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NEW OSTROWSKI-LIKE TYPE INEQUALITIES FOR DIFFERENTIABLE (s, m)-CONVEX MAPPINGS

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Abstract: In this article a general integral identity for a twice differentiable mapping is derived. By using this result, the author establish some new Hermite-Hadamard-like type inequalities for differentiable s-convex mappings in the second sense.

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1. Introduction

Let $f : I \subseteq [0, \infty) \to R$ be a differentiable mapping on I^0, the interior of the interval I, such that $f' \in L([a, b])$ where $a, b \in I$ with $a < b$. If $\mid f'(x) \mid \leq M$, then the following inequality holds [2]:

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(u)du \right| \leq \frac{M}{b-a} [\frac{(x-a)^2 + (b-x)^2}{2}]. \tag{1}$$

This inequality is well known in the literature as the Ostrowski inequality. For some results which generalize, improve, and extend the above inequality, see [2, 5, 7] and [8], the references therein.

In sequel denote by I^0 the interior of an interval I.

We recall that the notion of s-convex and (s, m)-convex mappings generalized that of convex mappings [1, 4, 9, 10, 11, 14]:

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Definition 1.1. For some fixed $s \in (0,1]$ and $m \in [0,1]$ a mapping $f : I \subset [0,\infty) \to R$ is said to be (s,m)-*convex in the first sense* on I if

$$f(tx + m(1-t)y) \le t^s f(x) + m(1-t^s)f(y)$$

holds, for all $x, y \in I$ and $t \in [0,1]$.

Let us denote the sets of (s,m)-convex mappings in the first sense on $[a,b]$ by $K^1_{s,m}([a,b])$.

Definition 1.2. For some fixed $s \in (0,1]$ and $m \in [0,1]$ a mapping $f : I \subset [0,\infty) \to R$ is said to be (s,m)-*convex in the second sense* on I if

$$f(tx + m(1-t)y) \le t^s f(x) + m(1-t)^s f(y)$$

holds, for all $x, y \in I$ and $t \in [0,1]$.

Let us denote the sets of (s,m)-convex mappings in the second sense on $[a,b]$ by $K^2_{s,m}([a,b])$. For the elementary and further properties of $K^2_{s,m}([a,b])$, see [7, 8].

In Definition 1.2, if we choose $s = 1$, then $f : [a,b] \subset [0,\infty) \to R$ is said to be m-*convex* on $[a,b]$, and if we choose $m = 1$, then $f : [a,b] \subset [0,\infty) \to R$ is said to be s-*convex in the second sense* on $[a,b]$.

In [8], the author proved the following theorem:

Theorem 1.1. *Let $f : I \subseteq [0,\infty) \to R$ be an (s,m)-convex mapping in the first sense on I, where $a, b \in I$ with $0 \le a < b < \infty$. If $f \in L^1([a,b])$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(u)du \le \min\left\{\frac{f(a) + smf(\frac{b}{m})}{s+1}, \frac{f(b) + smf(\frac{a}{m})}{s+1}\right\}.$$

Theorem 1.2. *Let $f : I \subseteq [0,\infty) \to R$ be an (s,m)-convex mapping in the second sense on I, where $a, b \in I$ with $0 \le a < b < \infty$. If $f \in L^1([a,b])$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(u)du \le \min\left\{\frac{f(a) + mf(\frac{b}{m})}{s+1}, \frac{f(b) + mf(\frac{a}{m})}{s+1}\right\}.$$

Havva Kavurmaci et. al. [5] eastablished the following theorems:

Theorem 1.3. *Let $f : I \subseteq [0,\infty) \to R$ be a differentiable mapping on I^0 such that $f' \in L([a,b])$, where $a, b \in I$ with $0 \le a < b < \infty$. If $\mid f' \mid^q$ is*

m-convex on $[a,b]$, for some fixed $m \in (0,1]$ and $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \leq \frac{1}{(p+1)^{\frac{1}{p}}}\left\{\begin{array}{c} \frac{(b-x)^2}{b-a}\left[\min\left\{\frac{|f(b)|^q+m|f(\frac{x}{m})|^q}{2},\frac{|f(x)|^q+m|f(\frac{b}{m})|^q}{2}\right\}\right]^{\frac{1}{q}} \\ +\frac{(x-a)^2}{b-a}\left[\min\left\{\frac{|f(a)|^q+m|f(\frac{x}{m})|^q}{2},\frac{|f(x)|^q+m|f(\frac{a}{m})|^q}{2}\right\}\right]^{\frac{1}{q}} \end{array}\right\} \quad (2)$$

for each $x \in [a,b]$.

Theorem 1.4. *Let $f : I \subseteq [0,\infty) \to R$ be a differentiable mapping on I^0 such that $f' \in L([a,b])$, where $a,b \in I$ with $0 \leq a < b < \infty$. If $| f' |^q$ is m-convex on $[a,b]$, for some fixed $m \in (0,1]$ and $q \geq 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(u)du\right| \leq (b-a)(\frac{1}{2})^{\frac{1}{p}} \times\left\{\begin{array}{c} (\frac{b-x}{b-a})^{\frac{2}{p}}\left\{\frac{1}{3}(\frac{b-x}{b-a})^3 \mid f'(a)\mid^q +m\frac{(b-x)^2(b-3a+2x)}{6(b-a)^3}\mid f'(\frac{b}{m})\mid^q\right\}^{\frac{1}{q}} \\ +(\frac{x-a}{b-a})^{\frac{2}{p}}\left\{(\frac{1}{6}+\frac{(b-x)^2(3a-b-2x)}{6(b-a)^3})\mid f'(a)\mid^q +m\frac{1}{3}(\frac{x-a}{b-a})^3\mid f'(\frac{b}{m})\mid^q\right\}^{\frac{1}{q}} \end{array}\right\}$$

for each $x \in [a,b]$.

In recent years many authors have studied error estimations for Hermite-Hadamard's inequality, Simpson's inequality and Ostrowski's inequality; for refinements, counterparts, generalizations and new inequalities for them, see [1, 2, 3, 4, 5, 7, 8, 9, 10, 11, 13, 14, 15].

To establish new generalizations and refinements of Ostrowski-like type inequalities based on (s,m)-convex mappings in the first sense and in the second sense, we need the following lemma:

Lemma 1. If $f : I \subset [0,\infty) \to R$ is a differentiable mapping such that f' is integrable, where $a,b \in I$ with $a < b$. then the following equality holds:

$$\int_a^b f(x)dx-(b-a)f(ha+(1-h)b) = (b-a)^2\int_0^1 k(t,h)f'(ta+(1-t)b)dt, \quad (3)$$

where

$$k(t,h) = \begin{cases} t & t \in [0,h] \\ t-1 & t \in (h,1]. \end{cases}$$

In this article, a general integral equality for a differentiable mapping such that f' is integrable is derived. By using this result, the author establish some new Ostrowski-like type inequalities for differentiable (s,m)-convex mappings in the first sense and in the second sense.

2. Ostrowski-Like Type Inequalities for (s,m)-Convex Mappings in the Second Sense

Theorem 2.1. *Let $f : I \subset [0,\infty) \to R$ be a differentiable mapping such that f' is integrable on $[a,b]$, where $a, b \in I$ with $a < b$. If $\mid f' \mid$ is an (s,m)-convex mapping in the second sense on $[a,b]$, for some fixed $m \in (0,1]$ and $s \in [0,1]$, then the following inequality holds:*

$$\begin{aligned} &\frac{1}{b-a}\Big|\frac{1}{b-a}\int_a^b f(u)du - f(ha+(1-h)b)\Big| \\ &\le \min\Big\{\mu_1(h,s) \mid f'(a) \mid + \mu_1(1-h,s)m \mid f'(\frac{b}{m}) \mid, \\ &\qquad \mu_1(h,s) \mid f'(b) \mid + \mu_1(1-h,s)m \mid f'(\frac{a}{m}) \mid \Big\}, \end{aligned} \tag{4}$$

where

$$\mu_1(h,s) = \frac{1 - h^{s+1}(s+2-2h(s+1))}{(s+1)(s+2)}$$

for any $h \in [0,1]$.

Proof. From Lemma 1 and by definition of $k(t,h)$, we get:

$$\begin{aligned} &\frac{1}{b-a}\Big|\frac{1}{b-a}\int_a^b f(u)du - f(ha+(1-h)b)\Big| \\ &\le \int_0^h t \mid f'(ta+(1-t)b) \mid dt \\ &\qquad + \int_h^1 (1-t) \mid f'(ta+(1-t)b) \mid dt. \end{aligned} \tag{5}$$

Since $| f' |$ is (s, m)-convex in the second sense on $[a, b]$, we know that, for any $t \in [0, 1]$,

$$| f'(ta + (1-t)b) | \leq t^s | f'(a) | + m(1-t)^s | f'(\frac{b}{m}) | . \tag{6}$$

Hence by (5) and (6), we have

$$\begin{aligned} &\frac{1}{b-a}\Big|\frac{1}{b-a}\int_a^b f(u)du - f(ha+(1-h)b)\Big| \\ &\leq \Big\{\int_0^h t^{s+1}dt + \int_h^1 (1-t)t^s dt\Big\} | f'(a) | \\ &\quad + \Big\{\int_0^h t(1-t)^s dt + \int_h^1 (1-t)^{s+1}dt\Big\} m | f'(\frac{b}{m}) | \\ &= \mu_1(h,s) | f'(a) | + \mu_1(1-h,s) m | f'(\frac{b}{m}) |, \end{aligned} \tag{7}$$

and analogously

$$\begin{aligned} &\frac{1}{b-a}\Big|\frac{1}{b-a}\int_a^b f(u)du - f(ha+(1-h)b)\Big| \\ &\leq \mu_1(h,s) | f'(b) | + \mu_1(1-h,s) m | f'(\frac{a}{m}) | . \end{aligned} \tag{8}$$

By (7) and (8) the assertion in this theorem is proved.

Theorem 2.2. *Let $f : I \subset [0, \infty) \to R$ be a differentiable mapping on I^0 such that f' is integrable on $[a, b]$, where $a, b \in I$ with $a < b$. If $| f' |^q$ is an (s, m)-convex mapping in the second sense on $[a, b]$, for some fixed $m \in (0, 1]$, $s \in [0, 1]$ and $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then the following inequality holds:*

$$\begin{aligned} &\frac{1}{b-a}\Big|\frac{1}{b-a}\int_a^b f(u)du - f(ha+(1-h)b)\Big| \\ &\leq \Big\{\frac{h^2}{(p+1)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}}\Big\} \\ &\quad \times \Big[\min\Big\{\begin{array}{c} | f'(ha+(1-h)b) |^q + m | f'(\frac{b}{m}) |^q, \\ m | f'(\frac{ha+(1-h)b}{m}) |^q + m | f'(b) |^q \end{array}\Big\}\Big]^{\frac{1}{q}} \\ &+ \Big\{\frac{(1-h)^2}{(p+1)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}}\Big\} \\ &\quad \times \Big[\min\Big\{\begin{array}{c} | f'(a) |^q + m | f'(\frac{ha+(1-h)b}{m}) |^q, \\ m | f'(\frac{a}{m}) |^q + m | f'(ha+(1-h)b) |^q \end{array}\Big\}\Big]^{\frac{1}{q}} \end{aligned} \tag{9}$$

for any $h \in [0,1]$.

Proof. From Lemma 1 and using the Hölder inequality, we get:

$$\begin{aligned}
&\frac{1}{b-a}\Big|\frac{1}{b-a}\int_a^b f(u)du - f(ha+(1-h)b)\Big| \\
&\le \Big\{\int_0^h t^p dt\Big\}^{\frac{1}{p}}\Big\{\int_0^h \mid f'(ta+(1-t)b)\mid^q dt\Big\}^{\frac{1}{q}} \\
&\quad + \Big\{\int_h^1 (1-t)^p dt\Big\}^{\frac{1}{p}}\Big\{\int_h^1 \mid f'(ta+(1-t)b)\mid^q dt\Big\}^{\frac{1}{q}} \\
&\le \Big\{\frac{h^{p+1}}{p+1}\Big\}^{\frac{1}{p}}\Big\{\int_0^h \mid f'(ta+(1-t)b)\mid^q dt\Big\}^{\frac{1}{q}} \\
&\quad + \Big\{\frac{(1-h)^{p+1}}{p+1}\Big\}^{\frac{1}{p}}\Big\{\int_h^1 \mid f'(ta+(1-t)b)\mid^q dt\Big\}^{\frac{1}{q}}. \qquad (10)
\end{aligned}$$

By Theorem 1.2, we have

$$\begin{aligned}
&\int_0^h \Big|f'(ta+(1-t)b)\Big|^q dt \\
&= \frac{h}{s+1}\min\left\{\begin{array}{c} \mid f'(ha+(1-h)b)\mid^q + m\mid f'(\frac{b}{m})\mid^q, \\ m\mid f'(\frac{ha+(1-h)b}{m})\mid^q + m\mid f'(b)\mid^q \end{array}\right\} \qquad (11)
\end{aligned}$$

and

$$\begin{aligned}
&\int_h^1 \Big|f'(ta+(1-t)b)\Big|^q dt \\
&= \frac{1-h}{s+1}\min\left\{\begin{array}{c} \mid f'(a)\mid^q + m\mid f'(\frac{ha+(1-h)b}{m})\mid^q, \\ m\mid f'(\frac{a}{m})\mid^q + m\mid f'(ha+(1-h)b)\mid^q \end{array}\right\}. \qquad (12)
\end{aligned}$$

By (10)-(12), the assertion (9) is proved.

Remark 1. In Theorem 2.2, (i) if we choose $s=1$, $h=\frac{b-x}{b-a}$ for $x \in [a,b]$, then we get Theorem 1.4,

Theorem 2.3. *Let $f : I \subset [0,\infty) \to R$ be a differentiable mapping on I^0 such that f' is integrable on $[a,b]$, where $a,b \in I$ with $a<b$. If $\mid f'\mid^q$ is an (s,m)-convex mapping in the second sense on $[a,b]$, for some fixed $m \in (0,1]$, $s \in [0,1]$ and $q \ge 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then, for any $h \in [0,1]$ the following inequality holds:*

$$\frac{1}{b-a}\Big|\frac{1}{b-a}\int_a^b f(u)du - f(ha+(1-h)b)\Big|$$

$$
\begin{aligned}
&\le \left(\frac{h^2}{2}\right)^{\frac{1}{p}} \\
&\quad \times \min\Big[\big\{\mu_2(h,s) \mid f'(a) \mid^q + \nu_2(h,s)m \mid f'(\frac{b}{m}) \mid^q \big\}^{\frac{1}{q}}, \\
&\qquad \big\{\mu_2(h,s)m \mid f'(\frac{a}{m}) \mid^q + \nu_2(h,s) \mid f'(b) \mid^q \big\}^{\frac{1}{q}}\Big] \\
&\quad + \left(\frac{(1-h)^2}{2}\right)^{\frac{1}{p}} \\
&\quad \times \min\Big[\big\{\nu_2(1-h,s) \mid f'(a) \mid^q + \mu_2(1-h,s)m \mid f'(\frac{b}{m}) \mid^q \big\}^{\frac{1}{q}}, \\
&\qquad \big\{\nu_2(1-h,s)m \mid f'(\frac{a}{m}) \mid^q + \mu_2(1-h,s) \mid f'(b) \mid^q \big\}^{\frac{1}{q}}\Big], \qquad (13)
\end{aligned}
$$

where

$$
\begin{aligned}
\mu_2(h,s) &= \frac{h^{s+2}}{s+2}, \\
\nu_2(h,s) &= \frac{1-(1-h)^{s+1}(1+h(s+1))}{(s+1)(s+2)}.
\end{aligned}
$$

Proof. From Lemma 1 and using the power mean inequality, we get:

$$
\begin{aligned}
&\frac{1}{b-a}\left|\frac{1}{b-a}\int_a^b f(u)du - f(ha+(1-h)b)\right| \\
&\le \Big\{\int_0^h t dt\Big\}^{\frac{1}{p}}\Big\{\int_0^h t \mid f'(ta+(1-t)b) \mid^q dt\Big\}^{\frac{1}{q}} \\
&\quad + \Big\{\int_h^1 (1-t)dt\Big\}^{\frac{1}{p}}\Big\{\int_h^1 (1-t) \mid f'(ta+(1-t)b) \mid^q dt\Big\}^{\frac{1}{q}} \\
&\le \left(\frac{h^2}{2}\right)^{\frac{1}{p}}\Big(\int_0^h t \mid f'(ta+(1-t)b) \mid^q dt\Big)^{\frac{1}{q}} \\
&\quad + \left(\frac{(1-h)^2}{2}\right)^{\frac{1}{p}}\Big(\int_h^1 (1-t) \mid f'(ta+(1-t)b) \mid^q dt\Big)^{\frac{1}{q}}. \qquad (14)
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\int_0^h t \mid f'(ta+(1-t)b) \mid^q dt \\
&\le \int_0^h t\big\{t^s \mid f'(a) \mid^q + m(1-t)^s \mid f'(\frac{b}{m}) \mid^q \big\}dt \\
&= \mu_2(h,s) \mid f'(a) \mid^q + \nu_2(h,s)m \mid f'(\frac{b}{m}) \mid^q \qquad (15)
\end{aligned}
$$

and

$$\int_h^1 (1-t) \mid f'(ta+(1-t)b) \mid^q dt$$
$$\leq \nu_2(1-h,s) \mid f'(a) \mid^q + \mu_2(1-h,s)m \mid f'(\frac{b}{m}) \mid^q . \quad (16)$$

By (14)-(16), the assertion (13) is proved.

Remark 2. In Theorem 2.3, (i) if we choose $s = 1$, $h = \frac{b-x}{b-a}$ for $x \in [a,b]$, then we get Theorem 1.4.

3. Ostrowski-Like Type Inequalities for (s,m)-Convex Mappings in the First Sense

By using Lemma 1, we establish new generalizations and refinements of Ostrowski-like type inequalities based on (s,m)-convex mappings in the first sense:

Theorem 3.1. *Let $f : I \subset [0,\infty) \to R$ be a differentiable mapping such that f' is integrable on $[a,b]$, where $a,b \in I$ with $a < b$. If $\mid f' \mid$ is an (s,m)-convex mapping in the first sense on $[a,b]$, for some fixed $m \in (0,1]$ and $s \in [0,1]$, then the following inequality holds:*

$$\frac{1}{b-a}\Big|\frac{1}{b-a}\int_a^b f(u)du - f(ha+(1-h)b)\Big|$$
$$\leq \min\Big\{ \big(\mu_4(h,s)+\nu_4(h,s)\big) \mid f'(a) \mid + \big(\frac{1}{2} - h + h^2 - \mu_4(h,s) - \nu_4(h,s)\big)m \mid f'(\frac{b}{m}) \mid,$$
$$\big(\mu_4(h,s)+\nu_4(h,s)\big)m \mid f'(\frac{a}{m}) \mid + \big(\frac{1}{2} - h + h^2 - \mu_4(h,s) - \nu_4(h,s)\big) \mid f'(b) \mid \Big\}, \quad (17)$$

where

$$\mu_4(h,s) = \frac{h^{s+2}}{s+2},$$
$$\nu_4(h,s) = \frac{1 + h^{s+1}(h-2-(1-h)s)}{(s+1)(s+2)}$$

for any $h \in [0,1]$.

Proof. From Lemma 1 and by definition of $k(t,h)$, we get:

$$\begin{aligned}
&\frac{1}{b-a}\Big|\frac{1}{b-a}\int_a^b f(u)du - f(ha+(1-h)b)\Big| \\
&\le \int_0^h t \mid f'(ta+(1-t)b) \mid dt \\
&\qquad + \int_h^1 (1-t) \mid f'(ta+(1-t)b) \mid dt. \qquad (18)
\end{aligned}$$

Since $\mid f' \mid$ is (s,m)-convex in the first sense on $[a,b]$, we know that, for any $t \in [0,1]$,

$$\mid f'(ta+(1-t)b) \mid \le t^s \mid f'(a) \mid + m(1-t^s) \mid f'(\frac{b}{m}) \mid . \qquad (19)$$

Hence by (6) and (7), we have

$$\begin{aligned}
&\frac{1}{b-a}\Big|\frac{1}{b-a}\int_a^b f(u)du - f(ha+(1-h)b)\Big| \\
&\le \Big\{\int_0^h t^{s+1}dt + \int_h^1 (1-t)t^s dt\Big\} \mid f'(a) \mid \\
&\quad + \Big\{\int_0^h t(1-t^s)dt + \int_h^1 (1-t)(1-t^s)dt\Big\} m \mid f'(\frac{b}{m}) \mid \\
&= \big(\mu_4(h,s) + \nu_4(h,s)\big) \mid f'(a) \mid \\
&\qquad + \big(\frac{1}{2} - h + h^2 - \mu_4(h,s) - \nu_4(h,s)\big) m \mid f'(\frac{b}{m}) \mid, \qquad (20)
\end{aligned}$$

and analogously

$$\begin{aligned}
&\frac{1}{b-a}\Big|\frac{1}{b-a}\int_a^b f(u)du - f(ha+(1-h)b)\Big| \\
&\le \big(\mu_4(h,s) + \nu_4(h,s)\big) m \mid f'(\frac{a}{m}) \mid \\
&\qquad + \big(\frac{1}{2} - h + h^2 - \mu_4(h,s) - \nu_4(h,s)\big) \mid f'(b) \mid . \qquad (21)
\end{aligned}$$

By (18), (20) and (21), the assertion (17) in this theorem is proved.

Theorem 3.2. *Let $f : I \subset [0,\infty) \to R$ be a differentiable mapping on I^0 such that f' is integrable on $[a,b]$, where $a,b \in I$ with $a < b$. If $\mid f' \mid^q$ is* *an (s,m)-convex mapping in the first sense on $[a,b]$, for some fixed $m \in (0,1]$, $s \in [0,1]$ and $q > 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:*

$$\begin{aligned}
&\frac{1}{b-a}\Big|\frac{1}{b-a}\int_a^b f(u)du - f(ha+(1-h)b)\Big| \\
&\le \Big\{\frac{h^{\frac{p+1}{p}}}{(p+1)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}}\Big\} \\
&\quad\times\Big[\min\Big\{\begin{array}{c} \mid f'(ha+(1-h)b)\mid^q + sm \mid f'(\frac{b}{m})\mid^q, \\ sm\mid f'(\frac{ha+(1-h)b}{m})\mid^q + m\mid f'(b)\mid^q \end{array}\Big\}\Big]^{\frac{1}{q}} \\
&\quad+\Big\{\frac{(1-h)^{\frac{p+1}{p}}}{(p+1)^{\frac{1}{p}}(s+1)^{\frac{1}{q}}}\Big\} \\
&\quad\times\Big[\min\Big\{\begin{array}{c} \mid f'(a)\mid^q + sm\mid f'(\frac{ha+(1-h)b}{m})\mid^q, \\ sm\mid f'(\frac{a}{m})\mid^q + m\mid f'(ha+(1-h)b)\mid^q \end{array}\Big\}\Big]^{\frac{1}{q}}
\end{aligned} \tag{22}$$

for any $h \in [0,1]$.

Proof. From Lemma 1 and using the Hölder inequality, we get:

$$\begin{aligned}
&\frac{1}{b-a}\Big|\frac{1}{b-a}\int_a^b f(u)du - f(ha+(1-h)b)\Big| \\
&\le \Big\{\int_0^h t^p dt\Big\}^{\frac{1}{p}}\Big\{\int_0^h \mid f'(ta+(1-t)b)\mid^q dt\Big\}^{\frac{1}{q}} \\
&\quad+\Big\{\int_h^1 (1-t)^p dt\Big\}^{\frac{1}{p}}\Big\{\int_h^1 \mid f'(ta+(1-t)b)\mid^q dt\Big\}^{\frac{1}{q}} \\
&=\Big\{\frac{h^{p+1}}{p+1}\Big\}^{\frac{1}{p}}\Big\{\int_0^h \mid f'(ta+(1-t)b)\mid^q dt\Big\}^{\frac{1}{q}} \\
&\quad+\Big\{\frac{(1-h)^{p+1}}{p+1}\Big\}^{\frac{1}{p}}\Big\{\int_h^1 \mid f'(ta+(1-t)b)\mid^q dt\Big\}^{\frac{1}{q}}.
\end{aligned} \tag{23}$$

By Theorem 1.1, we have

$$\begin{aligned}
&\int_0^h \Big|f'(ta+(1-t)b)\Big|^q dt \\
&= \frac{h}{s+1}\min\Big\{\begin{array}{c} \mid f'(ha+(1-h)b)\mid^q + sm\mid f'(\frac{b}{m})\mid^q, \\ sm\mid f'(\frac{ha+(1-h)b}{m})\mid^q + m\mid f'(b)\mid^q \end{array}\Big\},
\end{aligned} \tag{24}$$

and

$$\int_h^1 \Big|f'(ta+(1-t)b)\Big|^q dt$$

$$= \frac{1-h}{s+1} \min\left\{ \begin{array}{c} | f'(a) |^q + sm | f'(\frac{ha+(1-h)b}{m}) |^q, \\ sm | f'(\frac{a}{m}) |^q + m | f'(ha+(1-h)b) |^q \end{array} \right\}. \tag{25}$$

By (23)-(25), the assertion (22) is proved.

Theorem 3.3. *Let $f : I \subset [0,\infty) \to R$ be a differentiable mapping on I^0 such that f' is integrable on $[a,b]$, where $a,b \in I$ with $a < b$. If $| f' |^q$ is an (s,m)-convex mapping in the first sense on $[a,b]$, for some fixed $m \in (0,1]$, $s \in [0,1]$ and $q \geq 1$ with $\frac{1}{p}+\frac{1}{q}=1$, then, under the notations of μ_{ij} in Theorem 3.1, for any $h \in [0,1]$ the following inequality holds:*

$$\begin{aligned}
&\frac{1}{b-a}\Big|\frac{1}{b-a}\int_a^b f(u)du - f(ha+(1-h)b)\Big| \\
&\leq \big(\frac{h^2}{2}\big)^{\frac{1}{p}} \\
&\quad\times\Big[\min\big\{\mu_4(h,s) \mid f'(a) \mid^q + \big(\frac{h^2}{2} - \mu_4(h,s)\big)m \mid f'(\frac{b}{m}) \mid^q, \\
&\qquad \mu_4(h,s)m \mid f'(\frac{a}{m}) \mid^q + \big(\frac{h^2}{2} - \mu_4(h,s)\big)m \mid f'(b) \mid^q \big\}\Big]^{\frac{1}{q}} \\
&\quad+\big(\frac{(1-h)^2}{2}\big)^{\frac{1}{p}} \\
&\quad\times\Big[\min\big\{\nu_4(h,s) \mid f'(a) \mid^q + \big(\frac{(1-h)^2}{2} - \nu_4(h,s)\big)m \mid f'(\frac{b}{m}) \mid^q, \\
&\qquad \nu_4(h,s)m \mid f'(\frac{a}{m}) \mid^q + \big(\frac{(1-h)^2}{2} - \nu_4(h,s)\big)m \mid f'(b) \mid^q \big\}\Big]^{\frac{1}{q}}.
\end{aligned} \tag{26}$$

Proof. From Lemma 1 and using the power mean inequality, we get:

$$\begin{aligned}
&\frac{1}{b-a}\Big|\frac{1}{b-a}\int_a^b f(u)du - f(ha+(1-h)b)\Big| \\
&\leq \Big\{\int_0^h tdt\Big\}^{\frac{1}{p}}\Big\{\int_0^h t \mid f'(ta+(1-t)b) \mid^q dt\Big\}^{\frac{1}{q}} \\
&\quad+\Big\{\int_h^1 (1-t)dt\Big\}^{\frac{1}{p}}\Big\{\int_h^1 (1-t) \mid f'(ta+(1-t)b) \mid^q dt\Big\}^{\frac{1}{q}} \\
&\leq \big(\frac{h^2}{2}\big)^{\frac{1}{p}}\Big(\int_0^h t \mid f'(ta+(1-t)b) \mid^q dt\Big)^{\frac{1}{q}} \\
&\quad+\big(\frac{(1-h)^2}{2}\big)^{\frac{1}{p}}\Big(\int_h^1 (1-t) \mid f'(ta+(1-t)b) \mid^q dt\Big)^{\frac{1}{q}}.
\end{aligned} \tag{27}$$

Note that

$$\begin{aligned}&\int_0^h t \mid f'(ta+(1-t)b) \mid^q dt\\&\le \int_0^h t\big\{t^s \mid f'(a) \mid^q + m(1-t^s) \mid f'(\frac{b}{m}) \mid^q \big\}dt\\&= \mu_4(h,s) \mid f'(a) \mid^q + \big(\frac{h^2}{2} - \mu_4(h,s)\big)m \mid f'(\frac{b}{m}) \mid^q \qquad (28)\end{aligned}$$

and

$$\begin{aligned}&\int_h^1 (1-t) \mid f'(ta+(1-t)b) \mid^q dt\\&\le \nu_4(h,s) \mid f'(a) \mid^q + \big(\frac{(1-h)^2}{2} - \nu_4(h,s)\big)m \mid f'(\frac{b}{m}) \mid^q \qquad (29)\end{aligned}$$

By (27)-(29), the assertion (26) in this theorem is proved.

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