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ON BE-HOMOMORPHISMS OF BE-SEMIGROUPS

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Abstract: In this paper, we introduce the notion of BE-homomorphism between BE-semigroups, and investigate some of their properties. Also, we establish construct the quotient self-distributive BE-semigroup via deductive system, and we have the fundamental theorem of homomorphisms for self-distributive BE-semigroups as a consequence.

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1. Introduction

Imai and Iséki introduced two classes of abstract algebras, namely, BCK-algebras and BCI-algebras [4], [5]. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [9], Neggers and Kim introduced the notion of d-algebras which is a generalization of BCK-algebras. Moreover, Jun et al [9] introduced a new notion, called a BH-algebra, which is a generaliza-

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tion of BCK/BCI-algebras. Recently, another generalization of BCK-algebras, the notion of a BE-algebra is introduced by Kim and Kim [8]. They provided an equivalent condition of the filters in BE-algebras using the notion of upper sets. In [2], [3], Ahn and So introduced the notion of ideals in BE-algebras and proved several characterizations of such ideals. In [1], Ahn and Kim combined BE-algebras and semigroups and introduced the notion of BE-semigroups. In this paper, we introduce the notion of BE-homomorphism of BE-semigroups, and investigate some of their properties. In addition, the notion of quotient self-distributive BE-semigroups is introduced.

2. Preliminaries

In this section, we cite some elementary aspects that will be used in the sequel of this paper.

Definition 2.1. (see [8]) An algebra $(X, *, 1)$ of type $(2, 0)$ is called a BE-algebra if:

$(BE1)$ $x * x = 1$ for all $x \in X$,

$(BE2)$ $x * 1 = 1$ for all $x \in X$,

$(BE3)$ $1 * x = x$ for all $x \in X$,

$(BE4)$ $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

We can define a relation " $\leq$ "on X by $x \leq y$ if and only if $x * y = 1$.

In an BE-algebra, the following identities are true (see [8]):

$(a1)$ $x * (y * x) = 1$.

$(a2)$ $x * ((x * y) * y)) = 1$.

Definition 2.2. (see [8]) A BE-algebra $(X, *, 1)$ is said to be self-distributive if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$.

Definition 2.3. (see [2]) A BE-algebra $(X; *, 1)$ is said to be transitive if for any $x, y, z \in X$, $y * z \leq (x * y) * (x * z)$.

Proposition 2.4. (see [2]) *If X is a self-distributive BE-algebra, then it is transitive.*

Definition 2.5. (see [10]) Let X, Y be two BE-algebras. A mapping $\psi : X \to Y$ is called a homomorphism from X into Y if for any $x, y \in X$, $\psi(x * y) = \psi(x) * \psi(y)$ holds.

Definition 2.6. (see [1]) An algebraic system $(X; \odot, *, 1)$ is called a BE-semigroup if it satisfies the following:

(I) $(X;\odot)$ is a semigroup,

(II) $(X;*,1)$ is a BE-algebra,

(III) the operation "$\odot$" is distributive (on both sides) over the operation "$*$", that is,

$$x\odot(y*z)=(x\odot y)*(x\odot z) \text{ and } (x*y)\odot z=(x\odot z)*(y\odot z) \text{ for all } x,y,z\in X.$$

Proposition 2.7. (see [1]) *Let $(X;\odot,*,1)$ be a BE-semigroup. Then*

(i) $(\forall x\in X)$ $(1\odot x=x\odot 1=1)$,

(ii) $(x,y,z\in X)$ $(x\leq y\Rightarrow x\odot z\leq y\odot z, z\odot x\leq z\odot y)$.

Definition 2.8. (see [1]) An element $a(\neq 1)$ in a BE-semigroup $(X;\odot,*,1)$ is said to be a left (resp. right) unit divisor if

$$(\exists b(\neq 1)\in X)\ (a\odot b=1 \text{ (resp. } b\odot a=1))$$

A unit divisor is an element of X which is both a left and a right unit divisors.

Definition 2.9. (see [1]) Let $(X;\odot,*,1)$ be a BE-semigroup. A nonempty subset D of X is called a left (resp. right) deductive system if it satisfies:

$(ds1)$ $X\odot D\subseteq D$ (resp. $(D\odot X\subseteq D)$),

$(ds2)$ $(\forall a\in D)$ $((\forall x\in X)$ $(a*x\in D\Rightarrow x\in D)$.

Both left and right deductive system is a two sided deductive system or simply deductive system.

Example 2.10. (see [1]) Let $X=\{1,x,y,z\}$ be the set with the following Cayley tables:

$\odot$	1	x	y	z
1	1	1	1	1
x	1	x	1	1
y	1	1	y	z
z	1	1	z	y

$*$	1	x	y	z
1	1	x	y	z
x	1	1	y	z
y	1	1	1	z
z	1	1	1	1

Then $(X;\odot,*,1)$ is a BE-semigroup. It is easy to show that $D=\{1,x\}$ is a deductive system of X.

Definition 2.11. A BE-semigroup $(X;\odot,*,1)$ is said to be self-distributive BE-semigroup if X is self-distributive BE-algebra.

3. Main Results

In this section we generalize the notion of a mapping from one BE-semigroup X into another BE-semigroup Y. Thus we define a BE-homomorphism in a usual way.

Definition 3.1. Let X and Y be two BE-semigroups. A mapping $\psi : X \to Y$ is called a BE-homomorphism if for all $a, b \in X$, $\psi(a * b) = \psi(a) * \psi(b)$ and $\psi(a \odot b) = \psi(a) \odot \psi(b)$.

A BE-homomorphism ψ is called a BE-monomorphism (resp. BE-epimorphism) if it is injective (resp. surjective). A bijective BE-homomorphism is called a BE-isomorphism. For any BE-homomorphism $\psi : X \to Y$, the set $\{x \in X \mid \psi(x) = 0\}$ is called the kernel of ψ, denoted by $Ker(\psi)$ and the set $\{\psi(x) \mid x \in X\}$ is called the image of ψ, denoted by $Im(\psi)$. We denote by $Hom(X, Y)$ the set of all BE-homomorphisms of BE-semigroups from X to Y.

In what follows let X and Y be BE-semigroups unless otherwise specified.

Example 3.2. Let $X = \{1, a, b, c\}$ and $Y = \{1, x, y, z\}$ be sets with cayley tables as follows:

$\odot$	1	a	b	c
1	1	1	1	1
a	1	1	1	1
b	1	1	1	1
c	1	a	b	c

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	1	1	1

$\odot$	1	x	y	z
1	1	1	1	1
x	1	1	1	1
y	1	1	1	y
z	1	1	y	z

$*$	1	x	y	z
1	1	x	y	z
x	1	1	y	z
y	1	x	1	z
z	1	1	1	1

Then $(X; \odot, *, 1)$ and $(Y; \odot, *, 1)$ are BE-semigroups (see [1], Example 3.2). Define a map $\psi : X \to Y$ by

$$\psi(x) = \begin{cases} 1, & \text{if } x = 1, a, b \\ z, & \text{if } x = c \end{cases} .$$

Then it is easy to check that ψ is a BE-homomorphism from X into Y.

Proposition 3.3. *Suppose that $\psi : X \to Y$ is a BE-homomorphism of BE-semigroups. Then*

(1) $\psi(1) = 1$.

(2) If $x * y = 1$, $x, y \in X$, then $\psi(x) * \psi(y) = 1$.

Proof. Since $\psi(1) = \psi(1 * 1) = \psi(1) * \psi(1) = 1$, (1) holds. If $x, y \in X$ and $x * y = 1$, then by (1), $\psi(x) * \psi(y) = \psi(x * y) = \psi(1) = 1$. □

Proposition 3.4. *Suppose that $\psi : X \to Y$ is a BE-homomorphism of BE-semigroups. Then ψ is a monomorphism if and only if $Ker(\psi) = \{1\}$.*

Proposition 3.5. *Let X, Y be BE-semigroups and $\psi \in Hom(X, Y)$. Then*

(1) $\psi(x \odot 1) = \psi(1 \odot x) = 1$, for all $x \in X$.

(2) $\psi(1 * x) = \psi(x)$, for all $x \in X$.

(3) $\psi(x * 1) = 1$, for all $x \in X$.

Proposition 3.6. *Let $\psi : X \to Y$ be a monomorphism of BE-semigroups. If $x \in X$ is a left (resp. right) unit divisor of X then $\psi(x)$ is a left (resp. right) unit divisor of Y.*

Proposition 3.7. *Let X, Y and Z be BE-semigroups. If $\psi \in Hom(X, Y)$ and $\omega \in Hom(Y, Z)$, then $\omega \circ \psi \in Hom(X, Z)$.*

Theorem 3.8. *Let X and Y be BE-semigroups and let B be a left (resp. right) deductive system of Y. Then for any $\psi \in Hom(X, Y)$, $\psi^{-1}(B)$ is a left (resp. right) deductive system of X containing $Ker(\psi)$.*

Proof. Let B be a left deductive system of Y. If $x \in X$ and $y \in \psi^{-1}(B)$, then $\psi(y) \in B$ and $\psi(x \odot y) = \psi(x) \odot \psi(y)$. It follows from the fact that B is a left deductive system that $\psi(x \odot y) \in B$, that is, $x \odot y \in \psi^{-1}(B)$. Hence $X \odot \psi^{-1}(B) \subseteq \psi^{-1}(B)$. Now let $x, y \in X$ be such that $y \in \psi^{-1}(B)$ and $y * x \in \psi^{-1}(B)$. Then $\psi(y) \in B$ and $\psi(y * x) = \psi(y) * \psi(x) \in B$. Since B is a left deductive system, we have $\psi(x) \in B$, that is, $x \in \psi^{-1}(B)$. Hence $\psi^{-1}(B)$ is a left deductive system of X. Since $\{1\} \subseteq B$, $Ker(\psi) = \psi^{-1}(\{1\}) \subseteq \psi^{-1}(B)$. Similarly we have the desired result for the right case. □

Theorem 3.9. *Let X and Y be BE-semigroups and let $\psi : X \to Y$ be a BE-epimorphism of BE-semigroups. If A is a left (resp. right) deductive system of X, then $\psi(A)$ is a left (resp. right) deductive system of Y.*

Proof. Let $x \in \psi(A)$ and $y \in Y$. Since ψ is onto, there exist $a \in X$ and $b \in A$ such that $\psi(a) = y$ and $\psi(b) = x$. Thus, $a \odot b \in A$ implies that $y \odot x \in \psi(A)$. Hence $Y \odot \psi(A) \subseteq \psi(A)$. Now, suppose $a \in \psi(A)$, $y \in Y$ and $a * y \in \psi(A)$. Since ψ is onto, there exist $b \in A$ and $x \in X$ such that $\psi(b) = a$ and $\psi(x) = y$.

Thus, $\psi(b * x) = \psi(b) * \psi(x) = a * y$, so $b * x \in A$. It follows from $(ds2)$ that $x \in A$. Hence $y = \psi(x) \in \psi(A)$. Therefore, $\psi(A)$ is a left deductive system of Y. Similarly we have the desired result for the right case. □

Theorem 3.10. *Let $\psi : X \to Y$ be a BE-homomorphism of BE-semigroups. Then $Ker(\psi)$ is a deductive system of X.*

Proof. Let $x \in X$ and $y \in Ker(\psi)$. Then we have $\psi(x \odot y) = \psi(x) \odot \psi(y) = \psi(x) \odot 1 = 1$ and $\psi(y \odot x) = \psi(y) \odot \psi(x) = 1 \odot \psi(x) = 1$. Hence $X \odot Ker(\psi) \subseteq Ker(\psi)$ and $Ker(\psi) \odot X \subseteq Ker(\psi)$. Now, let $a * x \in Ker(\psi)$ and $a \in Ker(\psi)$. Then $1 = \psi(a * x) = \psi(a) * \psi(x) = 1 * \psi(x) = \psi(x)$. Hence $x \in Ker(\psi)$. Therefore, $Ker(\psi)$ is a deductive system of X. □

Theorem 3.11. *Let $\psi \in Hom(X, Y)$. If X is transitive, then $\psi(X)$ is transitive.*

Proof. Let $\psi(x), \psi(y), \psi(z) \in \psi(X)$. Then

$$\begin{aligned}
(\psi(y) * \psi(z)) * ((\psi(x) * \psi(y)) * (\psi(x) * \psi(z))) &= \psi(y * z) * (\psi(x * y) * \psi(x * z)) \\
&= \psi(y * z) * \psi((x * y) * (x * z)) \\
&= \psi((y * z) * ((x * y) * (x * z))) \\
&= \psi(1) \\
&= 1.
\end{aligned}$$

Therefore, $\psi(X)$ is transitive. □

Theorem 3.12. *Let $\psi : X \to Y$ be a BE-semigroup monomorphism. If $\psi(X)$ is transitive, then X is transitive.*

Proof. Let $x, y, z \in X$. Then $(\psi(y)*\psi(z))*((\psi(x)*\psi(y))*(\psi(x)*\psi(z))) = 1$, and thus $\psi((y * z) *((x * y) * (x * z))) = 1$. Since ψ is a monomorphism, it follows from Proposition 3.4 that $(y * z) *((x * y) * (x * z)) = 1$. Therefore, X is transitive. □

Theorem 3.13. *Let X, Y and Z be BE-semigroups. Suppose that $\phi : X \to Y$ is a BE-epimorphism, and let $\psi : X \to Z$ be a BE-homomorphism. If $Ker(\phi) \subseteq Ker(\psi)$, then there exists a unique BE-homomorphism $\gamma : Y \to Z$ such that $\gamma \circ \phi = \psi$.*

Proof. Let $y \in Y$. Since ϕ is onto, there exists $x \in X$ such that $\phi(x) = y$. Define a mapping $\gamma : Y \to Z$ by $\gamma(y) = \psi(x)$. If $y = \phi(x_1) = \phi(x_2)$, $x_1, x_2 \in X$, then $1 = \phi(x_1) * \phi(x_2) = \phi(x_1 * x_2)$. Hence $x_1 * x_2 \in Ker(\phi)$. Since $Ker(\phi) \subseteq Ker(\psi)$, we have $1 = \psi(x_1) * \psi(x_2) = \psi(x_1 * x_2)$. Similarly, we get $\psi(x_2) * \psi(x_1) = 1$. Thus $\psi(x_1) = \psi(x_2)$. This means that γ is well-defined. Next we show that γ is a BE-homomorphism. Let $a, b \in Y$. Then there exist $x_1, x_2 \in X$ such that $a = \phi(x_1)$ and $b = \phi(x_2)$. Now we have

$$\gamma(a \odot b) = \gamma(\phi(x_1) \odot \phi(x_2)) = \gamma(\phi(x_1 \odot x_2)) = \psi(x_1) \odot \psi(x_2) = \gamma(a) \odot \gamma(b),$$

and

$$\gamma(a * b) = \gamma(\phi(x_1) * \phi(x_2)) = \gamma(\phi(x_1 * x_2)) = \psi(x_1) * \psi(x_2) = \gamma(a) * \gamma(b).$$

Hence γ is a BE-homomorphism. The uniqueness of γ follows directly from the fact that ϕ is a BE-epimorphism. □

Theorem 3.14. *Let X, Y and Z be BE-semigroups, and let $g : X \to Z$ be a BE-homomorphism and $h : Y \to Z$ be a BE-monomorphism with $Im(g) \subseteq Im(h)$, then there is a unique BE-homomorphism $f : X \to Y$ satisfying $h \circ f = g$.*

Proof. For each $x \in X$, $g(x) \in \mathrm{Im}(g) \subseteq \mathrm{Im}(h)$. Since h is a BE-monomorphism, there exists a unique $b \in Y$ such that $h(b) = g(a)$. Define a map $f : X \to Y$ by $f(a) = b$. Then $h \circ f = g$. To show that f is a BE-homomorphism. Let $c, d \in X$. Then

$$h(f(c * d)) = g(c * d) = g(c) * g(d) = h(f(c)) * h(f(d)) = h(f(c) * f(d)).$$

Since h is a BE-monomorphism, we have $f(c * d) = f(c) * f(d)$. Similarly, we can prove that $f(c \odot d) = f(c) \odot f(d)$. The uniqueness of f follows from the fact that h is a monomorphism. □

Let X be a self-distributive BE-semigroup and D be a deductive system of X. For any $x, y \in X$, we define a relation "$\sim_D$" on X as follows:

$$x \sim_D y \text{ if and only if } x * y \in D \text{ and } y * x \in D.$$

Proposition 3.15. *Let X be a self-distributive BE-semigroup and D be a deductive system of X. Then $\sim_D$ is an equivalence relation on X.*

Proof. Let $a \in D$. Since D is a deductive system of X and $1 \in X$, we have $1 \odot a = 1 \in D$. Let $x \in X$. Then $x * x = 1 \in D$, and thus $x \sim_D x$. This means that $\sim_D$ is reflexive. The symmetry of $\sim_D$ is immediate from the definition of the relation. For any $x, y, z \in X$, if $x \sim_D y$ and $y \sim_D z$, then $x * y \in D$,$y * x \in D$ and $y * z, z * y \in D$. Using $(BE2)$ and $(BE4)$, we obtain $(y*z)*((x*y)*(x*z)) = (y*z)*(x*(y*z)) = x*((y*z)*(y*z)) = x*1 = 1 \in D$, which implies that $(x * y) * (x * z) \in D$, and so $x * z \in D$ by $(ds2)$. Similarly, we have $z * x \in D$. Hence $x \sim_D z$. Therefore, $\sim_D$ is an equivalence relation on X. □

Let D be a deductive system of a self-distributive BE-semigroup X. Denote the equivalence class containing x by D^x and the set of all equivalence classes in X by X/D, that is, $D^x = \{y \in X \mid y \sim_D x\}$ and $X/D = \{D^x \mid x \in X\}$. Clearly, $D^1 = D$ and $D^x = D^y$ if and only if $x \sim_D y$.

Theorem 3.16. *If D is a deductive system of a self-distributive BE-semigroup X, then $(X/D, \circledcirc, \circledast, D^1)$ is a self-distributive BE-semigroup under the operations*

$$D^x \circledcirc D^y = D^{x \odot y} \text{ and } D^x \circledast D^y = D^{x*y}$$

Proof. Clearly $(X/D, \circledast, D^1)$ is a BE-algebra. Let $D^x = D^y$ and $D^u = D^v$. Then since D is a deductive system, we have $(x \odot u) * (x \odot v) = x \odot (u * v) \in D$ and $(x \odot v) * (x \odot u) = x \odot (v * u) \in D$. Thus $(x \odot u) \sim_D (x \odot v)$. On the other hand, $(x \odot v) * (y \odot v) = (x*y) \odot v \in D$ and $(y \odot v) * (x \odot v) = (y*x) \odot v \in D$. Hence $(x \odot v) \sim_D (y \odot v)$, and so $x \odot u \sim_D y \odot v$, that is, $D^{x \odot u} = D^{y \odot v}$. This shows that $\circledcirc$ is well-defined. Therefore, it is easy to prove that $(X/D, \circledcirc)$ is a semigroup. Moreover, for any $D^x, D^y, D^z \in X/D$, we obtain $D^x \circledcirc (D^y \circledast D^z) = D^x \circledcirc D^{y*z} = D^{x \odot (y*z)} = D^{(x \odot y)*(x \odot z)} = D^{(x \odot y)} \circledast D^{(x \odot z)} = (D^x \circledcirc D^y) \circledast (D^x \circledcirc D^z)$. Similarly, we have $(D^x \circledast D^y) \circledcirc D^z = (D^x \circledcirc D^z) \circledast (D^y \circledcirc D^z)$. Thus, X/D is a BE-semigroup. Let $D^x, D^y, D^z \in X/D$. Then $D^x \circledast (D^y \circledast D^z) = D^x \circledast D^{y*z} = D^{x*(y*z)} = D^{(x*y)*(x*z)} = D^{x*y} \circledast D^{x*z} = (D^x \circledast D^y) \circledast (D^x \circledast D^z)$. Therefore, $(X/D, \circledcirc, \circledast, D^1)$ is a self-distributive BE-semigroup. □

Proposition 3.17. *If I and J are deductive systems of a self-distributive BE-semigroup X and $I \subset J$, then*

(i) I is also a deductive system of J.

(ii) J/I is a deductive system of X/I.

Theorem 3.18. *Let $\psi : X \to Y$ be a BE-homomorphism of self-distributive BE-semigroups. Then for any deductive system D of X, $D/(Ker(\psi) \cap D) \cong \psi(D)$.*

Proof. Let $A = Ker(\psi) \cap D$. Clearly, A is a deductive system of D. Define a mapping $\xi : D/A \to Y$ by $\xi(A^x) = \psi(x)$, for all x in D. Then for any $A^x, A^y \in D/A$, we have

$$\begin{aligned} A^x &= A^y \Leftrightarrow x * y \in A, y * x \in A, \\ &\Leftrightarrow \psi(x * y) = 1, \psi(y * x) = 1, \\ &\Leftrightarrow \psi(x) \circledast \psi(y) = 1, \psi(y) \circledast \psi(x) = 1, \\ &\Leftrightarrow \psi(x) = \psi(y), \\ &\Leftrightarrow \xi(A^x) = \xi(A^y). \end{aligned}$$

Hence ξ is well-defined and one to one. For all $A^x, A^y \in D/A$, we have

$$\xi(A^x \circledast A^y) = \xi(A^{x*y}) = \psi(x * y) = \psi(x) * \psi(y) = \xi(A^x) * \xi(A^y),$$

and

$$\xi(A^x \circledcirc A^y) = \xi(A^{x \odot y}) = \psi(x \odot y) = \psi(x) \odot \psi(y) = \xi(A^x) \odot \xi(A^y).$$

Hence ξ is a BE-homomorphism of self-distributive BE-semigroups. Thus we obtain $\mathrm{Im}(\xi) = \{\xi(A^x) \mid x \in D\} = \{\psi(x) \mid x \in D\} = \psi(D)$. Therefore, $D/(Ker(\psi) \cap D) \cong \psi(D)$. □

Corollary 3.19. *If $\psi : X \to Y$ is a BE-epimorphism of self-distributive BE-semigroups, then $X/Ker(\psi) \cong Y$.*

Definition 3.20. A BE-semigroup X is said to be Noetherian if each deductive system of X is finitely generated.

Theorem 3.21. *Given two self-distributive BE-semigroups X, Y, if $\psi : X \to Y$ is a BE-epimorphism and X is Noetherian, then so is Y.*

Proof. By Theorem 3.19, $X/Ker(\psi) \cong Y$. It follows from Proposition 3.17(ii), that every deductive system of $X/Ker(\psi)$ is of the form $D/Ker(\psi)$, where D is a deductive system of X with $Ker(\psi) \subseteq D$. Let $D_1/Ker(\psi) \subseteq D_2/Ker(\psi) \subseteq ...$ be any ascending chain of deductive systems in Y. Then $Ker(\psi) \subseteq D_1 \subseteq D_2 \subseteq ...$ is an ascending chain of deductive systems of X. Since X is Noetherian, we have $D_n = D_{n+1} = ...$ for some natural number n. Hence we obtain $D_n/Ker(\psi) = D_{n+1}/Ker(\psi) = ...$. Therefore, Y is Noetherian. □

References

[1] S.S. Ahn, Y.H. Kim, On BE-semigroups, *International Journal of Mathematics and Mathematical Sciences*, **2011** (2011), Article ID 676020, 8 pages.

[2] S.S. Ahn, K.S. So, On ideals and upper sets in BE-algebras, *Scientiae Mathematicae Japonicae*, **68**, No. 2 (2008), 279-285.

[3] S.S. Ahn, K.S. So, On generalized upper sets in BE-algebras, *Bulletin of the Korean Mathematical Society*, **46**, No. 2 (2009), 281-287.

[4] K. Iséki, On BCI-algebras, *Math. Sem. Notes Kobe Univ.*, **8**, No. 1 (1980), 125-130.

[5] K. Iséki, S. Tanaka, An introduction to the theory of BCK-algebras, *Mathematica Japonica*, **23** (1978), 1-26.

[6] Y.B. Jun, E.H. Roh, H.S. Kim, On BH-algebras, *Scientiae Mathematicae Japonicae Online*, **1** (1998), 347-354.

[7] K.H. Kim, Multipliers in BE-algebras, *International Mathematical Forum*, **6**, No. 17 (2011), 815-820.

[8] H.S. Kim, Y.H. Kim, On BE-algebras, *Scientiae Mathematicae Japonicae*, **66**, No. 1 (2007), 113-116.

[9] J. Neggers, H.S. Kim, On d-algebras, *Mathematica Slovaca*, **49** (1999), 19-26.

[10] Y.H. Yon, S.M. Lee, K.H. Kim, On Congruences and BE-relations in BE-algebras, *International Mathematical Forum*, **5**, No. 46 (2010), 2263-2270.