) \\ \leq& [\delta^n + \delta^{n+1} + \cdots + \delta^{m-1}]F(d(y_1, y_0)) \\ \leq& \frac{\delta^n}{1-\delta} F(d(y_1, y_0)). \end{aligned}$$

Then $F(d(y_n, y_m)) \xrightarrow{\ll} \theta$ as $n, m \longrightarrow \theta$. Hence by $(F2)$, $\{y_n\}$ is a cauchy sequence, where $y_n = \{hx_n\}$. Therefore $\{hx_n\}$ is a cauchy sequence. Since $h(X)$ is complete, there exists q in $h(X)$ such that $hx_n \longrightarrow q$ as $n \longrightarrow \infty$. Consequently, we can find p in X such that $h(p) = q$. We shall show that $hp = fp = gp$.

Consider

$$F(d(fp, gx_{2n+1}) \leq \varphi(F(d(fp, hp)) + F(d(gx_{2n+1}, hx_{2n+1})))$$

$$\leq k(F(d(fp,hp)) + F(d(gx_{2n+1},hx_{2n+1})).$$

Letting $n \longrightarrow \infty$, we get

$$F(d(fp,q)) \leq k((F(d(fp,hp)) + F(d(q,q))).$$

That is

$$F(d(fp,q)) < \frac{1}{2}F(d(fp,q).$$

A contradiction. Therefore

$$hp = q = fp. \tag{2}$$

Similarly

$$\begin{aligned} F(d(gp,fx_{2n}) \leq &\varphi(F(d(gp,hp)) + F(d(fx_{2n+},hx_{2n}))) \\ \leq &k(F(d(gp,hp)) + F(d(fx_{2n},hx_{2n})). \end{aligned}$$

Letting $n \longrightarrow \infty$, we get

$$F(d(gp,q)) \leq k((F(d(gp,hp)) < F(d(gp,q))) \quad \text{with } k < 1,$$

A contradiction. Therefore

$$hp = q = gp. \tag{3}$$

From (2) and (3) it follows that

$$q = hp = fp = gp. \tag{4}$$

Therefore, p is a coincidence point of f,g,h.
Since,(f,h), (g,h) are (IT)-commuting at p.

$$\begin{aligned} F(d(ffp,fp)) = F(d(ffp,gp)) \leq &\varphi(F((d(ffp,hfp)) + F((d(gp,hp))) \\ \leq &k((F(d(ffp,hfp)) + F((d(gp,hp))) \\ < &\frac{1}{2}(F(d(ffp,fhp))) \\ = &F(d(ffp,ffp)) < \theta. \end{aligned}$$

A contradiction. Therefore $ffp = fp$, $fp = ffp = fhp = hfp$. This implies

$$ffp = hfp = fp = q. \tag{5}$$

Therefore

$$fp(=q) \quad \text{is a common fixed point of } f \text{ and } h. \tag{6}$$

Similarly we obtain

$$gp = ggp = ghp = hgp \quad \Rightarrow \quad ggp = hgp = gp = q. \tag{7}$$

Therefore

$$gp = fp(= q) \quad \text{is a common fixed point of } g \text{ and } h \tag{8}$$

In view of (7) and (8) it follows that f, g and h have a common fixed point namely q. The uniqueness of the common fixed point of q follows equation (1)and ϕ-triple. Indeed, let q_1 be another common fixed point of f, g and h. Consider,

$$\begin{aligned} d(q, q_1) = F(d(fq, gq_1)) &\leq \phi(F(d(fq, hq) + F(d(gq_1, hq_1))). \\ &\leq k(F(d(fq, hq) + F(d(gq_1, hq_1))). \\ &= k(F(d(hq, hq_1)) + F(d(hq_1, hq_1))) = k(\theta + \theta), \end{aligned}$$

$$\Longrightarrow \quad d(q, q_1) \leq \theta$$

Thus $q = q_1$. Therefore f, g and h have a unique common fixed point. □

If we let the mapping F be the identity mapping in Theorem 2.4, and let the φ -mapping be $\varphi(\omega) = k\omega$ where $k \in [0, \frac{1}{2})$a constant, then we obtain the following corollary.

Corollary 2.5. *Let (X, d) be a cone metric space, and let $f, g, h : X \longrightarrow X$ be self mappings such that such that*

$$d(fx, gy) \leq k(F(d(fx, hx) + F(d(gy, hy)), \quad \textit{for every } x, y \in X, \tag{9}$$

where $k \in [0, \frac{1}{2})$ constant. If $f(X) \cup g(X) \subset h(X)$ and $h(X)$ is a complete subspace of X. Then the maps f, g and h have a coincidence point p in X. Moreover, if (f, h) and (g, h) are $(IT)-$commuting at p, then f, g and h have a unique common fixed point.

Remark 2.6. If we take $g = f$ and $h = g$ in Theorem 2.4, then we obtain Theorem 2.3 in [1]. Also if we let $g = f$, $h = g$ and g is identity map on X in Theorem 2.4, then we obtain Theorem 1 in [6]. It is a extension of Banach Fixed Point Theorem for metric spaces.

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