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CURVES WITH LOW GONALITY AND MAXIMAL RANK

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Abstract: For every integer $n \geq 2$ set $\gamma_n := (n^2 - n)/2$. Set $g_2 := 1$, $g_3 := 4$ and $g_n := \gamma_n + n - 2 + g_{n-2}$ for all $n \geq 4$. We prove the following result. Fix integers n, g, k such that $k \geq n \geq 3$. Fix any integer g such that $2k + 1 \leq g \leq g_n$. Let X be a general k-gonal curve of genus g and L a general element of $\mathrm{Pic}^{g+n}(X)$. Then L is normally generated, i.e. $h_L(X) \subset \mathbb{P}^n$ is projectively normal.

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1. k-Gonal Projectively Normal Curves

For every integer $n \geq 2$ set $\gamma_n := (n^2 - n)/2$. Set $g_2 := 1$ and $g_3 := 4$. Define inductively the integer g_n, $n \geq 4$, by the formula $g_n := \gamma_n + n - 2 + g_{n-2}$. Fix any integer $n \geq 3$ and any smooth curve X of genus g. A general $L \in \mathrm{Pic}^{g+n}(X)$ is very ample and $h^1(X, L) = 0$. For any very ample $A \in \mathrm{Pic}(X)$ let $h_A : X \to \mathbb{P}^n$, $n := h^0(X, A) - 1$, denote the embedding associated to the complete linear system $|L|$. For all integers n, g, k let $\mathcal{H}(g, n, k)$ denote the set of all smooth and linearly curves $C \subset \mathbb{P}^n$ with $h^1(C, \mathcal{O}_C(1)) = 0$, $h^1(C, \mathcal{O}_C(1)) = 0$, genus g and gonality $\leq k$. For all $g \geq 0$ the set $\mathcal{H}(g, n, k)$ is irreducible.

In this note we prove the following result.

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Theorem 1. *Fix integers n, g, k such that $k \geq n \geq 3$. Fix any integer g such that $2k+1 \leq g \leq g_n$. Let X be a general k-gonal curve of genus g and L a general element of $\mathrm{Pic}^{g+n}(X)$. Then $h_L(X)$ has maximal rank, i.e. for for each $t \in \mathbb{N}$ either $h^1(\mathbb{P}^n, \mathcal{I}_{h_L(X)}(t)) = 0$ or $h^0(\mathbb{P}^n, \mathcal{I}_{h_L(X)}(t)) = 0$.*

In the set-up of Theorem 1 we have $h^1(\mathbb{P}^n, \mathcal{I}_{h_L(X)}(t)) = 0$ for all $t \geq 3$. For general curves of genus g see [2], [3]. The range $g \leq \gamma_n$ is true by [1], Theorem 2. Only the case $g > \gamma_n$ is new, as far as we know. Fix an integer $n \geq 3$ and take any integer $k \in \{2, \dots, n\}$. Let C be a smooth curve of genus g with a base point free $R \in \mathrm{Pic}^k(C)$ with $h^0(C, R) = 2$. Fix any very ample $L \in \mathrm{Pic}^{g+n}(C)$ with $h^1(C, L) = 0$. It is easy to show that $h_L(C)$ is contained in a k-dimensional scroll over $\mathbb{P}^1$. Hence if $g \gg n$, then $h_L(C)$ has not maximal rank. For very low k we may even compute the possible postulations of the curves $h_L(C)$. As an example we compute the cases $k = 2$ and $n = 3, 4$.

Remark 1. For non-linearly normal curves the obstructions coming from the gonality are less severe and disappears for very non-linearly normal embeddings (see [4]). For low gonality it boils down to the postulation of general linear projections of minimal degree scrolls (e.g. minimal degree surfaces for hyperelliptic curves) See [**?**] for some cases and Example 1 on how to use it..

Proof of Theorem 1. Since the closure in $\mathcal{M}_g$ of all smooth curves with gonality k contains all curves with gonality $< k$, it is sufficient to do the case $k = n$. The case $g \leq \gamma_n$ is true by [1], Theorem 2. Hence we may assume $\gamma_n < g \leq g_n$. Let $C \subset \mathbb{P}^n$ be a general non-special, n-gonal and linearly normal curve of genus γ_n. We have $\deg(C) = n + \gamma_n$. By [1], Theorem 2, we have $h^i(\mathcal{I}_C(2)) = 0$, $i = 0, 1$. It is for this part that we use that $k \geq n$ (in the second step we will only use that $k \geq n-1$). Let $R \in \mathrm{Pic}^k(C)$ the line bundle computing the gonality of C. Fix $g - \gamma_n$ general $D_i \in |L|$. For each i fix $S_i \subset D_i$ such that $\sharp(S_i) = 2$. Let $\langle S_i \rangle$ denote the line spanned by S_i. For general S_i the nodal curve $E := C \cup \bigcup_{i \geq 1} \langle S_i \rangle$ has arithmetic genus g. As in [1], proof of Theorem 2, we see that $E \in \overline{\mathcal{H}(g, n, n)}$. Since $h^0(\mathbb{P}^n, \mathcal{I}_C(2)) = 0$ and $E \supset C$, we have $h^0(\mathbb{P}^n, \mathcal{I}_C(2)) = 0$. By semicontinuity we have $h^0(\mathbb{P}^n, \mathcal{I}_X(2)) = 0$ for a general $X \in \mathcal{H}(n, g, n)$. Hence to prove Theorem 2 for the integers n and g it is sufficient to prove $h^1(\mathbb{P}^n, \mathcal{I}_X(t)) = 0$ for a general $X \in \mathcal{H}(n, g, n)$ and every $t \geq 3$. By Castelnuovo-Mumford's lemma it is sufficient to prove $h^1(\mathbb{P}^n, \mathcal{I}_X(3)) = 0$.

(a) For a general $L \in \mathrm{Pic}^{n+\gamma-n}(C)$ we have $h^1(C, L \otimes R^*) = 0$, because $\deg(L \otimes R^*) \geq \gamma_n - 1$. Fix a general $D \in |R|$. Since $h^1(C, L \otimes R^*) = 0$, D is formed by n distinct points spanning a hyperplane of $\mathbb{P}^{n-1}$. Fix $S \subset D$

such that $\sharp(S) = n-1$ and call V the linear span of S. Since D is linearly independent, V has codimension 2. We have $S = V \cap C$ (scheme-theoretic intersection). The proof of [?], Claim at top of page 247, says that the restriction map $\rho' : H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(3)) \to H^0(C \cup V, \mathcal{O}_{C\cup V}(3))$ is surjective. Since S is linearly independent, the restriction map $H^0(C, \mathcal{O}_C(3)) \to H^0(S, \mathcal{O}_S(3))$ is surjective. From the Mayer-Vietoris exact sequence

$$0 \to \mathcal{O}_{C\cup V}(3) \to \mathcal{O}_C(3) \oplus \mathcal{O}_V(3) \to \mathcal{O}_S(3) \to 0$$

we get the surjectivity of the restriction map

$$H^0(C \cup V, \mathcal{O}_{C\cup V}(3)) \to H^0(V, \mathcal{O}_V(3)).$$

(b) Assume $n = 3$. We have $g_3 = 4 = \gamma_3 + 1$. In this case V is a secant line to V. Since ρ' is surjective, we have $h^1(\mathcal{I}_{C\cup V}(3)) = 0$. As in [1], Proof of Theorem 2, we have $C \cup V \in \mathcal{H}(3, 4, 3)$.

(c) In this step we assume $g \geq \gamma_n + n - 2$. Assume for the moment $n \geq 5$. Since $g - \gamma_n \leq \gamma_{n-2} + n - 2$, [1], Theorem 2, applied to $V \cong \mathbb{P}^{n-2}$ gives that a general element $Y \in \mathcal{H}(n-2, g-\gamma_n-n-2, n-2)$ satisfies $h^1(V, \mathcal{I}_Y(3)) = 0$. Since $\mathrm{Aut}(V)$ is transitive on the set of all linearly independent subsets of V with cardinality $n-1$, we may find a curve Y such that $h^1(V, \mathcal{I}_{Y,V}(3)) = 0$ and $S \subset Y$. Notice that $C \cup Y$ is a nodal curve with arithmetic genus g and degree $g+n$. A Mayer-Vietoris exact sequence gives $h^1(C \cup Y, \mathcal{O}_{C\cup Y}(1)) = 0$. Let $N_{Y,V}$ and N_Y denote the normal bundle of Y in V and $\mathbb{P}^n$, respectively. We have $h^1(Y, N_{Y,V}) = 0$ and $N_Y \cong N_{Y,V} \oplus \mathcal{O}_Y(1)^{\oplus 2}$. Since $h^1(C, \mathcal{O}_C(1)) = 0$, we have $h^1(N_C) = 1$. Since $h^1(Y, \mathcal{O}_Y(1)) = 0$, the Euler sequence of $T\mathbb{P}^n$ shows that N_Y is a quotient of $\mathcal{O}_C(1)^{\oplus(n+1)}$. Hence the restriction map $H^0(Y, N_Y) \to H^0(S, N_Y|S)$ is surjective. Hence $C \cup Y$ is smoothable inside $\mathbb{P}^n$ (see [7]). As in [1], Proof of Theorem 2, using the admissible coverings we see that $C \cup Y \in \mathcal{H}(n, g, n)$.

Now assume $n = 4$, i.e. assume $\dim(V) = 2$. We have $\gamma_4 = 6$ and $g_4 = 8$. In this case the previous proof works taking as Y a smooth elliptic curve.

(d) Now assume $\gamma_n < g < \gamma_n + n - 2$. We take as Y a general smooth rational curve of V containing $g - \gamma_n + 1$ points of S and of degree $g - \gamma_n$. Notice that Y is a rational normal curve in its linear span $\langle Y \rangle$. Hence $h^1(\langle Y \rangle, \mathcal{I}_Y(t)) = 0$ for all $t > 0$. Since V is spanned by S and $\sharp(S \setminus S \cap Y) = \dim(V) - \dim(\langle Y \rangle)$, we have $h^1(V, \mathcal{I}_{S\cup Y,V}(3)) = 0$. Hence we may repeat the proof given in step (c). □

Example 1. Fix integers $n \geq 3$ and $g \geq 2$. Let $\Delta(n, g)$ be the set of all smooth and linearly normal hyperelliptic curves $X \subset \mathbb{P}^n$. Fix $X \in \Delta(n, g)$ and call T be the minimal degree surface containing X. There is an integer e such that $0 \le e \le n-1$, $e \equiv 0 \mod 2$ and either $e \le n-3$ and $T \cong F_e$ embedded by the complete linear system $|\mathcal{O}_{F_e}(h+((n-1+e)/2)f)|$ or $e = n-1$ and T is a cone over a rational normal curve of $\mathbb{P}^{n-1}$. In the latter case T is the image of $\mathbb{F}_{n-1}$ by the complete linear system $|\mathcal{O}_{F_e}(h+(n-1)f)|$ which contracts h to the vertex of T. If $e \ne n-1$, then set $Y := X$. If $e = n-1$, then call Y the strict transform of X in F_{n-1}. In the latter case we have $Y \cong X$, because X is smooth. Since the lines of T are spanned by the g^1_2 of X, we have $Y \in |\mathcal{O}_{F_e}(2h+xf)|$ for some $x \in \mathbb{Z}$. Since $\deg(X) = g+n$, we have $g+n = \mathcal{O}_{F_e}(2h+xf) \cdot \mathcal{O}_{F_e}(h+((n-1+e)/2)f) = -2e+x+n-1+e$, i.e. $x = g+1+e$. If T is a cone, then the smoothness of X implies $\mathcal{O}_{F_{n-1}}(h) \cdot \mathcal{O}_{F_e}(2h+(g+n)f)) \le 1$, i.e. $g \le n-1$. In this case X is projectively normal by a theorem of Castelnuovo (see [6]). From now on we assume $e \ne n-1$. Since T is projectively normal, for each integer $t \ge 2$ we have $h^0(\mathbb{P}^n, \mathcal{I}_X(t)) = h^0(\mathbb{P}^n, \mathcal{I}_T(t)) + h^0(T, \mathcal{O}_{X,T}(t))$. Fix an integer $t \ge 2$. Since T is projectively normal, we have $h^1(\mathbb{P}^n, \mathcal{I}_X(t)) = 0$ if and only if $h^1(T, \mathcal{I}_{X,T}(t)) = 0$, i.e. if and only if $h^1(F_e, \mathcal{O}_{F_e}((t-2)h + (t((n-1+e)/2) - g - 1 - e)f)) = 0$, i.e. if and only if $t((n-1+e)/2) - g - 1 - e) \ge -1 + e(t-2)$, i.e. if and only if

$$t(n-1-e) \ge 2g - 2e \tag{1}$$

For all $e \ne n-1$ X exists (in the prescribed $T = F_e$) if and only if $g+1+e \ge 2e$, i.e. if and only if $g \ge e-1$. All X contained in the same F_e have the same postulation. For each t the minimal integer g for which (1) holds is a function $\phi(e)$ strictly decreasing of e.

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