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ON THE SIMPSON-LIKE TYPE INEQUALITIES FOR TWICE DIFFERENTIABLE (α, m)-CONVEX MAPPINGS

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Abstract: In this article the author establish new results related to the Simpson-like inequalities for twice differentiable (α, m)-convex mappings whose second derivatives in absolute values aroused to the q-th ($q \geq 1$) power are (α, m)-convex. Some applications to special means of positive real numbers are also given.

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1. Introduction

The following definitions are well known in literature. In [15], Gh. Toader defined m-convexity: the mapping $f : [0, b] \to R, b > 0$ is said to be m-convex, where $m \in [0, 1]$, if we have

$$f(tx + m(1-t)y) \leq tf(x) + m(1-t)f(y) \tag{1}$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

The notion of m-convexity has been further generalized in [3] as it is stated in the following definition: the mapping $f : [0, b] \to R$, $b > 0$ is said to be (α, m)-*convex*, where $\alpha, m \in [0, 1]$, if we have

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$$f(tx+m(1-t)y) \leq t^{\alpha} f(x) + m(1-t^{\alpha}) f(y) \tag{2}$$

for all $x, y \in [0, b]$ and $t \in [0, 1]$.

Note that for $(\alpha, m) \in \{(0,0), (\alpha, 0), (1,0), (1,m), (1,1), (\alpha, 1)\}$ one obtains the following classes of mappings: increasing, α-starshaped, starshaped, convex, m-convex, α-convex and (α, m)-convex. For recent refinements, counterparts, generalizations and new Simpson's type inequalities for these classes, see [3, 5, 9, 10, 11].

Denote by $K_m^{\alpha}(I)$ the set of all (α, m)-convex mappings on I.

In the sequel, denote by I^0 the interior of an interval I .

Definition 1.1. The Gamma function Γ, the incomplete Beta function β and the hypergeometric function ${}_2F_1(a, b; c : x)$ are respectively defined by

$$\begin{aligned}
\Gamma(a) &= \int_0^{\infty} e^{-t} t^{a-1} dt, \quad a > 0, \\
\beta(x, a, b) &= \int_0^x t^{a-1}(1-t)^{b-1} dt, \quad a, b > 0, \\
{}_2F_1(a, b; c; x) &= \Sigma_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, 0 < x < 1,
\end{aligned}$$

where the Pochhammer symbol $(z)_n$ of z is defined by

$$(z)_n = \begin{cases} 1 & \text{if } \ n = 0 \\ z(z+1)\cdots(z+n-1) & \text{if } \ n > 0, \end{cases}$$

and c is not $0, -1, -2, \cdots$.

Note that the following properties hold:

$$\Gamma(x+1) = x\Gamma(x), \Gamma(1) = \Gamma(2) = 1, \Gamma(\frac{1}{2}) = \sqrt{\pi}, \beta(1, x, y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}.$$

S. S. Dragomir et. al.[2, 3], Mehmet Zeki Sarikaya et al.[12, 13] and the author [9, 10, 11] proved the following some recent developments on Hermite-Hadamard inequality for which the remainder is expressed in terms of lower derivatives than the twice.

In [6], M. E. Özdemir, M. Avci and H. Kavurmaci discussed the following results related to the right-hand side of the Hermite-Hadamard inequality for twice differentiable (α, m)-convex mappings:

Theorem 1.1. *Let $f : I \subset [0, b^*] \to R$ be a differentiable mapping on I^0 where $a, b \in I$ with $a < b$ and $b^* > 0$. If $\mid f'' \mid^q \in K_m^\alpha[a, b]$ for $(\alpha, m) \in [0, 1]^2$ and $q \geq 1$, then the following inequality holds:*

$$\Big|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)dx\Big| \leq \frac{(mb-a)^2}{2}\big(\frac{1}{6}\big)^{1-\frac{1}{q}}\{\mu_1 \mid f^{''}(a)\mid^q +\nu_1 m \mid f^{''}(b)\mid^q\}^{\frac{1}{q}},$$

where

$$\mu_1 = \frac{1}{(\alpha+2)(\alpha+3)}, \quad \nu_1 = \frac{1}{6} - \frac{1}{(\alpha+2)(\alpha+3)}.$$

Theorem 1.2. *Let $f : I \subset [0, b^*] \to R$ be a differentiable mapping on I^0 where $a, b \in I$ with $a < b$ and $b^* > 0$. If $\mid f'' \mid^q \in K_m^\alpha[a, b]$ for $(\alpha, m) \in [0, 1]^2$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$\Big|\frac{f(a)+f(mb)}{2}-\frac{1}{mb-a}\int_a^{mb} f(x)dx\Big| \leq \frac{(mb-a)^2}{2}\{\mu_2 \mid f^{''}(a)\mid^q +\nu_2 m \mid f^{''}(b)\mid^q\}^{\frac{1}{q}},$$

where

$$\mu_2 = \frac{q}{\alpha+q+1}\beta(\alpha+1, q), \quad \nu_2 = \frac{1}{q+1} - \mu_2.$$

The main purpose of this article is to establish the following new results related to the Simpson-like type inequalities for twice differentiable (α, m)-convex mappings.

2. The Simpson-Like Type Inequalities for Twice Differentiable (α, m)-Convex Mappings

To establish the new results related to the Simpson-like type inequalities based on twice differentiable (α, m)-convex mappings, we need the following lemma:

Lemma 1. *If $f : I \subset [0, b^*] \to R, b^* > 0$ is a twice differentiable mapping on I^0 such that $f^{''} \in L([a,b])$, where $a, b \in I$ with $0 \leq a < b < \infty$, then for $r > 1$ the following equality holds:*

$$\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1} f(\frac{a+b}{2}) - \frac{2}{r(b-a)} \int_a^b f(x)dx$$
$$= (b-a)^2 \int_0^1 k(t) f''(ta+(1-t)b)dt,$$

where

$$k(t) = \begin{cases} \frac{t}{r}(\frac{1}{r+1} - t) & t \in [0, \frac{1}{2}) \\ (1-t)(\frac{t}{r} - \frac{1}{r+1}) & t \in [\frac{1}{2}, 1]. \end{cases}$$

Proof. By the integration by parts, this identity is proved.

Theorem 2.1. *Let $f : I \subset [0, b^*] \to R$ be a twice differentiable mapping on I^0 such that $f^{''} \in L([a,b])$, where $a, b \in I$ with $a < b$ and $b^* > 0$. If $\mid f'' \mid \in K_m^{\alpha}[a,b]$ for $(\alpha, m) \in [0,1]^2$, then the following inequality holds:*

$$\frac{1}{(b-a)^2} \Big| \frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1} f(\frac{a+b}{2}) - \frac{2}{r(b-a)} \int_a^b f(x)dx \Big|$$
$$= \{\mu_{11} + \mu_{12} + \mu_{13} + \mu_{14}\} \mid f''(a) \mid$$
$$+ \{\nu_{11} + \nu_{12} + \nu_{13} + \nu_{14}\} m \mid f'(\frac{b}{m}) \mid \},$$

where

$$\mu_{11} = \frac{1}{r(r+1)^{\alpha+3}(\alpha+2)(\alpha+3)},$$
$$\nu_{11} = \frac{1}{6r(r+1)^3} - \mu_{11},$$
$$\mu_{12} = \frac{2^{\alpha+3} + (r+1)^{\alpha+2}\{(\alpha+2)r - (\alpha+4)\}}{2^{\alpha+3} r(r+1)^{\alpha+3}(\alpha+2)(\alpha+3)},$$
$$\nu_{12} = \frac{r^3 - 3r + 2}{24r(r+1)^3} - \mu_{12},$$
$$\mu_{13} = \frac{r^{\alpha+1}(3+\alpha+2r)}{(r+1)^{\alpha+3}(\alpha+1)(\alpha+2)(\alpha+3)}$$
$$+ \frac{4 + (1-r)\alpha^2 - 14r + (5-7r)\alpha}{2^{\alpha+3} r(r+1)(\alpha+1)(\alpha+2)(\alpha+3)},$$

$$\nu_{13} = \frac{r^3 - 3r + 2}{24r(r+1)^3} - \mu_{13},$$
$$\mu_{14} = \frac{(\alpha + 1 - 2r)(r+1)^{\alpha+2} + (\alpha + 3 + 2r)r^{\alpha+2}}{r(r+1)^{\alpha+3}(\alpha+1)(\alpha+2)(\alpha+3)},$$
$$\nu_{14} = \frac{1}{6r(r+1)^3} - \mu_{14}.$$

Proof. From Lemma 1 and using the (α, m)-convexity of $| f'' |$ we have:

$$\frac{1}{(b-a)^2}\Big|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}f(\frac{a+b}{2}) - \frac{2}{r(b-a)}\int_a^b f(x)dx\Big|$$
$$\leq \int_0^{\frac{1}{2}} \Big|\frac{t}{r}(\frac{1}{r+1} - t)\Big|\Big|f''(ta + (1-t)b)\Big|dt$$
$$+ \int_{\frac{1}{2}}^1 \Big|(1-t)(\frac{t}{r} - \frac{1}{r+1})\Big|\Big|f''(ta + (1-t)b)\Big|dt. \quad (3)$$

Note that

$$\int_0^{\frac{1}{2}} \Big|\frac{t}{r}(\frac{1}{r+1} - t)\Big|\Big|f''(ta + (1-t)b)\Big|dt$$
$$\leq \int_0^{\frac{1}{r+1}} \frac{t}{r}(\frac{1}{r+1} - t)\Big\{t^\alpha \mid f''(a) \mid + m(1-t^\alpha) \mid f''(\frac{b}{m}) \mid \Big\}dt$$
$$+ \int_{\frac{1}{r+1}}^{\frac{1}{2}} \frac{t}{r}(t - \frac{1}{r+1})\Big\{t^\alpha \mid f''(a) \mid + m(1-t^\alpha) \mid f''(\frac{b}{m}) \mid \Big\}dt$$
$$= \big\{\mu_{11} + \mu_{12}\big\} \mid f''(a) \mid + \big\{\nu_{11} + \nu_{12}\big\}m \mid f''(\frac{b}{m}) \mid \quad (4)$$

and

$$\int_{\frac{1}{2}}^1 \Big|(1-t)(\frac{t}{r} - \frac{1}{r+1})\Big|\Big|f''(ta + (1-t)b)\Big|dt$$
$$\leq \int_{\frac{1}{2}}^{\frac{r}{r+1}} (1-t)(\frac{1}{r+1} - \frac{t}{r})\{t^\alpha \mid f''(a) \mid + m(1-t^\alpha) \mid f''(\frac{b}{m}) \mid\}dt$$
$$+ \int_{\frac{r}{r+1}}^1 (1-t)(\frac{t}{r} - \frac{1}{r+1})\{t^\alpha \mid f''(a) \mid + m(1-t^\alpha) \mid f''(\frac{b}{m}) \mid\}dt$$
$$= \big\{\mu_{13} + \mu_{14}\big\} \mid f''(a) \mid + \big\{\nu_{13} + \mu_{14}\big\}m \mid f''(\frac{b}{m}) \mid . \quad (5)$$

By (3),(4) and (5), the assertion in this theorem is proved.

Theorem 2.2. *Let $f : I \subset [0, b^*] \to R$ be a twice differentiable mapping on I^0 such that $f'' \in L([a,b])$, where $a, b \in I$ with $a < b$ and $b^* > 0$. If $| f'' |^q \in K_m^\alpha[a,b]$ for $(\alpha, m) \in [0,1]^2$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$\frac{1}{(b-a)^2}\Big|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}f(\frac{a+b}{2}) - \frac{2}{r(b-a)}\int_a^b f(x)dx\Big|$$
$$= \Big\{\frac{r^3-3r+6}{24r(r+1)^3}\Big\}^{\frac{1}{p}}$$
$$\times\left\{\begin{array}{c} \{(\mu_{11}+\mu_{12}) \mid f''(a) \mid^q + (\nu_{11}+\nu_{12})m \mid f'(\frac{b}{m}) \mid^q\}^{\frac{1}{q}} \\ +\{(\mu_{13}+\mu_{14}) \mid f''(a) \mid^q + (\nu_{13}+\nu_{14})m \mid f'(\frac{b}{m}) \mid^q\}^{\frac{1}{q}} \end{array}\right\}, \tag{6}$$

where μ_{ij} and ν_{ij} are defined as in Theorem 4.

Proof. Suppose that $q > 1$. From Lemma 1 and using the Hölder's integral inequality for $q > 1$, we have

$$\frac{1}{(b-a)^2}\Big|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}f(\frac{a+b}{2}) - \frac{2}{r(b-a)}\int_a^b f(x)dx\Big|$$
$$\leq \Big\{\int_0^{\frac{1}{2}} \Big|\frac{t}{r}(\frac{1}{r+1} - t)\Big|dt\Big\}^{\frac{1}{p}}$$
$$\times \Big\{\int_0^{\frac{1}{2}} \Big|\frac{t}{r}(\frac{1}{r+1} - t)\Big|\Big|f''(ta+(1-t)b)\Big|^q dt\Big\}^{\frac{1}{q}}$$
$$+ \Big\{\int_{\frac{1}{2}}^1 \Big|(1-t)(\frac{t}{r} - \frac{1}{r+1})\Big|dt\Big\}^{\frac{1}{p}}$$
$$\times \Big\{\int_{\frac{1}{2}}^1 \Big|(1-t)(\frac{t}{r} - \frac{1}{r+1})\Big|\Big|f''(ta+(1-t)b)\Big|^q dt\Big\}^{\frac{1}{q}}$$
$$= \Big\{\frac{r^3-3r+6}{24r(r+1)^3}\Big\}^{\frac{1}{p}}$$
$$\times \Big\{\int_0^{\frac{1}{2}} \Big|\frac{t}{r}(\frac{1}{r+1} - t)\Big|\Big|f''(ta+(1-t)b)\Big|^q dt\Big\}^{\frac{1}{q}}$$
$$+ \Big\{\frac{r^3-3r+6}{24r(r+1)^3}\Big\}^{\frac{1}{p}}$$
$$\times \Big\{\int_{\frac{1}{2}}^1 \Big|(1-t)(\frac{t}{r} - \frac{1}{r+1})\Big|\Big|f''(ta+(1-t)b)\Big|^q dt\Big\}^{\frac{1}{q}}. \tag{7}$$

By the (α, m)-convexity of $\mid f'' \mid^q$ we have:

$$\int_0^{\frac{1}{2}} \Big|\frac{t}{r}(\frac{1}{r+1} - t)\Big|\Big|f''(ta + (1-t)b)\Big|^q dt$$
$$\leq (\mu_{11} + \mu_{12}) \mid f''(a) \mid^q + (\nu_{11} + \nu_{12})m \mid f'(\frac{b}{m}) \mid^q, \tag{8}$$

and

$$\int_{\frac{1}{2}}^1 \Big|(1-t)(\frac{t}{r} - \frac{1}{r+1})\Big|\Big|f''(ta + (1-t)b)\Big|^q dt$$
$$\leq (\mu_{13} + \mu_{14}) \mid f''(a) \mid^q + (\nu_{13} + \nu_{14})m \mid f'(\frac{b}{m}) \mid^q . \tag{9}$$

By the inequalities (7),(8) and (9), we get:

$$\frac{1}{(b-a)^2}\Big|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}f(\frac{a+b}{2}) - \frac{2}{r(b-a)}\int_a^b f(x)dx\Big|$$
$$= \Big(\frac{r^3 - 3r + 6}{24r(r+1)^3}\Big)^{\frac{1}{p}}$$
$$\times \left\{ \begin{array}{l} \big\{(\mu_{11} + \mu_{12}) \mid f''(a) \mid^q + (\nu_{11} + \nu_{12})m \mid f'(\frac{b}{m}) \mid^q\big\}^{\frac{1}{q}} \\ + \big\{(\mu_{13} + \mu_{14}) \mid f''(a) \mid^q + (\nu_{13} + \nu_{14})m \mid f'(\frac{b}{m}) \mid^q\big\}^{\frac{1}{q}} \end{array} \right\},$$

which implies that the assertion (6) in this theorem holds.

Corollary 2.3. *In Theorem 4,*
(a) if we choose $r = \alpha = 1$, then we get:

$$\Big|\frac{f(a)+f(b)}{2} + f(\frac{a+b}{2}) - \frac{2}{b-a}\int_a^b f(x)dx\Big|$$
$$\leq \frac{1}{48}(b-a)^2\Big\{ \mid f''(a) \mid + m \mid f''(\frac{b}{m}) \mid \Big\}.$$

(b) if we choose $r = 2, m = \alpha = 1$, then we get:

$$\Big|\frac{f(a)+f(b)}{6} + \frac{2}{3}f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)dx\Big|$$
$$\leq \frac{1}{162}(b-a)^2\Big\{ \mid f''(a) \mid + \mid f''(b) \mid \Big\}.$$

Corollary 2.4. *In Theorem 5,*
(a) if we choose $q > 1$, then we get:

$$\Big|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}f(\frac{a+b}{2}) - \frac{2}{r(b-a)}\int_a^b f(x)dx\Big|$$
$$\leq \Big(\frac{r^3-3r+6}{24r(r+1)^3}\Big)^{\frac{1}{p}}(b-a)^2$$
$$\times \left\{ \begin{array}{c} \{(\mu_{11}+\mu_{12}) \mid f''(a) \mid^q + (\nu_{11}+\nu_{12})m \mid f'(\frac{b}{m}) \mid^q\}^{\frac{1}{q}} \\ +\{(\mu_{13}+\mu_{14}) \mid f''(a) \mid^q + (\nu_{13}+\nu_{14})m \mid f'(\frac{b}{m}) \mid^q\}^{\frac{1}{q}} \end{array} \right\}.$$

(b) if we choose $r = \alpha = 1$ and $q > 1$, then we get:

$$\Big|\frac{f(a)+f(b)}{2} + f(\frac{a+b}{2}) - \frac{2}{b-a}\int_a^b f(x)dx\Big|$$
$$\leq \frac{(b-a)^2}{(3\cdot 2^4)^{\frac{1}{p}}}\Big\{(\frac{1}{192}) \mid f''(a) \mid^q + m(\frac{1}{64}) \mid f''(\frac{b}{m}) \mid^q \Big\}^{\frac{1}{q}}$$
$$+ \Big\{(\frac{1}{64}) \mid f''(a) \mid^q + m(\frac{1}{192}) \mid f''(\frac{b}{m}) \mid^q \Big\}^{\frac{1}{q}}.$$

Theorem 2.5. *Let $f : I \subset [0, b^*] \to R$ be a twice differentiable mapping on I^0 such that $f'' \in L([a,b])$, where $a, b \in I$ with $a < b$ and $b^* > 0$. If $\mid f'' \mid^q \in K_m^\alpha[a,b]$ for $(\alpha, m) \in [0,1]^2$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$\frac{1}{(b-a)^2}\Big|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}f(\frac{a+b}{2}) - \frac{2}{r(b-a)}\int_a^b f(x)dx\Big|$$
$$\leq M_1^{\frac{1}{p}}(r)\Big\{\mu_{21} \mid f''(a) \mid^q + \nu_{21} m \mid f''(\frac{b}{m}) \mid^q \Big\}^{\frac{1}{q}}$$
$$+ M_2^{\frac{1}{p}}(r)\Big\{\mu_{22} \mid f''(a) \mid^q + \nu_{22} m \mid f''(\frac{b}{m}) \mid^q \Big\}^{\frac{1}{q}}$$
$$+ M_3^{\frac{1}{p}}(r)\Big\{\mu_{23} \mid f''(a) \mid^q + \nu_{23} m \mid f''(\frac{b}{m}) \mid^q \Big\}^{\frac{1}{q}}$$
$$+ M_4^{\frac{1}{p}}(r)\Big\{\mu_{24} \mid f''(a) \mid^q + \nu_{24} m \mid f''(\frac{b}{m}) \mid^q \Big\}^{\frac{1}{q}}, \tag{10}$$

where

$$M_1(r) = \frac{\beta(1, \frac{1}{2}, p+1)}{2^{2p+1}r^p(r+1)^{2p+1}},$$

$$M_2(r) = \frac{\beta(1, -\frac{1}{2} - p, p+1) - 4^p \beta(\frac{2}{r+1}, -1-2p, p+1)}{4^{p+1} r^p (r+1)^{2p+1}},$$
$$M_3(r) = \frac{(-1)^{p+1} \beta(\frac{1-r}{2}, p+1, p+1)}{r^p (r+1)^{2p+1}},$$
$$M_4(r) = \frac{\beta(1, p+1, p+1)}{r^p (r+1)^{2p+1}},$$

and

$$\mu_{21} = \frac{1}{(r+1)^{\alpha+1}(\alpha+1)}, \qquad \nu_{21} = \frac{1}{r+1} - \mu_{21},$$
$$\mu_{22} = \frac{(r+1)^{\alpha+1} - 2^{\alpha+1}}{2^{\alpha+1}(r+1)^{\alpha+1}(\alpha+1)}, \quad \nu_{22} = \frac{r-1}{2(r+1)} - \mu_{22},$$
$$\mu_{23} = \frac{2^{\alpha+1} r^{\alpha+1} - (r+1)^{\alpha+1}}{2^{\alpha+1}(r+1)^{\alpha+1}(\alpha+1)}, \quad \nu_{23} = \frac{r-1}{2(r+1)} - \mu_{23},$$
$$\mu_{24} = \frac{(r+1)^{\alpha+1} - r^{\alpha+1}}{(r+1)^{\alpha+1}(\alpha+1)}, \qquad \nu_{24} = \frac{1}{r+1} - \mu_{24}.$$

Proof. From (3) in Lemma 1 and using the Hölder's integral inequality, we have:

$$\Big|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1} f(\frac{a+b}{2}) - \frac{2}{r(b-a)} \int_a^b f(x)dx\Big|$$
$$\leq (b-a)^2 \Big[\Big\{ \int_0^{\frac{1}{r+1}} \Big\{\frac{t}{r}(\frac{1}{r+1} - t)\Big\}^p dt\Big\}^{\frac{1}{p}}$$
$$\times \Big\{ \int_0^{\frac{1}{r+1}} \mid f''(ta+(1-t)b) \mid^q dt\Big\}^{\frac{1}{q}}$$
$$+ \Big\{ \int_{\frac{1}{r+1}}^{\frac{1}{2}} \Big\{\frac{t}{r}(t - \frac{1}{r+1})\Big\}^p dt\Big\}^{\frac{1}{p}}$$
$$\times \Big\{ \int_{\frac{1}{r+1}}^{\frac{1}{2}} \mid f''(ta+(1-t)b) \mid^q dt\Big\}^{\frac{1}{q}}$$
$$+ \Big\{ \int_{\frac{1}{2}}^{\frac{r}{r+1}} \Big\{(1-t)(\frac{1}{r+1} - \frac{t}{r})\Big\}^p dt\Big\}^{\frac{1}{p}}$$
$$\times \Big\{ \int_{\frac{1}{2}}^{\frac{r}{r+1}} \mid f''(ta+(1-t)b) \mid^q dt\Big\}^{\frac{1}{q}}$$
$$+ \Big\{ \int_{\frac{r}{r+1}}^{1} \Big\{(1-t)(\frac{t}{r} - \frac{1}{r+1})\Big\}^p dt\Big\}^{\frac{1}{p}}$$

$$\times \Big\{ \int_{\frac{r}{r+1}}^{1} \mid f''(ta + (1-t)b) \mid^q dt \Big\}^{\frac{1}{q}} \Big]. \tag{11}$$

By the (α, m)-convexity of $\mid f'' \mid^q$, we have:

$$(i) \int_0^{\frac{1}{r+1}} \mid f''(ta + (1-t)b) \mid^q dt$$
$$\leq \mu_{21} \mid f''(a) \mid^q + \nu_{21} m \mid f''(\frac{b}{m}) \mid^q, \tag{12}$$
$$(ii) \int_{\frac{1}{r+1}}^{\frac{1}{2}} \mid f''(ta + (1-t)b) \mid^q dt$$
$$\leq \mu_{22} \mid f''(a) \mid^q + \nu_{22} m \mid f''(\frac{b}{m}) \mid^q, \tag{13}$$
$$(iii) \int_{\frac{1}{2}}^{\frac{r}{r+1}} \mid f''(ta + (1-t)b) \mid^q dt$$
$$\leq \mu_{23} \mid f''(a) \mid^q + \nu_{23} m \mid f''(\frac{b}{m}) \mid^q, \tag{14}$$

$$(iv) \int_{\frac{r}{r+1}}^{1} \mid f''(ta + (1-t)b) \mid^q dt$$
$$\leq \mu_{24} \mid f''(a) \mid^q + \nu_{24} m \mid f''(\frac{b}{m}) \mid^q . \tag{15}$$

By (11)-(15), we get the inequality (10).

Corollary 2.6. *In Theorem 8,*
(a) if we choose $r = \alpha = m = 1$, then we get:

$$\Big|\frac{f(a)+f(b)}{2} + f(\frac{a+b}{2}) - \frac{2}{(b-a)} \int_a^b f(x)dx\Big|$$
$$\leq (b-a)^2 \Big[\Big\{ \frac{\beta(1, \frac{1}{2}, 1+p)}{4^{2p+1}} \Big\}^{\frac{1}{p}} \Big\{ \frac{1}{8} \mid f''(a) \mid^q + \frac{3}{8} \mid f''(b) \mid^q \Big\}^{\frac{1}{q}}$$
$$+ \Big\{ \frac{\beta(1, 1+p, 1+p)}{2^{2p+1}} \Big\}^{\frac{1}{p}} \Big\{ \frac{3}{8} \mid f''(a) \mid^q + \frac{1}{8} \mid f''(b) \mid^q \Big\}^{\frac{1}{q}} \Big].$$

(b) if we choose $r = 2$ and $\alpha = m = 1$, then we get:

$$\Big|\frac{f(a)+f(b)}{6} + \frac{2}{3} f(\frac{a+b}{2}) - \frac{1}{(b-a)} \int_a^b f(x)dx\Big|$$

$$\leq (b-a)^2\Big[\Big\{\frac{\beta(1,\frac{1}{2},1+p)}{2^{3p+1}3^{2p+1}}\Big\}^{\frac{1}{p}}\Big\{\frac{1}{18}\mid f''(a)\mid^q+\frac{5}{18}\mid f''(b)\mid^q\Big\}^{\frac{1}{q}}$$
$$+\Big\{\frac{2^{-2p-2}\beta(1,-\frac{1}{2}-p,1+p)-\beta(\frac{2}{3},-1-2p,1+p)}{2^p3^{2p+1}}\Big\}^{\frac{1}{p}}$$
$$\times\Big\{\frac{5}{72}\mid f''(a)\mid^q+\frac{7}{72}\mid f''(b)\mid^q\Big\}^{\frac{1}{q}}$$
$$+\Big\{\frac{{}_2F_1(p+1,-p;p+2;-\frac{1}{2})}{6^{2p+1}(p+1)}\Big\}^{\frac{1}{p}}\Big\{\frac{7}{72}\mid f''(a)\mid^q+\frac{5}{72}\mid f''(b)\mid^q\Big\}^{\frac{1}{q}}$$

$$+\Big\{\frac{\beta(1,1+p,1+p)}{2^p3^{2p+1}}\Big\}^{\frac{1}{p}}\Big\{\frac{5}{18}\mid f''(a)\mid^q+\frac{1}{18}\mid f''(b)\mid^q\Big\}^{\frac{1}{q}}\Big].$$

Theorem 2.7. *Let $f : I \subset [0,b^*] \to R$ be a differentiable mapping on I^0 such that $f'' \in L([a,b])$, where $a,b \in I$ with $a < b$ and $b^* > 0$. If $\mid f'' \mid^q \in K_m^\alpha[a,b]$ for $(\alpha,m) \in [0,1]^2$ and $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:*

$$\frac{1}{(b-a)^2}\Big|\frac{f(a)+f(b)}{r(r+1)}+\frac{2}{r+1}f(\frac{a+b}{2})-\frac{2}{r(b-a)}\int_a^b f(x)dx\Big|$$
$$\leq\Big\{\frac{1}{(r+1)^{p+1}(p+1)}\Big\}^{\frac{1}{p}}\Big\{\mu_{31}\mid f''(a)\mid^q+m\nu_{31}\mid f''(\frac{b}{m})\mid^q\Big\}^{\frac{1}{p}}$$
$$+\Big\{\frac{(r-1)^p}{2^{p+1}(p+1)(r+1)^{p+1}}\Big\}^{\frac{1}{p}}\Big\{\mu_{32}\mid f''(a)\mid^q+m\nu_{32}\mid f''(\frac{b}{m})\mid^q\Big\}^{\frac{1}{q}}$$
$$+\Big\{\frac{(r-1)^{p+1}}{2^p(p+1)r^{p+1}(r+1)^{p+1}}\Big\}^{\frac{1}{p}}\Big\{\mu_{33}\mid f''(a)\mid^q+m\nu_{33}\mid f''(\frac{b}{m})\mid^q\Big\}^{\frac{1}{q}}$$
$$+\Big\{\frac{1}{(p+1)r^p(r+1)^{p+1}}\Big\}^{\frac{1}{p}}\Big\{\mu_{34}\mid f''(a)\mid^q+m\nu_{34}\mid f''(\frac{b}{m})\mid^q\Big\}^{\frac{1}{q}},\qquad(16)$$

where

$$\mu_{31}=\frac{1}{r^q(r+1)^{\alpha+q+1}(\alpha+q+1)},$$
$$\nu_{31}=\frac{1}{r^q(r+1)^{q+1}(q+1)}-\mu_{31},$$
$$\mu_{32}=\frac{(r+1)^{\alpha+q+1}-2^{\alpha+q+1}}{2^{\alpha+q+1}(\alpha+q+1)r^q(r+1)^{\alpha+q+1}},$$
$$\nu_{32}=\frac{(r+1)^{q+1}-2^{q+1}}{2^{q+1}r^q(r+1)^{q+1}(q+1)}-\mu_{32},$$

$$\mu_{33} = \beta(\frac{r}{1+r}, 1+\alpha, 1+q) - \frac{{}_2F_1(\alpha+1, \alpha+q+1; \alpha+2; -1)}{(\alpha+1)},$$
$$\nu_{33} = \frac{(r+1)^{q+1} - 2^{q+1}}{2^{q+1}(r+1)^{q+1}(q+1)} - \mu_{33},$$
$$\mu_{34} = \beta(1, 1+\alpha, 1+q) - \beta(\frac{r}{1+r}, 1+\alpha, 1+q),$$
$$\nu_{34} = \frac{1}{(r+1)^{q+1}(q+1)} - \mu_{34}.$$

Proof. From (3) in Lemma 1 and using the well-known Hölder's integral inequality, we have:

$$\begin{aligned}
&\frac{1}{(b-a)^2}\Big|\frac{f(a)+f(b)}{r(r+1)} + \frac{2}{r+1}f(\frac{a+b}{2}) - \frac{2}{r(b-a)}\int_a^b f(x)dx\Big| \\
&\le \Big\{\int_0^{\frac{1}{r+1}} (\frac{1}{r+1} - t)^p dt\Big\}^{\frac{1}{p}} \Big\{\int_0^{\frac{1}{r+1}} (\frac{t}{r})^q \mid f''(ta+(1-t)b) \mid^q dt\Big\}^{\frac{1}{p}} \\
&+ \Big\{\int_{\frac{1}{r+1}}^{\frac{1}{2}} (t - \frac{1}{r+1})^p dt\Big\}^{\frac{1}{p}} \Big\{\int_{\frac{1}{r+1}}^{\frac{1}{2}} (\frac{t}{r})^q \mid f''(ta+(1-t)b) \mid^q dt\Big\}^{\frac{1}{q}} \\
&+ \Big\{\int_{\frac{1}{2}}^{\frac{r}{r+1}} (\frac{1}{r+1} - \frac{t}{r})^p dt\Big\}^{\frac{1}{p}} \Big\{\int_{\frac{1}{2}}^{\frac{r}{r+1}} (1-t)^q \mid f''(ta+(1-t)b) \mid dt\Big\}^{\frac{1}{q}} \\
&+ \Big\{\int_{\frac{r}{r+1}}^{1} (\frac{t}{r} - \frac{1}{r+1})^p dt\Big\}^{\frac{1}{p}} \Big\{\int_{\frac{r}{r+1}}^{1} (1-t)^q \mid f''(ta+(1-t)b) \mid^q dt\Big\}^{\frac{1}{q}} \\
&= \Big\{\frac{1}{(p+1)(r+1)^{p+1}}\Big\}^{\frac{1}{p}} \Big\{\int_0^{\frac{1}{r+1}} (\frac{t}{r})^q \mid f''(ta+(1-t)b) \mid^q dt\Big\}^{\frac{1}{p}} \\
&+ \Big\{\frac{(r-1)^{p+1}}{(p+1)2^{p+1}(r+1)^{p+1}}\Big\}^{\frac{1}{p}} \Big\{\int_{\frac{1}{r+1}}^{\frac{1}{2}} (\frac{t}{r})^q \mid f''(ta+(1-t)b) \mid^q dt\Big\}^{\frac{1}{q}} \\
&+ \Big\{\frac{(r-1)^{p+1}}{(p+1)2^p r^{p+1}(r+1)^{p+1}}\Big\}^{\frac{1}{p}} \Big\{\int_{\frac{1}{2}}^{\frac{r}{r+1}} (1-t)^q \mid f''(ta+(1-t)b) \mid^q dt\Big\}^{\frac{1}{q}} \\
&+ \Big\{\frac{1}{(p+1)r^p(r+1)^{p+1}}\Big\}^{\frac{1}{p}} \Big\{\int_{\frac{r}{r+1}}^{1} (1-t)^q \mid f''(ta+(1-t)b) \mid^q dt\Big\}^{\frac{1}{q}} \qquad (17)
\end{aligned}$$

where we have used the fact that $\frac{1}{2} < (\frac{1}{p+1})^{\frac{1}{p}} < 1$.

By the (α, m)-convexity of $\mid f'' \mid^q$, we have:

$$(i) \int_0^{\frac{1}{r+1}} (\frac{t}{r})^q \mid f''(ta+(1-t)b) \mid^q dt$$

$$\leq \mu_{31} \mid f''(a) \mid^q + m\nu_{31} \mid f''(\frac{b}{m}) \mid^q, \tag{18}$$

$$(ii) \int_{\frac{1}{r+1}}^{\frac{1}{2}} (\frac{t}{r})^q \mid f''(ta + (1-t)b) \mid^q dt$$

$$\leq \mu_{32} \mid f''(a) \mid^q + m\nu_{32} \mid f''(\frac{b}{m}) \mid^q, \tag{19}$$

$$(iii) \int_{\frac{1}{2}}^{\frac{r}{r+1}} (1-t)^q \mid f''(ta + (1-t)b) \mid^q dt$$

$$\leq \mu_{33} \mid f''(a) \mid^q + m\nu_{33} \mid f''(\frac{b}{m}) \mid^q, \tag{20}$$

$$(iv) \int_{\frac{r}{r+1}}^{1} (1-t)^q \mid f''(ta + (1-t)b) \mid^q dt$$

$$\leq \mu_{34} \mid f''(a) \mid^q + m\nu_{34} \mid f''(\frac{b}{m}) \mid^q. \tag{21}$$

By (17)-(21), the assertion (16) in this theorem is proved.

Corollary 2.8. *In Theorem 10,*
(a) if we choose $r = m = \alpha = 1$, then we get:

$$\frac{2^{1+\frac{1}{p}}}{(b-a)^2}\Big|\frac{f(a)+f(b)}{2} + f(\frac{a+b}{2}) - \frac{2}{(b-a)}\int_a^b f(x)dx\Big|$$
$$\leq \Big\{\frac{1}{2^{q+2}(q+2)} \mid f''(a) \mid^q + \frac{q+3}{2^{q+1}(q+1)(q+2)} \mid f''(b) \mid^q \Big\}^{\frac{1}{p}}$$
$$+ \Big\{\frac{q+3}{2^{q+2}(q+1)(q+2)} \mid f''(a) \mid^q + \frac{1}{2^{q+2}(q+2)} \mid f''(b) \mid^q \Big\}^{\frac{1}{q}}.$$

(b) if we choose $r = 2$ and $m = \alpha = 1$, then we get:

$$\frac{1}{(b-a)^2}\Big|\frac{f(a)+f(b)}{6} + \frac{2}{3}f(\frac{a+b}{2}) - \frac{1}{b-a}\int_a^b f(x)dx\Big|$$
$$\leq \Big\{\frac{1}{3^{p+1}}\Big\}^{\frac{1}{p}} \{\mu_{41} \mid f''(a) \mid^q + \nu_{41} \mid f''(b) \mid^q\}^{\frac{1}{p}}$$
$$+ \Big\{\frac{1}{6^{p+1}}\Big\}^{\frac{1}{p}} \{\mu_{42} \mid f''(a) \mid^q + \nu_{42} \mid f''(b) \mid^q\}^{\frac{1}{q}}$$
$$+ \Big\{\frac{1}{2^p 6^{p+1}}\Big\}^{\frac{1}{p}} \{\mu_{43} \mid f''(a) \mid^q + \nu_{43} \mid f''(b) \mid^q\}^{\frac{1}{q}}$$

$$+\left\{\frac{1}{3\cdot 6^p}\right\}^{\frac{1}{p}}\{\mu_{44}\mid f''(a)\mid^q+\nu_{44}\mid f''(b)\mid^q\}^{\frac{1}{q}}.$$

where

$$\mu_{41}=\frac{1}{2^q3^{q+2}(q+2)},\qquad \nu_{41}=\frac{1}{2^q3^{q+1}(q+1)}-\mu_{41},$$

$$\mu_{42}=\frac{3^{q+2}-2^{q+2}}{2^{2q+2}3^{q+2}(q+2)},\qquad \nu_{42}=\frac{3^{q+1}-2^{q+1}}{2^q6^{q+1}(q+1)}-\mu_{42},$$

$$\mu_{43}=\frac{3^{q+2}(q+3)-2^{q+2}(2q+5)}{6^{q+2}(q+1)(q+2)},\nu_{43}=\frac{3^{q+1}-2^{q+1}}{6^{q+1}(q+1)}-\mu_{43},$$

$$\mu_{44}=\frac{2q+5}{3^{q+2}(q+1)(q+2)},\qquad \nu_{44}=\frac{1}{3^{q+1}(q+1)}-\mu_{44}.$$

3. Applications to Special Means

Now using the results in Section 2, we give some applications to the following special means of positive real numbers $a, b \in R^+$ with $b \geq a$:

(1) The arithmetic mean: $A(a,b)=\frac{a+b}{2}$.

(2) The generalized logarithmic mean:

$$L_n(a,b)=\left[\frac{b^{n+1}-a^{n+1}}{(b-a)(n+1)}\right]^{\frac{1}{n}},\quad a\neq b.$$

(3) The identric mean:

$$I(a,b)=\begin{cases} a & a=b\\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & a\neq b.\end{cases}$$

(4) The logarithmic mean:

$$L(a,b)=\frac{b-a}{\ln b-\ln a}.$$

(5) The geometric mean:

$$G(a,b)=\sqrt{ab}.$$

(6) The harmonic mean:

$$H(a,b)=\frac{2ab}{a+b}.$$

Proposition 1. *Let $n \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$, $[a, b] \subset [0, b^*]$ and $q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then we have the following inequality:*

$$\begin{aligned} & \mid A(a^n, b^n) + A^n(a, b) - 2L_n^n(a, b) \mid \\ & \leq \frac{(b-a)^2}{48} (\frac{1}{2})^{\frac{1}{q}} \mid n(n-1) \mid \\ & \quad \times \left\{ A^{\frac{1}{q}} \left(a^{(n-2)q}, 3b^{(n-2)q} \right) + A^{\frac{1}{q}} \left(3a^{(n-2)q}, b^{(n-2)q} \right) \right\}. \end{aligned}$$

Proof. The assertion follows from Corollary 6 (a) and Corollary 7 (a) for $m = 1$, $f(x) = x^n$ and n as specified as above.

Proposition 2. *Let $n \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$, $[a, b] \subset [0, b^*]$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then we have the following inequality:*

$$\begin{aligned} & \mid A(a^n, b^n) + A^n(a, b) - 2L_n^n(a, b) \mid \\ & \leq \frac{(b-a)^2}{24} (\frac{1 + 3^{\frac{1}{q}}}{4^{\frac{1}{q}}}) \mid n(n-1) \mid A(a^{n-2}, b^{n-2}). \end{aligned}$$

Proof. The assertion follows from Corollary 7 (a) for $m = 1$, $f(x) = x^n$ for $x \in [a, b] \subset [0, b^*]$ and n as specified as above.

Proposition 3. *Let $n \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$, $[a, b] \subset [0, b^*]$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then we have the following inequality:*

$$\begin{aligned} & \mid \frac{1}{3} A(a^n, b^n) + \frac{2}{3} A^n(a, b) - L_n^n(a, b) \mid \\ & \leq \frac{(b-a)^2}{3 \cdot 2^4} \mid n(n-1) \mid \\ & \quad \times \left[\left\{ (\frac{59}{192}) a^{(n-2)q} + (\frac{133}{192}) b^{(n-2)q} \right\}^{\frac{1}{q}} + \left\{ (\frac{133}{192}) a^{(n-2)q} + (\frac{59}{192}) b^{(n-2)q} \right\}^{\frac{1}{q}} \right]. \end{aligned}$$

Proof. The assertion follows from Corollary 7 (b) for $m = 1$, $f(x) = x^n$ for $x \in [a, b] \subset [0, b^*]$ and n as specified as above.

Proposition 4. *Let $n \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$, $[a, b] \subset [0, b^*]$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then we have the following inequality:*

$$\begin{aligned} & \mid A(a^n, b^n) + A^n(a, b) - 2L_n^n(a, b) \mid \\ & \leq \frac{(b-a)^2}{3^{1+\frac{3}{q}}} 2^{1-\frac{3}{p}} \mid n(n-1) \mid A(a^{n-2}, b^{n-2}). \end{aligned}$$

Proof. The assertion follows from Corollary 7 (b) for $m = 1$, $f(x) = x^n$ for $x \in [a, b] \subset [0, b^*]$ and n as specified as above.

Proposition 5. *Let $[a, b] \subset [0, b^*]$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then we have the following inequality:*

$$
\begin{aligned}
&| \ln I(a,b) - \frac{1}{3}\ln G(a,b) - \frac{2}{3}\ln A(a,b) | \\
&\leq \frac{(b-a)^2}{3\cdot 2^4}\Big[\Big\{(\frac{59}{192})^{\frac{1}{q}} + (\frac{133}{192})^{\frac{1}{q}}\Big\}\frac{2}{a^2} + \Big\{(\frac{133}{192})^{\frac{1}{q}} + (\frac{59}{192})^{\frac{1}{q}}\Big\}\frac{2}{b^2}\Big].
\end{aligned}
$$

Proof. The assertion follows from Corollary 7 (b) for $f(x) = \ln(\frac{1}{x})$.

Proposition 6. *Let $n \in (-\infty, 0) \cup [1, \infty) \setminus \{-1\}$, $m = 1$, $[a, b] \in [0, b^*]$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then we have the following inequality:*

$$
\begin{aligned}
&| \frac{1}{3}A(a^n, b^n) + \frac{2}{3}A^n(a,b) - L_n^n(a,b) | \\
&\leq \frac{(b-a)^2}{3\cdot 2^4} \mid n(n-1) \mid \\
&\quad \times \Big[\Big\{(\frac{59}{192})^{\frac{1}{q}} + (\frac{133}{192})^{\frac{1}{q}}\Big\}a^{n-2} + \Big\{(\frac{133}{192})^{\frac{1}{q}} + (\frac{59}{192})^{\frac{1}{q}}\Big\}b^{n-2}\Big].
\end{aligned}
$$

Proof. The assertion follows from Corollary 7 (a) for $m = 1$, $f(x) = x^n$ for $x \in [a, b] \subset [0, b^*]$ and n as specified as above.

Proposition 7. *Let $[a, b] \subset [0, b^*]$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$. Then we have the following inequality:*

$$
\begin{aligned}
&| H^{-1}(a,b) + A^{-1}(a,b) - 2L^{-1}(a,b) | \\
&\leq (b-a)^2 \Big\{\frac{\sqrt{\pi}p\Gamma(p)}{\Gamma(p+\frac{3}{2})}\Big\}^{\frac{1}{p}} \Big\{\frac{1+3^{\frac{1}{q}}}{2^{4+\frac{1}{q}}}\Big\} A(a^{-3}, b^{-3}).
\end{aligned}
$$

Proof. The assertion follows from Corollary 9 (a) for $f(x) = \frac{1}{x}$ for $x \in [a, b] \subset [0, b^*]$.

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