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NUMERICAL STUDIES FOR SOLVING FRACTIONAL-ORDER LOGISTIC EQUATION

N.H. Sweilam[1 §], M.M. Khader[2], A.M.S. Mahdy[3]

[1]Department of Mathematics
Faculty of Science
Cairo University
Giza, EGYPT

[2]Department of Mathematics
Faculty of Science
Benha University
Benha, EGYPT

[3]Department of Mathematics
Faculty of Science
Zagazig University
Zagazig, EGYPT

Abstract: In this paper, finite difference method (FDM) and variational iteration method (VIM) have been successfully implemented for solving non-linear fractional-order Logistic equation (FOLE). We have apply the concepts of fractional calculus to the well known population growth model in chaotic dynamic. The fractional derivative is described in the Caputo sense. The result is generalized of the classical population growth model to arbitrary order. The resulted non-linear system of algebraic equations using FDM is solved with the well know Newton iteration method. Where the condition of convergence is verified. Using initial value, the explicit solutions of population size for different particular cases have been derived. Numerical results show that the proposed methods are extremely efficient to solve this complicated biological model.

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[§]Correspondence author

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1. Introduction

The fractional order Logistic model can obtain by applying the fractional derivative operator on the Logistic equation. The model is initially published by Pierre Verhulst in 1838 [4]. The continuous Logistic model is described by first order ordinary differential equation. The discrete Logistic model is simple iterative equation that reveals the chaotic property in certain regions [1]. There are many variations of the population modeling. The Verhulst model is the classic example to illustrate the periodic doubling and chaotic behavior in dynamical system [1]. The model is described the population growth may be limited by certain factors like population density (see [2], [15]).

Ordinary and partial fractional differential equations have been the focus of many studies due to their frequent appearance in various applications in fluid mechanics, viscoelasticity, biology, physics and engineering [3]. Recently, a large amount of literatures developed concerning the application of fractional differential equations in non-linear dynamics. Consequently, considerable attentions have been given to the solutions of fractional differential equations of physical interest. Most fractional differential equations do not have exact solutions, so approximate and numerical techniques (see [5], [6], [10]-[13]), must be used. Recently, several numerical and approximate methods to solve the fractional differential equations have been given such as variational iteration method [7], homotopy perturbation method [21], Adomian decomposition method, homotopy analysis method and collocation method (see [9], [25]).

Among them of these methods is the variational iteration method which is proposed by He [7] as a modification of the general Lagrange multiplier method. This method is based on the use of restricted variations and correction functionals which has found a wide application for the solution of non-linear differential equations [22]. This method does not require the presence of small parameters in the differential equation, and does not require that the nonlinearities be differentiable with respect to the dependent variable and its derivatives. This technique provides a sequence of functions which converges to the exact solution of the problem. This procedure is a powerful tool for solving various kinds of problems, for example, VIM is used to solve the delay differential equations [8]. This technique solves the problem without any need to discretization of the variables, therefore, in some problems, it is not affected by computation round off errors and one is not faced with necessity of large computer memory and time. The proposed scheme provides the solution of the problem in a closed form while the mesh point techniques, such as finite difference method (see [14], [19], [23]-[25]) provide the approximation at mesh points only.

The solution of Logistic equation is explained the constant population growth rate which not includes the limitation on food supply or spread of diseases [15]. The solution curve of the model is increase exponentially from the multiplication factor up to saturation limit which is maximum carrying capacity [15], $\frac{dN}{dt} = \rho\, N(1 - \frac{N}{K})$ where N is the population with respect to time, ρ is the rate of maximum population growth and K is the carrying capacity. The solution of continuous Logistic equation is in the form of constant growth rate as in formula $N(t) = N_0 e^{\rho t}$ where N_0 is the initial population [20].

We describe some necessary definitions and mathematical preliminaries of the fractional calculus theory required for our subsequent development.

Definition 1. The Caputo fractional derivative operator D^α of order α is defined in the following form

$$D^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x \frac{f^{(m)}(t)}{(x-t)^{\alpha-m+1}} dt, \quad \alpha > 0,$$

where $m - 1 < \alpha \leq m,\ m \in \mathbb{N},\ x > 0$ and $\Gamma(.)$ is the Gamma function.

Similar to integer-order differentiation, Caputo fractional derivative operator is a linear operation

$$D^\alpha \left(\lambda\, f(x) + \mu\, g(x)\right) = \lambda\, D^\alpha\, f(x) + \mu\, D^\alpha\, g(x),$$

where λ and μ are constants. Recall that for $\alpha \in \mathbb{N}$, the Caputo differential operator coincides with the usual differential operator of integer order.

For more details on fractional derivatives definitions and its properties (see [16], [18]).

In this article, we consider FOLE of the form

$$\frac{d^\alpha u(t)}{dt^\alpha} = \rho u(t)(1 - u(t)), \quad t > 0, \quad \rho > 0, \tag{1}$$

the parameter α refers to the fractional order of time derivative with $0 < \alpha \leq 1$.

We also assume an initial condition

$$u(0) = u^0, \qquad u^0 > 0. \tag{2}$$

For $\alpha = 1$, Eq.(1) is the standard Logistic equation

$$\frac{du(t)}{dt} = \rho u(t)(1 - u(t)).$$

The exact solution to this problem is

$$u(t) = \frac{u^0}{(1-u^0)e^{-\rho t} + u^0}.$$

Applications of Logistic Equation

A typical application of the Logistic equation is a common model of population growth. Let $u(t)$ represents the population size and t represents the time where the constant $\rho > 0$ defines the growth rate.

Another application of Logistic curve is in medicine, where the Logistic differential equation is used to model the growth of tumors. This application can be considered an extension of the above mentioned use in the framework of ecology. Denoting with $u(t)$ the size of the tumor at time t.

The existence and the uniqueness of the proposed problem (1) are introduced in details in [6]. The main idea of this work is to derive an approximate formula of the fractional derivative $D^{\alpha}u(t)$ and use it to discretize (1) to get a non-linear system of algebraic equations thus greatly simplifying the problem to solve the resulting system. And comparing with the variational iteration method.

This paper is arranged as follows: Section 2, is assigned to introduce an approximate formula of the fractional derivative. Section 3, is assigned to obtain the numerical scheme using FDM of the fractional-order Logistic equation. Section 4, the procedure of the solution using variational iteration method is given. Finally in section 5, the report ends with a brief conclusion.

2. Approximate Formula of Fractional Derivative

In this section, we present a discrete approximation to the fractional derivative $D^{\alpha}u(t)$. For a positive integer M, the grid in time for the finite difference algorithm is defined by $k = \frac{T_f}{M}$. The grid points in the time interval $[0, T_f]$ are labeled $t_n = nk,\ n = 0, 1, 2, ..., M$.

Now, the discrete approximation of the fractional derivative $D^{\alpha}u(t)$ can be obtained by a simple quadrature formula as follows (see [17], [24])

$$\begin{aligned}
\frac{d^\alpha u(t_n)}{dt^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} (t_n - s)^{-\alpha} \frac{d}{ds} u(s) ds \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{n} \int_{(j-1)k}^{jk} \left[\frac{u_j - u_{j-1}}{k} + o(k) \right] (nk - s)^{-\alpha} ds \\
&= \frac{1}{\Gamma(1-\alpha)(1-\alpha)} \sum_{j=1}^{n} \left[\frac{u_j - u_{j-1}}{k} + o(k) \right] \\
&\quad \times [(n-j+1)^{1-\alpha} - (n-j)^{1-\alpha}][k^{1-\alpha}] \\
&= \frac{1}{\Gamma(1-\alpha)(1-\alpha)k^\alpha} \sum_{j=1}^{n} (u_j - u_{j-1}) \left[(n-j+1)^{1-\alpha} - (n-j)^{1-\alpha} \right] \\
&+ \frac{1}{\Gamma(1-\alpha)(1-\alpha)} \sum_{j=1}^{n} \left[(n-j+1)^{1-\alpha} - (n-j)^{1-\alpha} \right] o(k^{2-\alpha}).
\end{aligned}$$

Setting and shifting indices, we have

$$\sigma_{\alpha,k} = \frac{1}{\Gamma(1-\alpha)(1-\alpha)k^\alpha} \quad \text{and} \quad \omega_j^{(\alpha)} = j^{1-\alpha} - (j-1)^{1-\alpha}, \tag{3}$$

Setting and shifting indices, we have

$$\sigma_{\alpha,k} = \frac{1}{\Gamma(1-\alpha)(1-\alpha)k^\alpha} \quad \text{and} \quad \omega_j^{(\alpha)} = j^{1-\alpha} - (j-1)^{1-\alpha}, \tag{4}$$

and

$$\begin{aligned}
\frac{d^\alpha u(t_n)}{dt^\alpha} &= \sigma_{\alpha,k} \sum_{j=1}^{n} \omega_j^{(\alpha)} (u_{n-j+1} - u_{n-j}) + \frac{1}{\Gamma(1-\alpha)(1-\alpha)} n^{1-\alpha} o(k^{2-\alpha}) \\
&= \sigma_{\alpha,k} \sum_{j=1}^{n} \omega_j^{(\alpha)} (u_{n-j+1} - u_{n-j}) + o(k).
\end{aligned}$$

Here

$$\frac{d^\alpha u(t_n)}{dt^\alpha} = D^\alpha u_n + o(k),$$

and the first-order approximation method for the computation of Caputo's fractional derivative is given by the expression

$$D^\alpha u_n \cong \sigma_{\alpha,k} \sum_{j=1}^{n} \omega_j^{(\alpha)} (u_{n-j+1} - u_{n-j}), \qquad n = 1, 2, ..., M. \tag{5}$$

3. Discretization for Fractional Logistic Equation

In this section, finite difference method with the discrete formula (5) is used to estimate the time α-order fractional derivative to solve numerically, the fractional-order Logistic equation (1). Using (5) the restriction of the exact solution to the grid points centered at $t_n = nk$, in Eq.(1)

$$\sigma_{\alpha,k} \sum_{j=1}^{n} \omega_j^{(\alpha)} (u_{n-j+1} - u_{n-j}) + o(k) = \rho u_n (1 - u_n),$$

$$\sigma_{\alpha,k} \sum_{j=1}^{n} \omega_j^{(\alpha)} (u_{n-j+1} - u_{n-j}) = \rho u_n (1 - u_n) + T(t), \tag{6}$$

where $T(t)$ is the truncation term. Thus, according to Eq.(6), the numerical method is consistent, first order correct in time.

The resulting finite difference equations are defined by

$$\sigma_{\alpha,k} \sum_{j=1}^{n} \omega_j^{(\alpha)} (u_{n-j+1} - u_{n-j}) = \rho u_n (1 - u_n), \qquad n = 1, 2, ..., M. \tag{7}$$

This scheme presents a non-linear system of algebraic equations. In our calculation, we used the Newton iteration method to solve this system.

In view of Newton iteration method, we can write the non-linear system (7) in the following iteration formula

$$U^{m+1} = U^m - J^{-1}(U^m)\, F(U^m), \tag{8}$$

where $F(U^m)$ is the vector which represents the non-linear equations and $J^{-1}(U^m)$ is the inverse of the Jacobian matrix, where

$$J(U^m) = \text{Diagonal}(r\omega_1^{(\alpha)} - \rho + 2\rho u_1^{(m)}, \, ..., \, r\omega_M^{(\alpha)} - \rho + 2\rho u_M^{(m)}),$$

$$\text{and} \quad |J(U^m)| = \prod_{i=1}^{M} (r\omega_i^{(\alpha)} - \rho + 2\rho u_i^{(m)}).$$

Since, it is easy to see that the quantity $\prod_{i=1}^{M} (r\omega_i^{(\alpha)} - \rho + 2\rho u_i^{(m)}) \neq 0$, we can deduce that the Jacobian matrix which corresponding to Newton's formula (8) is nonsingular, therefore, we can show that the numerical scheme (7) is convergent.

4. Procedure Solution using VIM

The VIM gives the possibility to write the solution of Eq.(1), $(0 < \alpha \le 1)$ with the aid of the correction functionals in the form

$$u_{n+1}(t) = u_n(t) + \int_0^t \lambda(\tau)\Big[\frac{du_n}{d\tau} - \rho\tilde{u}_n(1-\tilde{u}_n)\Big]d\tau, \tag{9}$$

where λ is a general Lagrange multiplier, which can be identified optimally via the variational theory [7]. The function $\tilde{u}_n$ is a restricted variation, which means that $\delta\tilde{u}_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations u_n, $n \ge 0$, of the solution $u(t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The initial values of the solution are usually used for selecting the zeroth approximation u_0. With λ determined, then several approximations u_n, $n \ge 0$, follow immediately. Making the above correction functional stationary

$$\begin{aligned}
\delta u_{n+1}(t) &= \delta u_n(t) + \delta\int_0^t \lambda(\tau)\Big[\frac{du_n}{d\tau} - \rho\tilde{u}_n(1-\tilde{u}_n)\Big]d\tau \\
&= \delta u_n(t) + \int_0^t \delta\lambda(\tau)\Big[\frac{du_n}{d\tau}\Big]d\tau \\
&= \delta u_n(t) + [\lambda(\tau)\delta u_n(\tau)]_{\tau=t} - \int_0^t \delta u_n\dot{\lambda}(\tau)d\tau = 0,
\end{aligned} \tag{10}$$

where $\delta\tilde{u}_n$ is considered as a restricted variation, i.e., $\delta\tilde{u}_n = 0$, yields the following stationary conditions (by comparison the two sides in the above equation)

$$\dot{\lambda}(\tau) = 0, \qquad 1 + \lambda(\tau)|_{\tau=t} = 0. \tag{11}$$

The equation in (11) is called Lagrange-Euler equation with the natural boundary condition. The solution of this equation gives the Lagrange multiplier $\lambda(\tau) = -1$.

Now, by substituting in Eq.(9), the following variational iteration formula can be obtained

$$u_{n+1}(t) = u_n(t) - \int_0^t \Big[\frac{d^\alpha u_n}{d\tau^\alpha} - \rho u_n(1-u_n)\Big]d\tau, \qquad n \ge 0. \tag{12}$$

We start with initial approximation $u_0(t) = 0.85$, and by using the above iteration formula (12), we can directly obtain the components of the solution. Consequently, the exact solution may be obtained by using

$$u(t) = \lim_{n\to\infty} u_n(t). \tag{13}$$

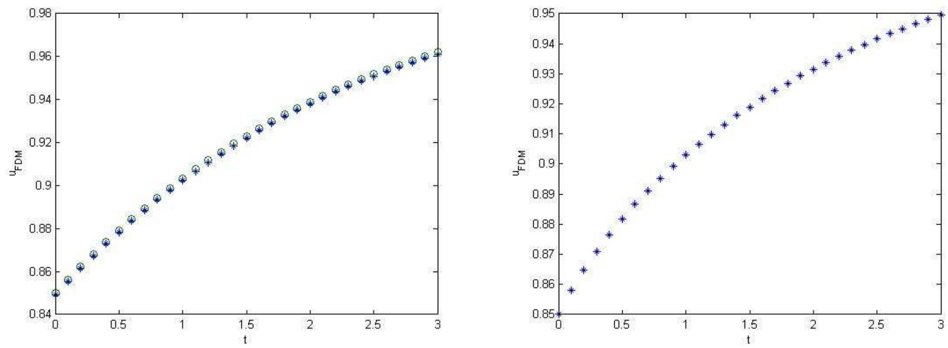

Figure 1: A comparison between the numerical solution and the exact solution at $\alpha = 1$ (left). The behavior of the numerical solution at $\alpha = 0.85$ (right) using FDM

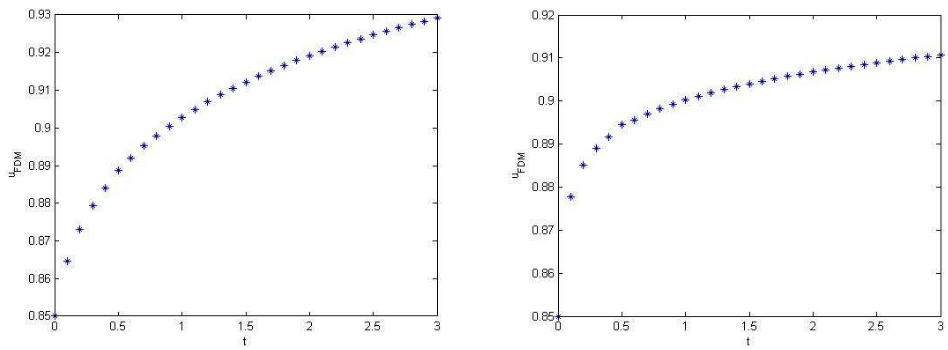

Figure 2: The behavior of the numerical solution using FDM at $\alpha = 0.55$ (left) and at $\alpha = 0.25$ (right)

Now, the first three components of the solution $u(t)$ of the Logistic equation by using (12)

$$u_0(t) = 0.85,$$

$$u_1(t) = 0.85 + 0.06375\, t,$$

$$u_2(t) = 0.85 + 0.06375\, t - t\,(-0.06375 + 0.05941\, t^{0.15} + 0.01115\, t + 0.00067\, t^2).$$

The behavior of the approximation solutions using variational iteration method, u_{VIM}, with $n = 4$, compared with the numerical solution using FDM, u_{FDM}, with $k = 0.1$ are presented in Figures 1-4 in the interval [0, 3] for different values of α ($\alpha = 1, 0.85, 0.55, 0.25$).

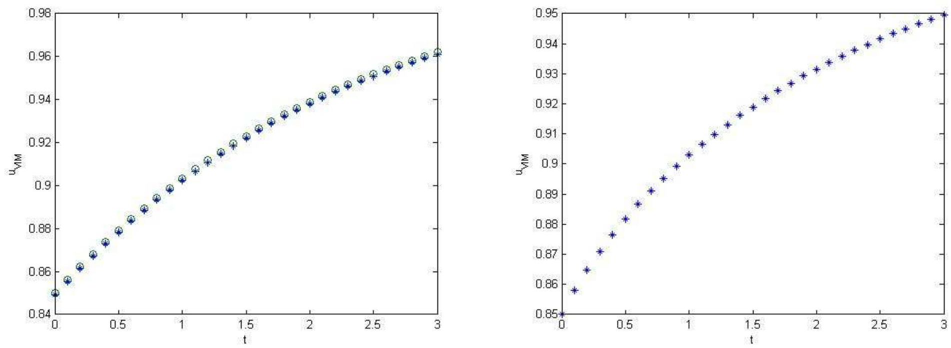

Figure 3: A comparison between the approximate solution and the exact solution $\alpha = 1$ (left). The behavior of the numerical solution at $\alpha = 0.85$ (right) using VIM

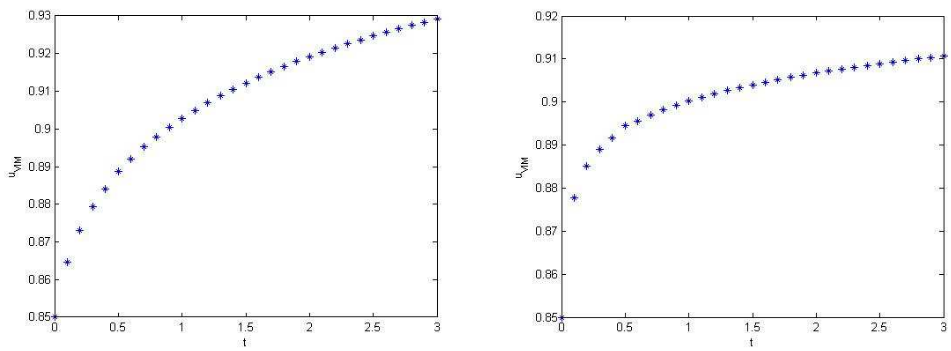

Figure 4: The behavior of the approximate solution using VIM at $\alpha = 0.55$ (left) and at $\alpha = 0.25$ (right)

From the numerical results in all Figures 1-4, we can conclude that the two proposed methods, FDM and VIM are in excellent agreement with the exact solution. Also, we can find that these two methods are extremely efficient to solve this complicated biological model.

5. Conclusion

In this article, we used two computational methods, FDM and VIM for solving the fractional-order Logistic equation. We derived an approximate formula of the fractional derivative. Using finite difference scheme the FOLE reduced to a non-linear system of algebraic equations which solved by Newton iteration method. The results obtained by FDM is comparied with VIM. Also, it is evident that the overall errors can be made smaller by adding new terms from the obtained sequence of solutions using VIM. From the obtained numerical results we can conclude that these two methods give results with excellent agreement with the exact solution. All numerical results are obtained using Matlab 8.

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