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ON STRATIFIED L-FILTER STRUCTURE

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Abstract: In this paper, we construct stratified L-filter (respectively, stratified L-filterbase) structure from a given L-filter (respectively, L-filterbase) structure, where L is a strictly two-sided, commutative quantale lattice. Also, we study the images and preimages of L-filter structure and stratification of it. Then, we investigate some properties of these concepts.

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1. Introduction

It is well known that, the notion of a filter on a set is a basic concept in topology and plays an important role in it. J. G. Garcia et al., [6] introduced the notion of L-filter structure and proved many theorems corresponding to the usual theorems, where L is a strictly two-sided, comutative quantale lattice.

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Höhle et al., [10], [11] introduced the notion of L-filter on a complete quasi-monoidal lattice (including GL-monoid [6], [7]) L-instead of completely distributive lattice [1]-[5] or the unit interval [2], [3] as an extension of fuzzy filters [8], [9], [13]. The notion of L-filter facilitated to study L-fuzzy topologies [11], [14] and L-fuzzy uniform spaces [1], [17]. Let L be a complete lattice and $\phi : X \to Y$ a function. The Zadeh image and preimage operators $\phi_L^{\rightarrow} : L^X \to L^Y$ and $\phi_L^{\leftarrow} : L^X \leftarrow L^Y$ are defined by

$$\phi_L^{\rightarrow}(f)(y) = \bigvee\{f(x) \mid \phi(x) = y\}, \qquad \phi_L^{\leftarrow}(g) = g \circ \phi.$$

Rodabaugh [15], [16] gives two different proofs for all cqml's (complete lattices) L vindicating Zadeh's definitions, first, using AFT (adjoint functor theorem) to lift the Zadeh operators from traditional operators, and second, classes of naturality diagrams indexed by L to generate Zadeh operators directly from the original function.

In this paper, we construct the notions of stratified L-filter structure and stratified L-filterbase structure from a given L-filter structure and L-filterbase structure, respectively, where L is a strictly two-sided, commutative quantale lattice and investigate some of their properties. We consider the Zadeh preimage operator $\phi_1^{\Leftarrow}$ of the Zadeh image operator and the Zadeh preimage operator $\phi_2^{\Rightarrow}$ of the Zadeh preimage operator. Then, we investigate the images and preimages of stratified L-filters induced by functions. Moreover, we investigate the relationships between L-filter structure and stratification of it.

2. Preliminaries

Throughout this paper, let X be a nonempty set. Let $L = (L, \leq, \vee, \wedge, 0, 1)$ be a complete lattice which satisfies the infinitely distributive laws where 0 and 1 denote the least and the greast elements in L. For $\alpha \in L$, $\underline{\alpha}(x) = \alpha$ for all $x \in X$.

Definition 2.1. (see [8], [16], [19]) A triple $(L, \leq, \odot)$ is called a strictly two-sided, commutative quantale lattice (scq-lattice, for short) iff it satisfies the following properties:

(L1) $(L, \odot)$ is a commutative semigroup.

(L2) $x = x \odot 1$, for each $x \in L$.

(L3) $\odot$ is distributive over arbitrary joins, i.e.,

$$(\bigvee_{i\in\Gamma} r_i) \odot s = \bigvee_{i\in\Gamma}(r_i \odot s).$$

In this paper, we always assume that $(L, \leq, \odot)$ is an scq-lattice.

Remark. (see [6], [9], [12], [13], [18]) (1) A complete lattice satisfying the infinite distributive law is an scq-lattice. In particular, the unit interval $([0,1], \leq, \wedge, 0, 1)$ is an scq-lattice.

(2) Every left continuous t-norm T on $([0,1], \leq, t)$ with $\odot = t$ is an scq-lattice.

(3) Every GL-monoid is an scq-lattice.

Lemma 2.2. (see [19], [20]) *For each $x, y, z \in L$, we have the following properties:*

(1) If $y \leq z$, then $(x \odot y) \leq (x \odot z)$.

(2) $x \odot y \leq x \wedge y$.

Definition 2.3. (see [6], [9], [10], [13]) A mapping $\mathcal{F} : L^X \to L$ is called an L-filter on X if it satisfies the following conditions:

(LF1) $\mathcal{F}(\underline{0}) = 0$ and $\mathcal{F}(\underline{1}) = 1$.

(LF2) $\mathcal{F}(f \wedge g) \geq \mathcal{F}(f) \odot \mathcal{F}(g)$, for each $f, g \in L^X$.

(LF3) If $f \leq g$, then $\mathcal{F}(f) \leq \mathcal{F}(g)$.

An L-filter $\mathcal{F}$ is said to be stratified iff $\mathcal{F}$ satisfies the following condition:

(LFS) $\mathcal{F}(f \wedge \underline{\alpha}) \geq \mathcal{F}(f) \odot \alpha$, for each $f \in L^X$ and $\alpha \in L$.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be L-filters on X. We say $\mathcal{F}_1$ is finer than $\mathcal{F}_2$ ($\mathcal{F}_2$ is coarser than $\mathcal{F}_1$) if $\mathcal{F}_2(f) \leq \mathcal{F}_1(f)$ for all $f \in L^X$.

Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be L-filters on X and Y respectively. A mapping $\phi : (X, \mathcal{F}_1) \to (Y, \mathcal{F}_2)$ is called an L-filter map iff $\mathcal{F}_2 \leq \mathcal{F}_1 \circ \phi_L^{\leftarrow}$.

3. Stratified L-Filter Structure

In this section, we construct a stratified L-filter (resp., L-filterbase) from a given L-filter (resp., L-filterbase). Then, we induce an L-filter structure from a given family of L-filter structures and we study the stratification of it.

Theorem 3.1. *Let $\mathcal{F}$ be an L-filter on X. We define for all $f \in L^X$,*

$$\mathcal{F}^{\mathsf{st}}(f) = \bigvee \{\mathcal{F}(g) \odot \alpha \mid f \geq g \wedge \underline{\alpha}\}.$$

Then $\mathcal{F}^{\mathsf{st}}$ is the coarsest stratified L-filter on X which is finer than $\mathcal{F}$.

Proof. First, we will prove $\mathcal{F}^{\mathrm{st}}$ is stratified L-filter on X.

(LF1) and (LF3) are obvious.

(LF2) Suppose that there exist $f_1, f_2 \in L^X$, such that

$$\mathcal{F}^{\mathrm{st}}(f_1 \wedge f_2) \not\geq \mathcal{F}^{\mathrm{st}}(f_1) \odot \mathcal{F}^{\mathrm{st}}(f_2).$$

By the definition of $\mathcal{F}^{\mathrm{st}}$, there exist $g_i \in L^X$ and $\alpha_i \in L$ with $f_i \geq g_i \wedge \underline{\alpha_i}$, $i = 1, 2$ such that

$$\mathcal{F}^{\mathrm{st}}(f_1 \wedge f_2) \not\geq \mathcal{F}(g_1) \odot \mathcal{F}(g_2) \odot \alpha_1 \odot \alpha_2.$$

On the other hand, since

$$f_1 \wedge f_2 \geq (g_1 \wedge g_2) \wedge (\underline{\alpha_1} \wedge \underline{\alpha_2}) = (g_1 \wedge g_2) \wedge \underline{\sigma}, \text{ where } \underline{\sigma} = \underline{\alpha_1} \wedge \underline{\alpha_2}.$$

Then we have

$$\begin{aligned} \mathcal{F}^{\mathrm{st}}(f_1 \wedge f_2) &\geq \mathcal{F}(g_1 \wedge g_2) \odot \sigma \\ &\geq \mathcal{F}(g_1) \odot \mathcal{F}(g_2) \odot \sigma \\ &\geq \mathcal{F}(g_1) \odot \mathcal{F}(g_2) \odot \alpha_1 \odot \alpha_2, \end{aligned}$$

and this is a contradiction. Thus (LF2) holds.

(LFS) Suppose that there exist $f \in L^X$ and $\alpha \in L$, such that

$$\mathcal{F}^{\mathrm{st}}(f \wedge \underline{\alpha}) \not\geq \mathcal{F}^{\mathrm{st}}(f) \odot \alpha.$$

By the definition of $\mathcal{F}^{\mathrm{st}}$, there exist $g \in L^X$ and $\beta \in L$ with $f \geq g \wedge \underline{\beta}$ such that

$$\mathcal{F}^{\mathrm{st}}(f \wedge \underline{\alpha}) \not\geq \mathcal{F}(g) \odot \alpha \odot \beta.$$

On the other hand, since $f \wedge \underline{\alpha} \geq g \wedge \underline{\alpha} \wedge \underline{\beta} = g \wedge \underline{\sigma}$, where $\underline{\sigma} = \underline{\alpha} \wedge \underline{\beta}$. Then we have

$$\mathcal{F}^{\mathrm{st}}(f \wedge \underline{\alpha}) \geq \mathcal{F}(g) \odot \sigma \geq \mathcal{F}(g) \odot \alpha \odot \beta,$$

and this is a contradiction. Hence (LFS) holds.

Second, for each $f \in L^X$, $1 \in L$ with $f \geq f \wedge \underline{1}$ such that $\mathcal{F}^{\mathrm{st}}(f) \geq \mathcal{F}(f)$. Hence $\mathcal{F}^{\mathrm{st}}$ is finer than $\mathcal{F}$.

Finally, consider $\mathcal{F}^*$ is a stratified L-filter on X which is finer than $\mathcal{F}$, then $\mathcal{F}^*(f) \geq \mathcal{F}(f)$ for each $f \in L^X$. And we will show $\mathcal{F}^{\mathrm{st}} \leq \mathcal{F}^*$. Suppose there exists f such that

$$\mathcal{F}^*(f) \not\geq \mathcal{F}^{\mathrm{st}}(f).$$

By the definition of $\mathcal{F}^{\mathrm{st}}$, there exist $g \in L^X$ and $\alpha \in L$ with $f \geq g \wedge \underline{\alpha}$ such that

$$\mathcal{F}^*(f) \ngeq \mathcal{F}(g) \odot \alpha.$$

On the other hand, since $\mathcal{F}^*$ is stratified, then we have

$$\mathcal{F}^*(f) \geq \mathcal{F}^*(g \wedge \underline{\alpha}) \geq \mathcal{F}^*(g) \odot \alpha \geq \mathcal{F}(g) \odot \alpha,$$

and this is a contradiction. Thus $\mathcal{F}^{\mathrm{st}}$ is the coarsest stratified L-filter on X which is finer than $\mathcal{F}$. □

Notation. Let $\mathcal{B} : L^X \to L$ be a mapping and $f \in L^X$. We denote

$$\langle \mathcal{B} \rangle (f) = \bigvee_{g \leq f} \mathcal{B}(g).$$

Definition 3.2. A mapping $\mathcal{B} : L^X \to L$ is called an L- filterbase on X if it satisfies the following conditions:

(LB1) $\mathcal{B}(\underline{0}) = 0$ and $\mathcal{B}(\underline{1}) = 1$.

(LB2) $\langle \mathcal{B} \rangle (f \wedge g) \geq \mathcal{B}(f) \odot \mathcal{B}(g)$, for each $f, g \in L^X$.

An L-filterbase $\mathcal{B}$ is said to be stratified iff $\mathcal{B}$ satisfies the following condition:

(LBS) $\langle \mathcal{B} \rangle (f \wedge \underline{\alpha}) \geq \mathcal{B}(f) \odot \alpha$, for each $f \in L^X$ and $\alpha \in L$.

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be L-filterbases on X. We say $\mathcal{B}_1$ is finer than $\mathcal{B}_2$ ($\mathcal{B}_2$ is coarser than $\mathcal{B}_1$) if $\langle \mathcal{B}_1 \rangle (f) \geq \langle \mathcal{B}_2 \rangle (f)$, for all $f \in L^X$.

Theorem 3.3. (see [19]) *If $\mathcal{B} : L^X \to L$ is an L-filterbase, then $\langle \mathcal{B} \rangle$ is the coarsest L-filter satisfying $\mathcal{B} \leq \langle \mathcal{B} \rangle$.*

Theorem 3.4. (see [19]) *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be L-filterbases on X and Y, respectively. Let $\phi : X \to Y$ be a function. Then we have the following properties:*

(1) $\phi : (X, \langle \mathcal{B}_1 \rangle) \to (Y, \langle \mathcal{B}_2 \rangle)$ is an L-filter map iff $\langle \mathcal{B}_1 \rangle \circ \phi_L^{\leftarrow} \geq \mathcal{B}_2$.

(2) If $\mathcal{B}_2 \leq \mathcal{B}_1 \circ \phi_L^{\leftarrow}$, then $\phi : (X, \langle \mathcal{B}_1 \rangle) \to (Y, \langle \mathcal{B}_2 \rangle)$ is an L-filter map.

Theorem 3.5. *Let $\mathcal{B}$ be an L-filterbase on X. We define the mapping $\mathcal{B}^{\mathsf{st}} : L^X \to L$ as follows: for all $f \in L^X$,*

$$\mathcal{B}^{\mathsf{st}}(f) = \bigvee \{ \mathcal{B}(g) \odot \alpha \mid f \geq g \wedge \underline{\alpha} \}.$$

Then we have:

(1) $\mathcal{B}^{\mathsf{st}}$ *is the coarsest stratified* L*-filterbase on* X *which is satisfying* $\mathcal{B} \leq \mathcal{B}^{\mathsf{st}}$.

(2) $\langle \mathcal{B}^{\mathsf{st}} \rangle = \langle \mathcal{B} \rangle^{\mathsf{st}}$.

Proof. (1) (LB1) and (LBS) are similar to the proof of Theorem 3.1.

(LB2) Suppose that there exist $f_1, f_2 \in L^X$ such that

$$\langle \mathcal{B}^{\mathrm{st}} \rangle (f_1 \wedge f_2) \not\geq \mathcal{B}^{\mathrm{st}}(f_1) \odot \mathcal{B}^{\mathrm{st}}(f_2).$$

By the definition of $\mathcal{B}^{\mathrm{st}}$, there exist $g_i \in L^X$ and $\alpha_i \in L$ with $f_i \geq g_i \wedge \underline{\alpha_i}$, $i = 1, 2$ such that

$$\langle \mathcal{B}^{\mathrm{st}} \rangle (f_1 \wedge f_2) \not\geq \mathcal{B}(g_1) \odot \mathcal{B}(g_2) \odot \alpha_1 \odot \alpha_2.$$

Since, $\langle \mathcal{B} \rangle (g_1 \wedge g_2) \geq \mathcal{B}(g_1) \odot \mathcal{B}(g_2)$, then we have

$$\langle \mathcal{B}^{\mathrm{st}} \rangle (f_1 \wedge f_2) \not\geq \langle \mathcal{B} \rangle (g_1 \wedge g_2) \odot \alpha_1 \odot \alpha_2.$$

From the definition of $\langle \mathcal{B} \rangle$, there exists $h \in L^X$ with $h \leq g_1 \wedge g_2$ such that

$$\langle \mathcal{B}^{\mathrm{st}} \rangle (f_1 \wedge f_2) \not\geq \mathcal{B}(h) \odot \alpha_1 \odot \alpha_2.$$

On the other hand, since

$$f_1 \wedge f_2 \geq (g_1 \wedge g_2) \wedge (\underline{\alpha_1} \wedge \underline{\alpha_2}) = (g_1 \wedge g_2) \wedge \underline{\sigma} \geq h \wedge \underline{\sigma}, \text{ where } \underline{\sigma} = \underline{\alpha_1} \wedge \underline{\alpha_2},$$

then by the definition of $\langle \mathcal{B}^{\mathrm{st}} \rangle$, we have

$$\langle \mathcal{B}^{\mathrm{st}} \rangle (f_1 \wedge f_2) \geq \mathcal{B}^{\mathrm{st}}(f_1 \wedge f_2) \geq \mathcal{B}(h) \odot \sigma \geq \mathcal{B}(h) \odot \alpha_1 \odot \alpha_2,$$

and this is a contradiction. Thus (LB2) holds. Therefore, $\mathcal{B}^{\mathrm{st}}$ is a stratified L-filterbase on X. It can be easily seen that $\mathcal{B} \leq \mathcal{B}^{\mathrm{st}}$ and also it can be easily seen that $\mathcal{B}^{\mathrm{st}}$ is finer than $\mathcal{B}$. Now let us show that $\mathcal{B}^{\mathrm{st}}$ is the coarsest stratified L-filterbase satisfying $\mathcal{B} \leq \mathcal{B}^{\mathrm{st}}$. Let $\mathcal{B}^*$ be an stratified L-filterbase on X which satisfies $\mathcal{B}^* \geq \mathcal{B}$. Suppose that there exists $f \in L^X$ such that

$$\langle \mathcal{B}^* \rangle (f) \not\geq \langle \mathcal{B}^{\mathrm{st}} \rangle (f).$$

By the definition of $\langle \mathcal{B}^{\mathrm{st}} \rangle$, there exists $g \in L^X$ with $g \leq f$ such that

$$\langle \mathcal{B}^* \rangle (f) \not\geq \mathcal{B}^{\mathrm{st}}(g).$$

By the definition of $\mathcal{B}^{\rm st}$, there exist $h \in L^X$ and $\alpha \in L$ with $g \geq h \wedge \underline{\alpha}$ such that $\langle \mathcal{B}^* \rangle(f) \not\geq \mathcal{B}(h) \odot \alpha$. Since $f \geq g \geq h \wedge \underline{\alpha}$, then

$$\langle \mathcal{B}^* \rangle(f) \geq \langle \mathcal{B}^* \rangle(h \wedge \underline{\alpha}) \geq \mathcal{B}^*(h) \odot \alpha \geq \mathcal{B}(h) \odot \alpha.$$

So, we have that $\langle \mathcal{B}^{\rm st}(f) \leq \langle \mathcal{B}^* \rangle(f)$. It contradicts with the assumption.

(2) From (1), we can easily prove $\langle \mathcal{B}^{\rm st} \rangle \geq \langle \mathcal{B} \rangle^{\rm st}$.

Conversely, suppose that there exists $f \in L^X$, such that

$$\langle \mathcal{B}^{\rm st} \rangle(f) \not\leq \langle \mathcal{B} \rangle^{\rm st}(f).$$

By the definition of $\langle \mathcal{B}^{\rm st} \rangle$, there exists $g \in L^X$ with $g \leq f$ such that

$$\mathcal{B}^{\rm st}(g) \not\leq \langle \mathcal{B} \rangle^{\rm st}(f).$$

By the definition of $\mathcal{B}^{\rm st}$, there exist $h \in L^X$ and $\alpha \in L$ with $g \geq h \wedge \underline{\alpha}$ such that

$$\langle \mathcal{B} \rangle^{\rm st}(f) \not\geq \mathcal{B}(h) \odot \alpha.$$

On the other hand, since $f \geq h \wedge \underline{\alpha}$, then we have

$$\langle \mathcal{B} \rangle^{\rm st}(f) \geq \langle \mathcal{B} \rangle(h) \odot \alpha \geq \mathcal{B}(h) \odot \alpha,$$

and this is a contradiction. Thus, $\langle \mathcal{B}^{\rm st} \rangle(f) \leq \langle \mathcal{B} \rangle^{\rm st}(f)$. □

Theorem 3.6. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be L-filters on X and Y, respectively.*

If $\phi : (X, \mathcal{F}_1) \to (Y, \mathcal{F}_2)$ is an L-filter map, then $\phi : (X, \mathcal{F}_1^{\rm st}) \to (Y, \mathcal{F}_2^{\rm st})$ is an L-filter map.

Proof. Suppose there exists $f \in L^Y$ such that

$$\mathcal{F}_2^{\rm st}(f) \not\leq \mathcal{F}_1^{\rm st} \circ \phi_L^{\leftarrow}(f).$$

By the definition of $\mathcal{F}_2^{\rm st}$, there exist $g \in L^Y$ and $\alpha \in L$ with $f \geq g \wedge \underline{\alpha}$ such that

$$\mathcal{F}_1^{\rm st} \circ \phi_L^{\leftarrow}(f) \not\geq \mathcal{F}_2(g) \odot \alpha.$$

Since $f : (X, \mathcal{F}_1) \to (Y, \mathcal{F}_2)$ is L-filter map, then $\mathcal{F}_1 \circ \phi_L^{\leftarrow}(g) \geq \mathcal{F}_2(g)$. On the other hand, since $\phi_L^{\leftarrow}(f) \geq \phi_L^{\leftarrow}(g) \wedge \underline{\alpha}$, we have

$$\mathcal{F}_1^{\rm st} \circ \phi_L^{\leftarrow}(f) \geq \mathcal{F}_1 \circ \phi_L^{\leftarrow}(g) \odot \alpha \geq \mathcal{F}_2(g) \odot \alpha,$$

and this is a contradiction. Thus, $\mathcal{F}_2^{\rm st}(f) \leq \mathcal{F}_1^{\rm st} \circ \phi_L^{\leftarrow}(f), \forall f \in L^Y$. □

Following example shows that the converse of Theorem 3.6 need not be true.

Example. Let X be any set, $L=[0,1]$ and $\odot=\wedge$. Define L-filterbases $\mathcal{B}_1$ and $\mathcal{B}_2$ on X as follows

$$\mathcal{B}_1(f)=\begin{cases}1, & \text{if } f=\underline{1}\\ 0, & \text{otherwise,}\end{cases} \quad \text{and } \mathcal{B}_2(f)=\begin{cases}1, & \text{if } f=\underline{1}\\ \frac{1}{3}, & \text{if } f=\underline{0.5}\\ 0, & \text{otherwise.}\end{cases}$$

By Theorem 3.3, we obtain

$$\mathcal{B}_1^{\text{st}}(f)=\mathcal{B}_2^{\text{st}}(f)=\begin{cases}\alpha, & \text{if } f=\underline{\alpha} \text{ for each } \alpha\in L\\ 0, & \text{otherwise.}\end{cases}$$

Clearly the identity mapping $id_X:(X,\mathcal{B}_1^{\text{st}})\to(X,\mathcal{B}_2^{\text{st}})$ is an L-filter map, but $id_X:(X,\mathcal{B}_1)\to(X,\mathcal{B}_2)$ is not an L-filter map.

Notation. We denote

$$\mathcal{F}^\circ=\{f\in L^X \mid \mathcal{F}(f)\neq 0\}.$$

Theorem 3.7. *Let $\{\mathcal{F}_i\}_{i\in\Gamma}$ be a family of L-filters on X satisfying the following condition:*

(F) If $f_i\in(\mathcal{F}_i)^\circ$ for all $i\in\Gamma$, then we have $\bigwedge_{i\in K} f_i\neq\underline{0}$ for every finite index subset K of Γ. We define a function $\bigsqcup_{i\in\Gamma}\mathcal{F}_i:L^X\to L$ as

$$\bigsqcup_{i\in\Gamma}\mathcal{F}_i(g)=\begin{cases}\bigvee\{\bigodot_{i\in K}\mathcal{F}_i(g_i)\}, & \text{if } g=\bigwedge_{i\in K} g_i,\, g_i\in(\mathcal{F}_i)^\circ,\\ 0, & \text{otherwise,}\end{cases}$$

where the first $\bigvee$ is taken over all finite index subset K of Γ such that $g=\bigwedge_{i\in K} g_i$. If there exists $k_0\in\Gamma$ such that $\overline{\mathcal{F}_{k_0}}=\mathcal{F}_{k_0}^{\text{st}}$ and $\overline{\mathcal{F}_i}=\mathcal{F}_i$ for each $i\in\Gamma-\{k_0\}$. Then:

(1) $\bigsqcup_{i\in\Gamma}\mathcal{F}_i$ is the coarsest L-filter on X which is finer than $\mathcal{F}_i$ for each $i\in\Gamma$.

(2) $\bigsqcup_{i\in\Gamma}\overline{\mathcal{F}_i}$ is the coarsest stratified L-filter on X which is finer than $\mathcal{F}_i$ for each $i\in\Gamma$.

(3) $\bigsqcup_{i\in\Gamma}\overline{\mathcal{F}_i}=(\bigsqcup_{i\in\Gamma}\mathcal{F}_i)^{\text{st}}$.

Proof. (1) Firstly, we will show that $\mathcal{H} = \bigsqcup_{i\in\Gamma} \mathcal{F}_i$ is an L-filter on X.

(LF1) It is trivial that $\mathcal{H}(\underline{0}) = 0$. Since $\underline{1} = \underline{1} \wedge \underline{1}$, we have $\mathcal{H}(\underline{1}) = 1$.

(LF2) For every finite index subsets J and K of Γ such that $f = \bigwedge_{j\in J} f_j$ and $g = \bigwedge_{k\in K} g_k$, we have $f \wedge g = (\bigwedge_{j\in J} f_j) \wedge (\bigwedge_{k\in K} g_k)$. Furthermore, put for $m \in J \cup K$, $f \wedge g = \bigwedge_{m\in J\cup K} h_m$, where

$$h_m = \begin{cases} f_m & \text{if } m \in J - (K \cap J) \\ g_m & \text{if } m \in K - (K \cap J) \\ f_m \wedge g_m & \text{if } m \in (K \cap J). \end{cases}$$

We have

$$\mathcal{H}(f \wedge g) \geq \bigodot_{m\in J\cup K} \mathcal{F}_m(h_m) \geq (\bigodot_{j\in J} \mathcal{F}_j(f_j)) \odot (\bigodot_{k\in K} \mathcal{F}_k(g_k)).$$

If we take $\vee$ over $f = \bigwedge_{j\in J} f_j$ and $g = \bigwedge_{k\in K} g_k$, then by (L3) of Definition 2.1, we have $\mathcal{H}(f \wedge g) \geq \mathcal{H}(f) \odot \mathcal{H}(g)$.

(LF3) Suppose that $f \leq g$, by the definition of $\mathcal{H}$, there exists a finite index set J of Γ with $f = \bigwedge_{j\in J} f_j$ such that

$$\mathcal{H}(f) \geq \bigodot_{j\in J} \mathcal{F}_j(f_j).$$

On the other hand, since $g = f \vee g = \bigwedge_{j\in J}(f_j \vee g)$, we have

$$\mathcal{H}(g) \geq \bigodot_{j\in J} \mathcal{F}_j(f_j \vee g) \geq \bigodot_{j\in J} \mathcal{F}_j(f_j).$$

Then, by (L3) of Definition 2.1, we have $\mathcal{H}(f) \leq \mathcal{H}(g)$.

Second, we will show that $\mathcal{H}(f) \geq \mathcal{F}_i(f)$ for each $i \in \Gamma$ from the following:

If $\mathcal{F}_i(f) = 0$, it is trivial.

If $\mathcal{F}_i(f) \neq 0$, for $f = f \wedge \underline{1}$, we have $\mathcal{H}(f) \geq \mathcal{F}_i(f) \odot \mathcal{F}_i(\underline{1}) = \mathcal{F}_i(f)$.

Finally, suppose that $\mathcal{F} \geq \mathcal{F}_i$ for each $i \in \Gamma$, we will show that $\mathcal{F} \geq \mathcal{H}$. By the definition of $\mathcal{H}$, there exists a finite index set J of Γ with $f = \bigwedge_{j\in J} f_j$ such that

$$\mathcal{H}(f) \geq \bigodot_{j\in J} \mathcal{F}_j(f_j).$$

On the other hand, since $\mathcal{F} \geq \mathcal{F}_i$ for all $i \in J$, we have

$$\mathcal{F}(f) \geq \bigodot_{j\in J} \mathcal{F}(f_j) \geq \bigodot_{j\in J} \mathcal{F}_j(f_j).$$

Then, by (L3) of Definition 2.1, we have $\mathcal{F} \geq \mathcal{H}$.

(2) First, we will show the family $\{\overline{\mathcal{F}_i}\}_{i\in\Gamma}$ satisfies the condition (F). Let $g_i \in (\overline{\mathcal{F}_i})^\circ$ for all $i \in \Gamma$. If $k_0 \notin M$, then we have $\bigwedge_{i\in M} g_i \neq \underline{0}$ for every finite index subset M of Γ. If $k_0 \in M$ and $g_{k_0} \in (\overline{\mathcal{F}_{k_0}})^\circ = (\mathcal{F}_{k_0}^{\text{st}})^\circ$, then there exist $g_k \in L^X$ and $\alpha \in L$ with $g_{k_0} \geq g_k \wedge \underline{\alpha}$ such that

$$\mathcal{F}_{k_0}^{\text{st}}(g_{k_0}) \geq \mathcal{F}_{k_0}(g_k) \odot \alpha \neq 0.$$

Then $\mathcal{F}_{k_0}(g_k) \neq 0$ and $\alpha \neq 0$, for each $k \in K$. From the condition of (F), $g_k \wedge (\bigwedge_{i\in M-\{k_0\}} g_i) \neq \underline{0}$, for each $k \in K$. So we have

$$\bigwedge_{i\in M} g_i = g_{k_0} \wedge (\bigwedge_{i\in M-\{k_0\}} g_i) \geq g_k \wedge (\bigwedge_{i\in M-\{k_0\}} g_i) \wedge \underline{\alpha} \neq \underline{0}.$$

Thus, $\bigwedge_{i\in M} g_i \neq \underline{0}$. Hence $\bigsqcup_{i\in\Gamma} \overline{\mathcal{F}_i}$ is the coarsest L-filter on X which is finer than $\overline{\mathcal{F}_i}$ for each $i \in \Gamma$. Since $\mathcal{F}_{k_0}^{\text{st}}$ is finer than $\mathcal{F}_{k_0}$, then $\bigsqcup_{i\in\Gamma} \overline{\mathcal{F}_i}$ is the coarsest L-filter on X which is finer than $\mathcal{F}_i$ for each $i \in \Gamma$. Now, we will show that $\bigsqcup_{i\in\Gamma} \overline{\mathcal{F}_i}$ is stratified. Let $f \in L^X$ and $\alpha \in L$. Then we have

$$\begin{aligned}
\alpha \odot (\bigsqcup_{i\in\Gamma} \overline{\mathcal{F}_i})(f) &= \alpha \odot (\bigvee\{ \bigodot_{i\in K-\{k_0\}} \overline{\mathcal{F}_i}(f_i) \odot \mathcal{F}_{k_0}^{\text{st}}(f_{k_0})\}) \\
&= \bigvee\{ \bigodot_{i\in K-\{k_0\}} \overline{\mathcal{F}_i}(f_i) \odot (\alpha \odot \mathcal{F}_{k_0}^{\text{st}}(f_{k_0}))\} \\
&\leq \bigvee\{ \bigodot_{i\in K-\{k_0\}} \overline{\mathcal{F}_i}(f_i) \odot \mathcal{F}_{k_0}^{\text{st}}(\underline{\alpha} \wedge f_{k_0})\} \\
&= \bigvee\{\bigodot_{i\in K} \overline{\mathcal{F}_i}(p_i)\} \\
&\leq \bigsqcup_{i\in\Gamma} \overline{\mathcal{F}_i}(\underline{\alpha} \wedge f),
\end{aligned}$$

where $p_i = f_i$ if $i \in K - \{k_0\}$ and $p_i = \underline{\alpha} \wedge f_i$ if $i = k_0$. Thus, $\bigsqcup_{i\in\Gamma} \overline{\mathcal{F}_i}$ is stratified.

(3) Clearly $\bigsqcup_{i\in\Gamma} \overline{\mathcal{F}_i}$ is stratified finer than $\bigsqcup_{i\in\Gamma} \mathcal{F}_i$. By Theorem 3.1 , $\bigsqcup_{i\in\Gamma} \overline{\mathcal{F}_i} \geq (\bigsqcup_{i\in\Gamma} \mathcal{F}_i)^{\text{st}}$. We only show that $\bigsqcup_{i\in\Gamma} \overline{\mathcal{F}_i} \leq (\bigsqcup_{i\in\Gamma} \mathcal{F}_i)^{\text{st}}$. Suppose

there exists $f \in L^X$ such that

$$\bigsqcup_{i\in\Gamma} \overline{\mathcal{F}_i}(f) \not\leq (\bigsqcup_{i\in\Gamma} \mathcal{F}_i)^{\mathrm{st}}(f).$$

By the definition of $\bigsqcup_{i\in\Gamma} \overline{\mathcal{F}_i}$, there exists a finite family $\{f_i \mid f = \bigwedge_{i\in K} f_i\}$ such that

$$\bigodot_{i\in K} \overline{\mathcal{F}_i}(f_i) \not\leq (\bigsqcup_{i\in\Gamma} \mathcal{F}_i)^{\mathrm{st}}(f).$$

If $k_0 \notin K$, then by the definition of $\bigsqcup_{i\in\Gamma} \mathcal{F}_i$, we have

$$(\bigsqcup_{i\in\Gamma} \mathcal{F}_i)^{\mathrm{st}}(f) \geq (\bigsqcup_{i\in\Gamma} \mathcal{F}_i)(f) \geq \bigodot_{i\in K} \mathcal{F}_i(f_i) = \bigodot_{i\in K} \overline{\mathcal{F}_i}(f_i).$$

It is a contradiction. If $k_0 \in K$, then

$$(\bigodot_{i\in K-\{k_0\}} \overline{\mathcal{F}_i}(f_i)) \odot \overline{\mathcal{F}_{k_0}}(f_{k_0}) = (\bigodot_{i\in K-\{k_0\}} \overline{\mathcal{F}_i}(f_i)) \odot \mathcal{F}^{\mathrm{st}}_{k_0}(f_{k_0}).$$

By the definition of $\mathcal{F}^{\mathrm{st}}_{k_0}(f_{k_0})$, there exist $g \in L^X$ and $\alpha \in L$ with $f_{k_0} \geq g \wedge \underline{\alpha}$ such that

$$(\bigsqcup_{i\in\Gamma} \mathcal{F}_i)^{\mathrm{st}}(f) \not\geq (\bigodot_{i\in K-\{k_0\}} \overline{\mathcal{F}_i}(f_i)) \odot (\mathcal{F}_{k_0}(g) \odot \alpha).$$

Since $f = (\bigwedge_{i\in K-\{k_0\}} f_i) \wedge f_{k_0} \geq (\bigwedge_{i\in K-\{k_0\}} f_i) \wedge g \wedge \underline{\alpha}$, then we have

$$\begin{aligned}(\bigsqcup_{i\in\Gamma} \mathcal{F}_i)^{\mathrm{st}}(f) &\geq \bigsqcup_{i\in\Gamma} \mathcal{F}_i((\bigwedge_{i\in K-\{k_0\}} f_i) \wedge g) \odot \alpha \\ &\geq \bigodot_{i\in K-\{k_0\}} (\mathcal{F}_i(f_i) \odot \mathcal{F}_{k_0}(g)) \odot \alpha \\ &= \bigodot_{i\in K-\{k_0\}} \overline{\mathcal{F}_i}(f_i) \odot (\mathcal{F}_{k_0}(g) \odot \alpha).\end{aligned}$$

It is a contradiction. Thus, $\bigsqcup_{i\in\Gamma} \overline{\mathcal{F}_i} \leq (\bigsqcup_{i\in\Gamma} \mathcal{F}_i)^{\mathrm{st}}$. □

Corollary 3.8. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be L-filters on X satisfying the following condition:*

(F) If $f_i \in (\mathcal{F}_i)^\circ$ for all $i = 1, 2$, then we have $f_1 \wedge f_2 \neq \underline{0}$. The function $\mathcal{F}_1 \sqcup \mathcal{F}_2 : L^X \to L$ defined as

$$\mathcal{F}_1 \sqcup \mathcal{F}_2(f) = \bigvee_{f=f_1\wedge f_2} \{\mathcal{F}_1(f_1) \odot \mathcal{F}_2(f_2)\},$$

is the coarsest L-filter on X which is finer than both of $\mathcal{F}_1$ and $\mathcal{F}_2$. Then:

(1) $\mathcal{F}_1^{\mathsf{st}} \sqcup \mathcal{F}_2$ and $\mathcal{F}_1 \sqcup \mathcal{F}_2^{\mathsf{st}}$ are the coarsest stratified L-filters on X, which finer than both of $\mathcal{F}_1$ and $\mathcal{F}_2$.

(2) $\mathcal{F}_1^{\mathsf{st}} \sqcup \mathcal{F}_2 = \mathcal{F}_1 \sqcup \mathcal{F}_2^{\mathsf{st}} = (\mathcal{F}_1 \sqcup \mathcal{F}_2)^{\mathsf{st}}$.

4. The Type $(\phi_1^{\Leftarrow}, \phi_2^{\Rightarrow})$ of the Preimages and Images of L-filter Structure

Basic scheme for second order preimage operator. Let $\phi : X \to Y$ be a function. Consider $[\phi_L^{\leftarrow}]_L^{\rightarrow} : L^{L^X} \leftarrow L^{L^Y}$. This is the Zadeh image operator of the Zadeh preimage operator. We denote it by $\phi_1^{\Leftarrow}$, that is, for all $\mathcal{V} \in L^{L^Y}$, $\forall f \in L^X$,

$$\phi_1^{\Leftarrow}(\mathcal{V})(f) = [\phi_L^{\leftarrow}]_L^{\rightarrow}(\mathcal{V})(f) = \bigvee\{\mathcal{V}(g) \mid f = \phi_L^{\leftarrow}(g)\}.$$

Basic scheme for second order image operator. Let $\phi : X \to Y$ be a function. Consider $[\phi_L^{\leftarrow}]_L^{\leftarrow} : L^{L^X} \to L^{L^Y}$. This is the Zadeh preimage operator of the Zadeh preimage operator. We denote it by $\phi_2^{\Rightarrow}$, that is, for all $\mathcal{U} \in L^{L^X}$, $\forall g \in L^Y$,

$$\phi_2^{\Rightarrow}(\mathcal{U})(g) = [\phi_L^{\leftarrow}]_L^{\leftarrow}(\mathcal{U})(g) = \mathcal{U} \circ \phi_L^{\leftarrow}(g).$$

In this section we consider the preimages and images of L-filter with respect to the pair $(\phi_1^{\Leftarrow}, \phi_2^{\Rightarrow})$.

Theorem 4.1. *If $\phi_L^{\leftarrow}(h) = \underline{0}$ implies $\mathcal{F}(h) = 0$, $\phi : X \to Y$ is injective function and $\mathcal{F}$ is an L-filter on Y. Then we have the following properties:*

(1) $\phi_1^{\Leftarrow}(\mathcal{F})$ is an L-filter on X.

(2) $(\phi_1^{\Leftarrow}(\mathcal{F}))^{\mathsf{st}} = \phi_1^{\Leftarrow}(\mathcal{F}^{\mathsf{st}})$.

Proof. (1) (LF1) Since $\phi_L^{\leftarrow}(\underline{1}) = \underline{1}$, $\phi_1^{\Leftarrow}(\mathcal{F})(\underline{1}) = 1$. By the assumption $\phi_1^{\Leftarrow}(\mathcal{F})(\underline{0}) = 0$.

(LF2) Suppose there exist $f_1, f_2 \in L^X$ such that

$$\phi_1^{\Leftarrow}(\mathcal{F})(f_1 \wedge f_2) \not\geq \phi_1^{\Leftarrow}(\mathcal{F})(f_1) \odot \phi_1^{\Leftarrow}(\mathcal{F})(f_2).$$

By the definition of $\phi_1^{\Leftarrow}(\mathcal{F})(f_i)$ for $i \in \{1,2\}$, there exists $h_i \in L^Y$ with $f_i = \phi_L^{\leftarrow}(h_i)$ such that

$$\phi_1^{\Leftarrow}(\mathcal{F})(f_1 \wedge f_2) \not\geq \mathcal{F}(h_1) \odot \mathcal{F}(h_2).$$

On the other hand, $f_1 \wedge f_2 = \phi_L^{\leftarrow}(h_1) \wedge \phi_L^{\leftarrow}(h_2) = \phi_L^{\leftarrow}(h_1 \wedge h_2)$,

$$\phi_1^{\Leftarrow}(\mathcal{F})(f_1 \wedge f_2) \geq \mathcal{F}(h_1 \wedge h_2) \geq \mathcal{F}(h_1) \odot \mathcal{F}(h_2).$$

It is a contradiction. Hence $\phi_1^{\Leftarrow}(\mathcal{F})(f_1 \wedge f_2) \geq \phi_1^{\Leftarrow}(\mathcal{F})(f_1) \odot \phi_1^{\Leftarrow}(\mathcal{F})(f_2)$.

(LF3) Let $f \leq g$ for $f, g \in L^X$. Since ϕ is injective, there exists $h \in L^Y$ with $h \circ \phi = f$ and $g = \phi_L^{\leftarrow}(h \vee \phi_L^{\rightarrow}(g))$. It implies

$$\phi_1^{\Leftarrow}(\mathcal{F})(g) \geq \mathcal{F}(h \vee \phi_L^{\rightarrow}(g)) \geq \mathcal{F}(h).$$

Hence $\phi_1^{\Leftarrow}(\mathcal{F})(g) \geq \phi_1^{\Leftarrow}(\mathcal{F})(f)$.

(2) We will show that $\phi_1^{\Leftarrow}(\mathcal{F}^{\mathrm{st}})$ is stratified. Let $f \in L^X$ and $\alpha \in L$. Then

$$\begin{aligned}
\alpha \odot \phi_1^{\Leftarrow}(\mathcal{F}^{\mathrm{st}})(f) &= \alpha \odot \Big(\bigvee_{f=\phi_L^{\leftarrow}(g)} \mathcal{F}^{\mathrm{st}}(g)\Big) \\
&= \bigvee_{f=\phi_L^{\leftarrow}(g)} \alpha \odot \mathcal{F}^{\mathrm{st}}(g) \\
&\leq \bigvee_{f\wedge\underline{\alpha}=\phi_L^{\leftarrow}(g\wedge\underline{\alpha})} \mathcal{F}^{\mathrm{st}}(g \wedge \underline{\alpha}) \\
&= \phi_1^{\Leftarrow}(\mathcal{F}^{\mathrm{st}})(f \wedge \underline{\alpha}).
\end{aligned}$$

Hence $\phi_1^{\Leftarrow}(\mathcal{F}^{\mathrm{st}})$ is stratified. By Theorem 3.1, $\phi_1^{\Leftarrow}(\mathcal{F}^{\mathrm{st}}) \geq (\phi_1^{\Leftarrow}(\mathcal{F}))^{\mathrm{st}}$.

Finally we will show $\phi_1^{\Leftarrow}(\mathcal{F}^{\mathrm{st}}) \leq (\phi_1^{\Leftarrow}(\mathcal{F}))^{\mathrm{st}}$. Suppose there exists $f \in L^X$ such that

$$\phi_1^{\Leftarrow}(\mathcal{F}^{\mathrm{st}})(f) \not\leq (\phi_1^{\Leftarrow}(\mathcal{F}))^{\mathrm{st}}(f).$$

By the definition of $\phi_1^{\Leftarrow}(\mathcal{F}^{\mathrm{st}})$, there exists $g \in L^Y$ with $f = \phi_L^{\leftarrow}(g)$ such that $\mathcal{F}^{\mathrm{st}}(g) \not\leq (\phi_1^{\Leftarrow}(\mathcal{F}))^{\mathrm{st}}(f)$. By the definition of $\mathcal{F}^{\mathrm{st}}$, there exist $h \in L^X$ and $\alpha \in L$ with $g \geq h \wedge \underline{\alpha}$ such that

$$\mathcal{F}(h) \odot \alpha \not\leq (\phi_1^{\Leftarrow}(\mathcal{F}))^{\mathrm{st}}(f).$$

On the other hand, $f = \phi_L^{\leftarrow}(g) \geq (\phi_L^{\leftarrow}(h) \wedge \underline{\alpha})$, then

$$(\phi_1^{\Leftarrow}(\mathcal{F}))^{\mathrm{st}}(f) \geq \phi_1^{\Leftarrow}(\mathcal{F})(\phi_L^{\leftarrow}(h)) \odot \alpha \geq \mathcal{F}(h) \odot \alpha.$$

It is a contradiction. Hence $\phi_1^{\Leftarrow}(\mathcal{F}^{\mathrm{st}}) \leq (\phi_1^{\Leftarrow}(\mathcal{F}))^{\mathrm{st}}$. □

Theorem 4.2. *Let $\{\mathcal{F}_i\}_{i\in\Gamma}$ be a family of L-filters on X_i and $\phi_i : X \to X_i$ injective functions, for each $i \in \Gamma$. We define a function $\bigsqcup_{i\in\Gamma}(\phi_i)_1^{\Leftarrow}(\mathcal{F}_i) : L^X \to L$ as:*

$$\bigsqcup_{i\in\Gamma}(\phi_i)_1^{\Leftarrow}(\mathcal{F}_i)(f) = \begin{cases} \bigvee\{\bigodot_{i\in K} \mathcal{F}_i(f_i)\}, & \text{if } f = \bigwedge_{i\in K}(\phi_i)_L^{\leftarrow}(f_i), \quad f_i \in (\mathcal{F}_i)^{\circ}, \\ 0, & \text{otherwise,} \end{cases}$$

where $\bigvee$ *is taken over all finite index subset* K *of* Γ *such that* $f = \bigwedge_{i\in K}(\phi_i)_L^{\leftarrow}(f_i)$. *If there exists* $k_0 \in \Gamma$ *such that* $\overline{\mathcal{F}_{k_0}} = \mathcal{F}_{k_0}^{\mathsf{st}}$ *and* $\overline{\mathcal{F}_i} = \mathcal{F}_i$ *for each* $i \in \Gamma - \{k_0\}$. *Then:*

(1) $\bigsqcup_{i\in\Gamma}(\phi_i)_1^{\Leftarrow}(\mathcal{F}_i)$ *is the coarsest* L*- filter on* X *for which each function* $\phi_i : X \to X_i$ *is an* L*-filter map.*

(2) A function $\psi : Y \to X$ *is an* L*-filter map iff for each* $i \in \Gamma$, $\phi_i \circ \psi : Y \to X_i$ *is an* L*-filter map.*

(3) $\bigsqcup_{i\in\Gamma}\overline{\mathcal{F}_i} = \bigsqcup_{i\in\Gamma}(\phi_i)_1^{\Leftarrow}(\overline{\mathcal{F}_i})$.

(4) $\bigsqcup_{i\in\Gamma}\overline{\mathcal{F}_i}$ *is the coarsest stratified* L*-filter on* X *for which each function* $\phi_i : X \to X_i$ *is* L*-filter map.*

Proof. (1) From Theorem 4.1, each $(\phi_i)_1^{\Leftarrow}(\mathcal{F}_i)$ is an L-filter on X. Firstly, we will show that $\bigsqcup_{i\in\Gamma}(\phi_i)_1^{\Leftarrow}(\mathcal{F}_i)$ exists, that is, it satisfies the condition (F) of Theorem 3.7.

(F) If $g_i \in ((\phi_i)_1^{\Leftarrow}(\mathcal{F}_i))^{\circ}$ for all $i \in \Gamma$, then there exists $h_i \in L^{X_i}$ with $g_i = (\phi_i)_L^{\leftarrow}(h_i)$ such that $\mathcal{F}_i(h_i) \neq 0$. It implies $h_i \neq \underline{0}$, that is, there exists $x_i \in X_i$ with $h_i(x_i) \neq 0$. For every finite index subset K of Γ, put

$$x = \begin{cases} \phi_i(x) = x_i, \text{ if } x_i \in X_i \text{ for each } i \in K \\ \phi_j(x) = x_j, \text{ if } x_j \in X_j \text{ for each } j \in \Gamma - K. \end{cases}$$

Then we have

$$\bigwedge_{i\in K} g_i(x) = \bigwedge_{i\in K}(\phi_i)_L^{\leftarrow}(h_i)(x) = \bigwedge_{i\in K} h_i(x_i) \neq 0.$$

Then, from Theorems 3.7 and 4.1, $\bigsqcup_{i\in\Gamma}(\phi_i)_1^{\Leftarrow}(\mathcal{F}_i)$ is an L-filter on X. For each $i \in \Gamma$ and $h_i \in L^{X_i}$, we have

$$\bigsqcup_{i\in\Gamma}(\phi_i)_1^{\Leftarrow}(\mathcal{F}_i)((\phi_i)_L^{\leftarrow}(h_i)) \geq \mathcal{F}_i(h_i).$$

Hence $\phi_i : (X, \bigsqcup_{i\in\Gamma}(\phi_i)_1^{\Leftarrow}(\mathcal{F}_i)) \to (X_i, \mathcal{F}_i)$ is an L-filter map.

Let $\phi_i : (X, \mathcal{F}) \to (X_i, \mathcal{F}_i)$ be an L-filter map for each $i \in \Gamma$, that is, $\mathcal{F}((\phi_i)_L^{\leftarrow}(h_i)) \geq \mathcal{F}_i(h_i)$. For all finite index subset K of Γ with

$$g = \bigwedge_{k\in K}(\phi_k)_L^{\leftarrow}(h_k),$$

we have

$$\mathcal{F}(g) \geq \bigodot_{k\in K}\mathcal{F}((\phi_k)_L^{\leftarrow}(h_k)) \geq \bigodot_{k\in K}\mathcal{F}_k(h_k).$$

It implies $\mathcal{F}(g) \geq \bigsqcup_{i\in\Gamma}((\phi_i)_1^{\Leftarrow}(\mathcal{F}_i))(g)$ for each $g \in L^X$.

(2) Necessity of the composition condition is clear since the composition of L-filter maps is an L-filter map.

Conversely, suppose $\psi : (Y, \mathcal{F}') \to (X, \bigsqcup_{i\in\Gamma}(\phi_i)_1^{\Leftarrow}(\mathcal{F}_i))$ is a map. For every finite index subset K with $g = \bigwedge_{k\in K}(\phi_k)_L^{\leftarrow}(h_k)$, since for each $i \in \Gamma$, $\phi_i \circ \psi : (Y, \mathcal{F}') \to (X_i, \mathcal{F}_i)$ is an L-filter map,

$$\mathcal{F}_i(h_i) \leq \mathcal{F}'((\phi_i \circ \psi)_L^{\leftarrow}(h_i)).$$

It follows $\mathcal{F}_k(h_k) \leq \mathcal{F}'((\phi_k \circ \psi)_L^{\leftarrow}(h_k))$ for all $k \in K$. Hence we have

$$\mathcal{F}' \circ \psi_L^{\leftarrow}(g) = \mathcal{F}'(\psi_L^{\leftarrow}(\bigwedge_{k\in K} \phi_k^{\leftarrow}(h_k))) \geq \bigodot_{k\in K} \mathcal{F}'((\phi_k \circ \psi)_L^{\leftarrow}(h_k)) \geq \bigodot_{k\in K} \mathcal{F}_k(h_k).$$

It implies $\mathcal{F}' \circ \psi_L^{\leftarrow}(g) \geq \bigsqcup_{i\in\Gamma}(\phi_i)_1^{\Leftarrow}(\mathcal{F}_i)(g)$ for all $g \in L^X$. Thus $\psi : (Y, \mathcal{F}') \to (X, \bigsqcup_{i\in\Gamma}(\phi_i)_1^{\Leftarrow}(\mathcal{F}_i))$ is an L-filter map.

(3) From Theorem 4.1, each $(\phi_i)_1^{\Leftarrow}(\overline{\mathcal{F}_i})$ is an L-filter on X. To prove (3), we only show that $\bigsqcup_{i\in\Gamma}(\phi_i)_1^{\Leftarrow}(\overline{\mathcal{F}_i})$ satisfies the condition (F) of Theorem 3.7.

(F) If $g_i \in ((\phi_i)_1^{\Leftarrow}(\overline{\mathcal{F}_i}))^\circ$ for all $i \in \Gamma$, then there exists $h_i \in L^{X_i}$ with $g_i = (\phi_i)_L^{\leftarrow}(h_i)$ such that $\overline{\mathcal{F}_i}(h_i) \neq 0$. It implies $h_i \neq \underline{0}$, that is, there exists $x_i \in X_i$ with $h_i(x_i) \neq 0$. For every finite index subset M of Γ, put

$$x = \begin{cases} \phi_i(x) = x_i, \text{ if } x_i \in X_i \text{ for each } i \in M \\ \phi_j(x) = x_j, \text{ if } x_j \in X_j \text{ for each } j \in \Gamma - M. \end{cases}$$

Then we have:

$$\bigwedge_{i\in M} g_i(x) = \bigwedge_{i\in M} (\phi_i)_L^{\leftarrow}(h_i)(x) = \bigwedge_{i\in M} h_i(x_i) \neq 0,$$

for every finite index subset M of Γ iff $k_0 \notin M$. If $k_0 \in M$, then by Theorem 4.1, $g_{k_0} \in ((\phi_{k_0})_1^{\Leftarrow}(\overline{\mathcal{F}_{k_0}}))^\circ = ((\phi_{k_0})_1^{\Leftarrow}(\mathcal{F}_{k_0}^{\mathrm{st}}))^\circ = (((\phi_{k_0})_1^{\Leftarrow}(\mathcal{F}_{k_0}))^{\mathrm{st}})^\circ$. By the definition of $((\phi_{k_0})_1^{\Leftarrow}(\mathcal{F}_{k_0}))^{\mathrm{st}}$, there exists $h \in L^X$ and $\alpha \in L$ with $g_{k_0} \geq h \wedge \underline{\alpha}$ such that

$$((\phi_{k_0})_1^{\Leftarrow}(\mathcal{F}_{k_0}))^{\mathrm{st}}(g_{k_0}) \geq ((\phi_{k_0})_1^{\Leftarrow}(\mathcal{F}_{k_0})(h) \odot \alpha) \neq 0.$$

Then $(\phi_{k_0})_1^{\Leftarrow}(\mathcal{F}_{k_0})(h) \neq 0$ and $\alpha \neq 0$. So there exists $p \in L^{X_{k_0}}$ with $(\phi_{k_0})_L^{\leftarrow}(p) =$

h such that $\mathcal{F}_{k_0}(p) \neq 0$ i.e., $p(x_{k_0}) \neq 0$. Hence we have

$$\begin{aligned}\bigwedge_{i\in M} g_i(x) &= g_{k_0}(x) \wedge (\bigwedge_{i\in M-\{k_0\}} (\phi_i)_L^{\leftarrow}(h_i))(x) \\ &\geq \{(\phi_{k_0})_L^{\leftarrow}(p)(x) \wedge \bigwedge_{i\in M-\{k_0\}} (\phi_i)_L^{\leftarrow}(h_i)(x) \wedge \alpha\} \\ &= \{p(x_{k_0}) \wedge \bigwedge_{i\in M-\{k_0\}} h_i(x_i) \wedge \alpha\} \neq 0.\end{aligned}$$

(4) From (3) and Theorems 3.7 and 4.1, we have

$$\begin{aligned}\bigsqcup_{i\in\Gamma} \overline{\mathcal{F}_i} = \bigsqcup_{i\in\Gamma} (\phi_i)_1^{\Leftarrow}(\overline{\mathcal{F}_i}) &= (\bigsqcup_{i\in\Gamma-\{k_0\}} (\phi_i)_1^{\Leftarrow}(\overline{\mathcal{F}_i})) \sqcup (\phi_{k_0})_1^{\Leftarrow}(\overline{\mathcal{F}_{k_0}}) \\ &= (\bigsqcup_{i\in\Gamma-\{k_0\}} (\phi_i)_1^{\Leftarrow}(\mathcal{F}_i)) \sqcup (\phi_{k_0})_1^{\Leftarrow}(\mathcal{F}_{k_0}^{\mathrm{st}}) \\ &= (\bigsqcup_{i\in\Gamma-\{k_0\}} (\phi_i)_1^{\Leftarrow}(\mathcal{F}_i)) \sqcup ((\phi_{k_0})_1^{\Leftarrow}(\mathcal{F}_{k_0}))^{\mathrm{st}} \\ &= (\bigsqcup_{i\in\Gamma} (\phi_i)_1^{\Leftarrow}(\mathcal{F}_i))^{\mathrm{st}}.\end{aligned}$$

By Theorem 3.1, $\bigsqcup_{i\in\Gamma} \overline{\mathcal{F}_i}$ is the coarsest stratified L-filter on X which each function $\phi_i : X \to X_i$ is an L-filter map. □

Theorem 4.3. *Let $\phi : X \to Y$ be a function and $\mathcal{F}$ be an L-filter on X. Then we have the following properties:*

(1) $\phi_2^{\Rightarrow}(\mathcal{F})$ is an L-filter on Y.

(2) $(\phi_2^{\Rightarrow}(\mathcal{F}))^{\mathrm{st}} = \phi_2^{\Rightarrow}(\mathcal{F}^{\mathrm{st}})$.

Proof. (1) (LF1) is obvious.

(LF2) For all $f, g \in L^Y$, we have

$$\begin{aligned}\phi_2^{\Rightarrow}(\mathcal{F})(f \wedge g) &= \mathcal{F} \circ \phi_L^{\leftarrow}(f \wedge g) \\ &= \mathcal{F}(\phi_L^{\leftarrow}(f) \wedge \phi_L^{\leftarrow}(g)) \\ &\geq \mathcal{F}(\phi_L^{\leftarrow}(f)) \odot \mathcal{F}(\phi_L^{\leftarrow}(g)) \\ &= \phi_2^{\Rightarrow}(\mathcal{F})(f) \odot \phi_2^{\Rightarrow}(\mathcal{F})(g).\end{aligned}$$

(LF3) If $f \leq g$ for all $f, g \in L^Y$, then

$$\phi_2^{\Rightarrow}(\mathcal{F})(f) = \mathcal{F} \circ \phi_L^{\leftarrow}(f) \leq \mathcal{F} \circ \phi_L^{\leftarrow}(g) = \phi_2^{\Rightarrow}(\mathcal{F})(g).$$

(2) We will show $\phi_2^{\Rightarrow}(\mathcal{F}^{\text{st}})$ is stratified. Let $f \in L^Y$ and $\alpha \in L$. Then

$$\begin{aligned} \alpha \odot \phi_2^{\Rightarrow}(\mathcal{F}^{\text{st}})(f) &= \alpha \odot (\mathcal{F}^{\text{st}} \circ \phi_L^{\leftarrow}(f)) \\ &= \alpha \odot \mathcal{F}^{\text{st}}(\phi_L^{\leftarrow}(f)) \\ &\leq \mathcal{F}^{\text{st}}(\phi_L^{\leftarrow}(f) \wedge \underline{\alpha}) \\ &= \mathcal{F}^{\text{st}} \circ \phi_L^{\leftarrow}(f \wedge \underline{\alpha}) \\ &= \phi_2^{\Rightarrow}(\mathcal{F}^{\text{st}})(f \wedge \underline{\alpha}). \end{aligned}$$

Hence $\phi_2^{\Rightarrow}(\mathcal{F}^{\text{st}})$ is stratified. By Theorem 3.1, $\phi_2^{\Rightarrow}(\mathcal{F}^{\text{st}}) \geq (\phi_2^{\Rightarrow}(\mathcal{F}))^{\text{st}}$.

Finally we will show $\phi_2^{\Rightarrow}(\mathcal{F}^{\text{st}}) \leq (\phi_2^{\Rightarrow}(\mathcal{F}))^{\text{st}}$. Suppose there exists $f \in L^Y$ such that

$$\phi_2^{\Rightarrow}(\mathcal{F}^{\text{st}})(f) \not\leq (\phi_2^{\Rightarrow}(\mathcal{F}))^{\text{st}}(f).$$

Then we have $\mathcal{F}^{\text{st}}(\phi_L^{\leftarrow}(f)) \not\leq (\phi_2^{\Rightarrow}(\mathcal{F}))^{\text{st}}(f)$. By the definition of $\mathcal{F}^{\text{st}}$, there exist $h \in L^X$ and $\alpha \in L$ with $\phi_L^{\leftarrow}(f) \geq h \wedge \underline{\alpha}$ such that

$$\mathcal{F}(h) \odot \alpha \not\leq (\phi_2^{\Rightarrow}(\mathcal{F}))^{\text{st}}(f).$$

On the other hand, $f \geq \phi_L^{\rightarrow}(\phi_L^{\leftarrow}(f)) \geq \phi_L^{\rightarrow}(h) \wedge \underline{\alpha}$, then

$$(\phi_2^{\Rightarrow}(\mathcal{F}))^{\text{st}}(f) \geq (\phi_2^{\Rightarrow}(\mathcal{F}))(\phi_L^{\rightarrow}(h)) \odot \alpha \geq \mathcal{F}(h) \odot \alpha.$$

It is a contradiction. Hence $\phi_2^{\Rightarrow}(\mathcal{F}^{\text{st}}) \leq (\phi_2^{\Rightarrow}(\mathcal{F}))^{\text{st}}$. □

5. Conclusion

In this study, we have constructed a stratified L-filter (resp., L-filterbase) from a given L-filter (resp., L-filterbase). Then, we have induced an L-filter structure from a given family of L-filter structures and we have studied the stratification of it. Moreover, we have investigated the images and preimages of stratified L-filters induced by functions and studied the relationships between L-filter structure and the stratification of it. Further, we hope to study this work in the framework of topogenous structures.

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