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THE $\mathcal{L}$-DUAL OF A GENERALIZED MATSUMOTO SPACE

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Abstract: One of the most well studied Finsler metrics are the (α, β)-metrics. Among them, Randers, Kropina and Matsumoto metrics are well known brand names in modern Finsler geometry. Recently a lot of work has been done on the generalized m- Kropina and generalized Matsumoto metrics.

In [4, 5, 7] the $\mathcal{L}$-dual of Randers, Kropina and Matsumoto space were introduced. Recently, in [13], the $\mathcal{L}$-dual of a generalized m-Kropina space has been introduced. In this paper we study the $\mathcal{L}$-dual of a generalized Matsumoto space.

AMS Subject Classification: 53B40, 53C60
Key Words: Finsler space, Cartan space, generalized Matsumoto space, the duality between Finsler and Cartan spaces, Legendre transformation

1. Introduction

The concept of $\mathcal{L}$-duality between Finsler and Cartan spaces was introduced by R. Miron [9] and was intensively studied by others [5, 6, 12]. One of the remarkable results obtained are the concrete $\mathcal{L}$-duals of Randers, Kropina and Matsumoto metrics [4, 5, 7]. However, the importance of $\mathcal{L}$-duality is by for limited to computing the dual of some Finsler fundamental functions.

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In [3], the complicated problem of classifying Randers metrics of constant flag curvature was solved by the means of duality. Other geometrical problems of (α, β)-metrics might be solved in future by considering not the metric itself, but its dual.

Recently, the $\mathcal{L}$-dual of a generalized m-Kropina space was introduced in [13]. A natural question arises: what is the $\mathcal{L}$-dual of a generalized Matsumoto space. In the present paper this is the question we are going to answer.

2. The Legendre Transformation

Let $F^n = (M, F)$ be an n-dimensional Finsler space. The fundamental function F(x, y) is called an (α, β)-metric if F is homogeneous of α and β of degree one, where $\alpha^2 = a(y,y) = a_{ij}y^iy^j,\ y = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$ is Riemannian metric, and $\beta = b_i(x)y^i$ is a 1-form on $\widetilde{TM} = TM - \{0\}$.

A Finsler space with fundamental function:

$$F(x,y) = \alpha(x,y) + \beta(x,y) \tag{1}$$

is called a Randers space, where as the space having the fundamental function:

$$F(x,y) = \frac{\alpha^2(x,y)}{\beta(x,y)} \tag{2}$$

is called a Kropina space [2, 8].

A Finsler space with fundamental function:

$$F(x,y) = \frac{\alpha^2(x,y)}{\alpha(x,y) - \beta(x,y)} \tag{3}$$

is called a Matsumoto space.

The generalized metrics:

$$F(x,y) = \frac{\alpha^{m+1}(x,y)}{\beta^m(x,y)},\ (m \neq 0, -1) \tag{4}$$

and

$$F(x,y) = \frac{\alpha^{m+1}(x,y)}{(\alpha(x,y) - \beta(x,y))^m},\ (m \neq 0, -1) \tag{5}$$

are called generalized Kropina and Matsumoto metrics respectively and the spaces equipped with the corresponding metrics are called generalized m-Kropina and generalized Matsumoto space respectively.

Definition 1. (see [1]) A Cartan space C^n is a pair (M, H) which consists of a real n-dimensional C^∞-manifold M and a Hamiltonian function $H : T^x M \setminus \{0\} \to \Re$, where $(T^m M, \pi^x, M)$ is the cotangent bundle of M such that $H(x, p)$ has the following properties:

1. It is two homogeneous with respect to $p_i \quad (i, j, k, \cdots = 1, 2, \ldots, n)$.
2. The tensor field $g^{ij}(x, p) = \frac{1}{2}\frac{\partial^2 H}{\partial p_i \partial p_j}$ is nondegenerate.

Let $C^n = (M, K)$ be an n-dimensional Cartan space having the fundamental function $K(x, p)$. We also consider Cartan spaces having the metric function of the following forms (see [4]):

$$K(x,p) = \sqrt{a^{ij}(x)p_i p_j} + b^i(x)p_i. \tag{6}$$

or

$$K(x,p) = \frac{a^{ij}p_i p_j}{b^i(x)p_i} \tag{7}$$

or

$$K(x,p) = \frac{a^{ij}p_i p_j}{\sqrt{a^{ij}(x)p_i p_j} - b^i(x)p_i} \tag{8}$$

with $a_{ij}a^{jk} = \delta_i^k$ and we will again call these spaces Randers, Kropina and Matsumoto spaces respectively on the cotangent bundle T^*M.

Definition 2. (see [1]) A regular Lagrangin (Hamiltonian) on a domain $D \subset TM(D^* \subset T^*M)$ is a real smooth function $L : D \to \Re(H : D^* \to \Re)$ such that the matrix with entries

$$g_{ab}(x,y) = \dot{\partial}_a \dot{\partial}_b L(x,y) \ \left(g^{*ab}(x,y) = \dot{\partial}^a \dot{\partial}^b H(x,y)\right)$$

is everywhere nondegenerate on $D(D^*)$, see [4].

A Lagrange (Hamilton) manifold is a pair (M, L) $((M, H))$, where M is a smooth manifold and $L(H)$ is regular Lagrangian (Hamiltonian) on $D(D^*)$.

Example 2.3. (a) Every Finsler space $F^n = (M, F(x, y))$ is a Lagrange manifold with $L = \frac{1}{2}F^2$.

(b) Every Cartan space $C^n = (M, \bar{F}(x, p))$ is a Hamilton manifold with $H = \frac{1}{2}\bar{F}^2$. (Here $\bar{F}$ is positively 1-homogeneous in p_i and the tensor $\bar{g}^{ab} = \frac{1}{2}\dot{\partial}_a\dot{\partial}_b\bar{F}^2$ is nondegenerate).

(c) (M, L) and (M, H) with

$$L(x, y) = \frac{1}{2} a_{ij}(x) y^i y^j + b_i(x) y^i + c(x)$$

and

$$H(x, y) = \frac{1}{2} \bar{a}^{ij}(x) p_i p_j + \bar{b}^i(x) p_i + \bar{c}(x)$$

are Lagrange and Hamilton manifolds respectively. (Here $a_{ij}(x), \bar{a}^{ij}$ are the fundamental tensors of Riemannian manifold, b_i are components of covector field, $\bar{b}^i$ are the components of a vector fields, C and $\bar{C}$ are the smooth functions on M).

Let $L(x, y)$ be a regular Lagrangian on a domain $D \subset TM$ and let $H(x, p)$ be a regular Hamiltonian on a domain $D^* \subset T^*M$. If L is a differential map, we can consider the fiber derivative of L, locally given by the diffeomorphism between the open set $U \subset D$ and $U^* \subset D^*$ (see [10, 11]):

$$\varphi(x, y) = (x^i, \dot{\partial}_a L(x, y)) \tag{9}$$

which is called the Legendre transformation. We can define, in this case, the function $H : U^* \mapsto R$:

$$H(x, y) = p_a y^a - L(x, y). \tag{10}$$

where $y = y^a$ is the solution of the equation:

$$p_a = \dot{\partial}_a L(x, y). \tag{11}$$

In the same manner, the fiber derivative of H is locally given by:

$$\varphi(x, p) = (x^i, \dot{\partial}^a H(x, p)) \tag{12}$$

where φ is a diffeomorphism between the same open sets $U^* \subset D^*$ and $U \subset D$ and we can consider the function $L : U \mapsto R$:

$$L(x, y) = p_a y^a - H(x, p). \tag{13}$$

where $p = (p_a)$ is the solution of the equations:

$$y^a = \dot{\partial}^a H(x, p). \tag{14}$$

The Hamiltonian given by (10) is the Legendre transformation of the Lagrangian L and the Lagrangian given by (13) is called the Legendre transformation of the Hamiltonian H.

If (M, K) is a Cartan space, then (M, H) is a Hamiltonian manifold (see [10, 11]), where $H(x,p) = \frac{1}{2}K^2(x,p)$ is 2-homogenous on a domain of T^*M. So we get the following transformation of H on U :

$$L(x,y) = p_a y^a - H(x,p) = H(x,p). \tag{15}$$

Theorem 3. *The scalar field $L(x,y)$ defined by* (2.15) *is a positively 2-homogeneous regular Lagrangian on U.*

Therefore, we get Finsler metric F of U, so that

$$L = \frac{1}{2}F^2 \tag{16}$$

Thus for the Cartan space (M, K) we always can locally associate a Finsler space (M, F) which will be called the $\mathcal{L}$-dual of a Cartan space $(M, C_{|U^*})$ vice versa, we can associate, locally, a Cartan space to every Finsler space which will be called the $\mathcal{L}$-dual of a Finsler space $(M, F_{|U})$.

3. The $\mathcal{L}$-Dual of a Generalized Matsumoto Space

In this case, we have $F = \frac{\alpha^{m+1}}{(\alpha-\beta)^m} (m \neq 0, -1)$.

We put $\alpha^2 = y_i y^i,\ \beta = b_i y^i,\ \beta^* = b^i p_i,\ p^i = a^{ij} p_j,\ \alpha^{*2} = p_i p^i = a^{ij} p_i p_j$.

We have :

$$p_i = \frac{1}{2}\dot{\partial}_i F^2 = F\left[\frac{(m+1)Fy_i}{\alpha^2} - \frac{mF}{\alpha(\alpha-\beta)}(y_i - b_i\alpha)\right] \tag{17}$$

Contracting (17) by p^i and b^i respectively, we get

$$\alpha^{*2} = F\left[\frac{(m+1)F^3}{\alpha^2} - \frac{mF}{\alpha(\alpha-\beta)}\left(F^2 - \alpha\beta^*\right)\right] \tag{18}$$

$$\beta^* = F\left[\frac{(m+1)F\beta}{\alpha^2} - \frac{mF}{\alpha(\alpha-\beta)}\left(\beta - b^2\alpha\right)\right]. \tag{19}$$

In [12], for a Finsler (α,β)-metric F on a manifold M, there is a positive function $\phi = \phi(s)$ on $(-b_0; b_0)$ with $\phi(0) = 1$ and $F = \alpha\phi(s),\ \ s = \frac{\beta}{\alpha}$, where $\alpha = \sqrt{a_{ij}y^i y^j}$ and $\beta = b_i y^i$ with $||\beta||_x < b_0, \forall x \in M$. ϕ satisfies $\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0, (|s| \leq b_0)$.

A generalized Matsumoto metric is a special (α, β)-metric with $\phi = \frac{1}{(1-s)^m}$. Using Shen's notation [14] $s = \frac{\beta}{\alpha}$, (18) and (19) become

$$\alpha^{*2} = F\left[\frac{(m+1)F}{(1-s)^{2m}} - \frac{m}{(1-s)^{2m+1}} + \frac{m\beta^*}{(1-s)^{m+1}}\right] \tag{20}$$

and

$$\beta^* = F\left[\frac{(m+1)s}{(1-s)^m} - \frac{m}{(1-s)^{m+1}}(s-b^2)\right]. \tag{21}$$

Putting $(1-s)^m = t$, so that $s = 1 - t^{\frac{1}{m}}$ in (20) and (21), we get

$$\alpha^{*2} = (m+1)\frac{F^2}{t^2} - \frac{mF^2}{t^{\frac{2m+1}{m}}} + \frac{m\beta^* F}{t^{\frac{m+1}{m}}} \tag{22}$$

and

$$\beta^* = F\left[\frac{(m+1)\left(1-t^{\frac{1}{m}}\right)}{t} - \frac{m}{t^{\frac{m+1}{m}}}\left(1-t^{\frac{1}{m}}-b^2\right)\right] \tag{23}$$

Now, we have following two cases:

Case I. For $b^2 = 1$, from (23), we get

$$F = \frac{\beta^* t}{(2m+1)-(m+1)t^{\frac{1}{m}}} \tag{24}$$

From (22) and (24), we get

$$(m+1)^2 s^3 + (m^2-1)s^2 + \left\{-(m^2+2m) + (m^2-1)K^2\right\} s - \left\{m^2 - (m^2+1)K^2\right\} = 0 \tag{25}$$

where $K = \frac{\beta^*}{\alpha^*}$

Solving (25) for s, using maple, we get

$$s = \frac{1}{3}\frac{1}{m+1}\left[a_1 - a_2K^2 + 3\sqrt{3}K\sqrt{a_3K^4 - a_4K^2 - a_1}(m+1)\right]^{\frac{1}{3}} - \frac{1}{3(m+1)}\frac{(a_5K^2 - a_6)}{\left[a_1 - a_2K^2 + 3\sqrt{3}K\sqrt{a_3K^4 - a_4K^2 - a_1}(m+1)\right]^{\frac{1}{3}}}$$

$$
\begin{aligned}
&-\frac{1}{3}\left(\frac{m-1}{m+1}\right)\\
&=\frac{1}{3}\frac{1}{m+1}\left\{\sqrt[3]{K_1+(m+1)K_2}-\frac{K_3}{\sqrt[3]{K_1+(m+1)K_2}}-(m-1)\right\}
\end{aligned}
$$

where

$$
\begin{aligned}
&K_1 = a_1 - a_2K^2, \quad K_2 = 3\sqrt{3}K\sqrt{a_3K^4 - a_4K^2 - a_1}, \quad K_3 = a_5K^2 - a_6,\\
&a_1 = 8m^3 + 12m^2 + 6m + 1, \quad a_2 = 9(m^3 + 2m^2 + 2m - 1),\\
&a_3 = m^4 - 2m^3 + 2m - 1,\\
&a_4 = m^4 - 10m^3 - 12m^2 - 4m - 2, \quad a_5 = 3m^2 - 3, \quad a_6 = 4m^2 + 4m + 1.
\end{aligned}
$$

From (24), we get

$$
F=\frac{3\beta^*\left[\frac{2(2m+1)-\sqrt[3]{K_1+(m+1)K_2}+\frac{(a_5K^2-a_6)}{\sqrt[3]{K_1+(m+1)K_2}}}{3(m+1)}\right]^m}{\sqrt[3]{K_1+(m+1)K_2}-\frac{(a_5K^2-a_6)}{\sqrt[3]{K_1+(m+1)K_2}}+(2m+1)}. \tag{26}
$$

From (15) and (16), we get

$$
H(x,p)=\frac{\frac{9}{2}\beta^{*2}\left[\frac{2(2m+1)-\sqrt[3]{K_1+(m+1)K_2}+\frac{(a_5K^2-a_6)}{\sqrt[3]{K_1+(m+1)K_2}}}{3(m+1)}\right]^{2m}}{\left[\sqrt[3]{K_1+(m+1)K_2}-\frac{(a_5K^2-a_6)}{\sqrt[3]{K_1+(m+1)K_2}}+(2m+1)\right]^2}. \tag{27}
$$

Hence we have the following theorem :

Theorem 4. *Let (M,F) be a generalized Matsumoto space and $b = (a_{ij}b^ib^j)^{\frac{1}{2}}$ the Riemannian length of b_i. Then if $b^2 = 1$, the $\mathcal{L}$-dual of (M,F) is the space having the fundamenta function:*

$$
H(x,p)=\frac{\frac{9}{2}\beta^{*2}\left[\frac{2(2m+1)-\sqrt[3]{K_1+(m+1)K_2}+\frac{(a_5K^2-a_6)}{\sqrt[3]{K_1+(m+1)K_2}}}{3(m+1)}\right]^{2m}}{\left[\sqrt[3]{K_1+(m+1)K_2}-\frac{(a_5K^2-a_6)}{\sqrt[3]{K_1+(m+1)K_2}}+(2m+1)\right]^2}. \tag{28}
$$

where

$$K_1 = a_1 - a_2K^2, \quad K_2 = 3\sqrt{3}K\sqrt{a_3K^4 - a_4K^2 - a_1},$$
$$K = \frac{\beta^*}{\alpha^*}, \quad \beta^* = b^i p_i, \quad \alpha^{*2} = a^{ij}p_ip_j,$$

and

$$a_1 = 8m^3 + 12m^2 + 6m + 1, \quad a_2 = 9(m^3 + 2m^2 + 2m - 1),$$
$$a_3 = m^4 - 2m^3 + 2m - 1,$$
$$a_4 = m^4 - 10m^3 - 12m^2 - 4m - 2, \quad a_5 = 3m^2 - 3, \quad a_6 = 4m^2 + 4m + 1.$$

Case 2. For $b^2 \neq 1$, from (23), we have

$$F = \frac{\beta^* t^{\frac{m+1}{m}}}{-(m+1)t^{\frac{2}{m}} + (2m+1)t^{\frac{1}{m}} - m(1-b^2)}. \tag{29}$$

Using (29) in (22), we get

$$s^4 + 2m_1s^3 + (m_2 + m_3K^2)s^2 + 2(m_4 + m_5K^2)s + (m_6 + m_7K^2) = 0, \tag{30}$$

where

$$m_1 = -\frac{1}{m+1}, \quad m_2 = \frac{-4m^2 + 2m^2b^2 + 2mb^2 - 4m + 1}{(m+1)^2}, \quad m_3 = \frac{m-1}{m+1},$$
$$m_4 = \frac{8m^2 + 8m - 8m^2b^2 - 6mb^2}{(m+1)^2}, \quad m_5 = \frac{2}{(m+1)^2},$$
$$m_6 = \frac{-4m^2 - 4m + 4m^2b^2 + 4mb^2 + m^2b^4}{(m+1)^2}, \quad m_7 = -\frac{(1 + m^2b^2)}{(m+1)^2}.$$

Using Farrai method for solving equation (30), we get

$$s = \frac{-(m_1 + \lambda) + \sqrt{(m_1 + \lambda)^2 - 4\upsilon + 4\mu}}{2},$$

where

$$\lambda = \sqrt{\frac{m_{14}^2 - 36m_{13} - 2m_8m_{14} + 3m_1^2}{3m_{14}}}, \quad \nu = \left(\frac{m_{14}^2 - 36m_{13} + m_8m_{14}}{6m_{14}}\right),$$
$$\mu = \frac{\sqrt{\left(m_{14}^2 - 36m_{13} + m_8m_{14}\right)^2 - 36m_{10}m_{14}^2}}{6m_{14}}.$$

Substituting the value of s in (29), we get

$$F = \frac{\beta^* \left(1 - (\frac{-(m_1+\lambda)\pm\sqrt{(m_1+\lambda)^2-4\upsilon+4\mu}}{2})\right)^{m+1}}{m(2-b^2) + (\frac{-(m_1+\lambda)\pm\sqrt{(m_1+\lambda)^2-4\upsilon+4\mu}}{2}) - (m+1)(\frac{-(m_1+\lambda)\pm\sqrt{(m_1+\lambda)^2-4\upsilon+4\mu}}{2})^2} \tag{31}$$

and finally $H(x,p) = \frac{1}{2}F^2$ gives the required $\mathcal{L}$-dual

$$H(x,p) = \frac{1}{2}\times \frac{\beta^{*2} \left(1 - (\frac{-(m_1+\lambda)\pm\sqrt{(m_1+\lambda)^2-4\upsilon+4\mu}}{2})\right)^{2(m+1)}}{\left[m(2-b^2) + (\frac{-(m_1+\lambda)\pm\sqrt{(m_1+\lambda)^2-4\upsilon+4\mu}}{2}) - (m+1)(\frac{-(m_1+\lambda)\pm\sqrt{(m_1+\lambda)^2-4\upsilon+4\mu}}{2})^2\right]^2}. \tag{32}$$

Hence we have the following theorem:

Theorem 5. *Let (M, F) be a generalized Matsumoto space and* $b = (a_{ij}b^ib^j)^{\frac{1}{2}}$ *the Riemannian length of* b_i*. Then if* $b^2 \neq 1$*, the* $\mathcal{L}$*-dual of (M, F) is the space having the fundamenta function:*

$$H(x,p) = \frac{1}{2}\times \frac{\beta^{*2} \left(1 - (\frac{-(m_1+\lambda)\pm\sqrt{(m_1+\lambda)^2-4\upsilon+4\mu}}{2})\right)^{2(m+1)}}{\left[m(2-b^2) + (\frac{-(m_1+\lambda)\pm\sqrt{(m_1+\lambda)^2-4\upsilon+4\mu}}{2}) - (m+1)(\frac{-(m_1+\lambda)\pm\sqrt{(m_1+\lambda)^2-4\upsilon+4\mu}}{2})^2\right]^2}, \tag{33}$$

where

$$K = \frac{\beta^*}{\alpha^*}, \quad \beta^* = b^ip_i, \quad \alpha^{*2} = a^{ij}p_ip_j,$$

$$\lambda = \sqrt{\frac{m_{14}^2 - 36m_{13} - 2m_8m_{14} + 3m_1^2}{3m_{14}}}, \quad \nu = \left(\frac{m_{14}^2 - 36m_{13} + m_8m_{14}}{6m_{14}}\right),$$

$$\mu = \frac{\sqrt{\left(m_{14}^2 - 36m_{13} + m_8m_{14}\right)^2 - 36m_{10}m_{14}^2}}{6m_{14}}, \quad m_1 = \frac{3}{m-1}.$$

$$
\begin{aligned}
m_2 &= \frac{-4m^2+2m^2b^2+2mb^2-4m+1}{(m+1)^2}, \quad m_3 = \frac{m-1}{m+1},\\
m_4 &= \frac{8m^2+8m-8m^2b^2-6mb^2}{(m+1)^2}, \quad m_5 = \frac{2}{(m+1)^2},\\
m_6 &= \frac{-4m^2-4m+4m^2b^2+4mb^2+m^2b^4}{(m+1)^2}, \quad m_7 = -\frac{(1+m^2b^2)}{(m+1)^2}.\\
m_8 &= m_2+m_3K^2, \quad m_9 = m_4+m_5K^2, \quad m_{10} = m_6+m_7K^2,\\
m_{11} &= -18m_8m_1m_9 - 36m_8m_{10} + 54m_1^2m_{10} + 54m_9^2 + m_8^3,\\
m_{12} &= 60m_1m_9m_{10}m_8^2 - 54m_8m_1^3m_9m_{10} + 24m_{10}^2m_8^2 - 3m_{10}m_8^4 + 81m_1^4m_{10}^2\\
&\quad + 3m_9^2m_8^3 + 144m_1m_9m_{10}^2 + 18m_1^2m_9^2m_{10} - 3m_1^2m_9^2m_8^2 - 54m_8m_1m_9^3\\
&\quad - 108m_8m_{10}^2m_1^2 - 108m_8m_{10}m_9^2 + 3m_1^2m_{10}m_8^3 - 48m_{10}^3 + 81m_9^4\\
&\quad + 48m_1^3m_9^3,\\
m_{13} &= \frac{1}{3}m_1m_9 - \frac{1}{3}m_{10} - \frac{1}{36}m_8^2, \quad m_{14} = \sqrt[3]{m_{11}+6\sqrt{m_{12}}}.
\end{aligned}
$$

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