International Journal of Pure and Applied Mathematics
Volume 78 No. 8 2012, 1241-1248
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

BOUNDARY INTEGRALS WITH VECTOR POTENTIALS APPLIED TO MATRIX INCLUSION PROBLEM

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Abstract: The inclusion problems arise in a natural way in some solid-state phase changes. These changes have metallurgical significance. For instance, the austenite-martensite transformation in steel manufacture can be regarded as an inclusion problem. Martensite plates can be considered as an elastic inclusion within the matrix, that generates misfit stresses similar to the generation of classical thermoelastic stresses by temperature gradients, and with equally significant physical effects. Eshelby [1] formulated the stress field generated by an ellipsoidal elastic inclusion within an infinite elastic continuum with the use of imagined operations of cutting, space filling and welding. These usages are essentially an intuitive application of Somiglianas formula which is a vector analogue of Greens formula as implied by Jaswon and Symm [2]. Boundary Integral method is well suited to achieve efficient numerical solutions for wide variety of problems involving a thin interface between a finite and infinite medium. Some numerical results are obtained.

AMS Subject Classification: 74B10
Key Words: linear elasticity, vector potentials, boundary integral equation

1. Introduction

The inclusion is a specified region within an infinite isotropic elastic medium.

Received: July 1, 2012

This tends to undergo spontaneously a prescribed homogeneous deformation E^o. However, it reaches an equilibrium configuration E^c due to the elastic constraints of the matrix (material that surrounds the region). In effect, the matrix equilibrium reactions T generate an elastic deformation $E^c - E^o$ that combines with the free deformation E^o to produce a net deformation E^c. Corresponding to these three deformations there exist three displacement fields $U^c - U^o$, U^o, U^c having the same respective significance. As regards the matrix, it has an internal boundary subject to tractions $-T$ that generate an elastic strain field e and displacement field u. The latter is necessarily continuous with U^c , and hence cannot be continuous with the purely elastic field $U^c - U^o$. We introduce stress fields P, p related by Hookes law to $E^c - E^o$,e respectively, i.e. in the usual tensor symbolism

$$\begin{aligned} P &= 2\mu(E^c - E^o) + \lambda(E^c - E^o) \\ p &= 2\mu e + \lambda e\delta \end{aligned} \tag{1}$$

The normal and tangential components of P are continuous with those of p at the boundary; as a corollary, the hoop stress components must be discontinuous or else the boundary would have no physical significance.

The mathematical problem here may be posed as follows in simple terms: What is E^c and the corresponding equilibrium elastic state of the system? An ingenious attack on this problem, utilizing the point-force concept, has been devised by Eshelby [1]. Due to the novelty and importance of his approach, we briefly go over the argument, which invokes a sequence of hypothetical operations. First cut out the inclusion from the matrix and allow it to achieve E^o. Next, impress upon it surface tractions that restore its original dimensions, i.e. that generate an elastic deformation $-E^o$. Insert the stress inclusion into the cavity left behind and rejoin the material across the cut; at this stage no stresses appear in the matrix. Finally, introduce a distribution of point-forces, equal and opposite to the impressed surface tensions. If the matrix were absent these forces would evidently nullify the latter, i.e. they would generate an elastic deformation E^o that exactly cancels the already impressed deformation $-E^o$. However, since they are actually acting within an infinite elastic solid, they generate a deformation that does not cancel $-E^o$, thereby producing the equilibrium configuration in question. This being recognized, a complete formal solution to the problem may be written down at once in terms of boundary integral equations.

2. Betti-Somigliana Formula

Following Jaswon and Symm [2], we write Somigliana's formula in the vector-dyadic form

$$\int_{\partial B} \left[\mathbf{g}(\mathbf{p},\mathbf{q})_i^* \phi(\mathbf{q}) - \mathbf{g}(\mathbf{p},\mathbf{q}) \phi_i^*(\mathbf{q}) \right] dq = 4\pi\phi(\mathbf{p}) \ ; \ \mathbf{p} \in B_i \tag{2}$$

This gives $4\pi\phi(\mathbf{p})$ in B_i as the superposition of a vector simple-layer potential

$$\int_{\partial B} \mathbf{g}(\mathbf{p},\mathbf{q}) \phi_i^*(\mathbf{q}) dq \ ; \ \mathbf{p} \in B_i \tag{3}$$

and a vector double-layer potential

$$\int_{\partial B} \mathbf{g}(\mathbf{p},\mathbf{q})_i^* \phi(\mathbf{q}) dq \ ; \ \mathbf{p} \in B_i \tag{4}$$

defined respectively by the vector source densities

$$\sigma = \phi_i^*, \quad \mu = \phi \ \text{ over } \partial B \tag{5}$$

Evidently Somigliana's fiormula constitutes the vector analogue of Green's formula, with the traction vector playing the role of the scalar normal derivative. The potential (3) stays continuous as $\mathbf{p}$ passes from B_i onto ∂B, but the potential (4) jumps by an amount $-2\pi\phi(\mathbf{p})$. Therefore the right hand side of (2) becomes $2\pi\phi(\mathbf{p})$ on ∂B, thereby yielding Somigliana's boundary formula

$$\int_{\partial B} \left[\mathbf{g}(\mathbf{p},\mathbf{q})_i^* \phi(\mathbf{q}) - \mathbf{g}(\mathbf{p},\mathbf{q}) \phi_i^*(\mathbf{q}) \right] dq = 2\pi\phi(\mathbf{p}) \quad ; \quad \mathbf{p} \in \partial B \tag{6}$$

As $\mathbf{p}$ passes from ∂B into B_e, there follows the interior reciprocal relation

$$\int_{\partial B} \left[\mathbf{g}(\mathbf{p},\mathbf{q})_i^* \phi(\mathbf{q}) - \mathbf{g}(\mathbf{p},\mathbf{q}) \phi_i^*(\mathbf{q}) \right] dq = 0 \quad ; \quad \mathbf{p} \in B_e \tag{7}$$

which may be viewed as a particular case of Bett's reciprocal relation, since both $\phi(\mathbf{p}), \mathbf{g}(\mathbf{p},\mathbf{q})$ are source-free fields in $B_i + \partial B$ if $\mathbf{p} \in B_e$.

Changing i into e yields Somigliana's exterior formula

$$\int_{\partial B} \left[\mathbf{g}(\mathbf{p},\mathbf{q})_e^* \phi(\mathbf{q}) - \mathbf{g}(\mathbf{p},\mathbf{q}) \phi_e^*(\mathbf{q}) \right] dq = 4\pi\phi(\mathbf{p}) \quad ; \quad \mathbf{p} \in B_e \tag{8}$$

which clearly holds only if $\phi = o(\mid \mathbf{p} \mid^{-\mathbf{1}})$ as $\mathbf{p} \to \infty$, i.e. if ϕ is regular at infinity. As $\mathbf{p}$ passes from B_e onto ∂B, we obtain the exterior boundary formula

$$\int_{\partial B} \left[\mathbf{g}(\mathbf{p},\mathbf{q})_e^* \phi(\mathbf{q}) - \mathbf{g}(\mathbf{p},\mathbf{q}) \phi_e^*(\mathbf{q}) \right] dq = 2\pi\phi(\mathbf{p}) \quad ; \quad \mathbf{p} \in \partial B \tag{9}$$

Finally, as $\mathbf{p}$ passes from ∂B into B_i, there follows the exterior reciprocal relation

$$\int_{\partial B} \left[\mathbf{g}(\mathbf{p},\mathbf{q})_e^* \phi(\mathbf{q}) - \mathbf{g}(\mathbf{p},\mathbf{q})\phi_e^*(\mathbf{q})\right] dq = 0 \quad ; \quad \mathbf{p} \in B_i \tag{10}$$

Possible connections between ϕ in (2) and ϕ in (8) will now be explored.

3. Arbitrariness in Somigliana's Formula

If ψ is an arbitrary regular displacement vector in B_e, it satisfies the exterior reciprocal relation

$$\int_{\partial B} \left[\mathbf{g}(\mathbf{p},\mathbf{q})_e^* \psi(\mathbf{q}) - \mathbf{g}(\mathbf{p},\mathbf{q})\psi_e^*(\mathbf{q})\right] dq = 0 \quad ; \quad \mathbf{p} \in B_i \tag{11}$$

Superposing this on Somigliana's interior formula (2) gives the more general representation formula

$$\int_{\partial B} \left[\mathbf{g}(\mathbf{p},\mathbf{q})_i^* (\phi(\mathbf{q}) - \psi(\mathbf{q}))\right] dq - \int_{\partial B} \left[\mathbf{g}(\mathbf{p},\mathbf{q})(\phi_i^*(\mathbf{q}) + \psi_e^*(\mathbf{q}))\right] dq = 4\pi\phi(\mathbf{p}); \quad \mathbf{p} \in B_i, \tag{12}$$

since $\mathbf{g}_i^* + \mathbf{g}_e^* = \mathbf{0}$. Assuming ϕ in $B_i + \partial B$ is given, i.e. if ϕ and ϕ_i^* have known continuous values over ∂B, two distinct possibilities arise for ψ:

(i) If $\psi = \phi$ over ∂B, (12) reduces to

$$-\int_{\partial B} \left[\mathbf{g}(\mathbf{p},\mathbf{q})(\phi_i^*(\mathbf{q}) + \psi_e^*(\mathbf{q}))\right] dq = 4\pi\phi(\mathbf{p}) \quad ; \quad \mathbf{p} \in B_i \tag{13}$$

showing that ϕ may always be generated as a vector simple-layer potential defined by the source density $\sigma = -(4\pi)^{-1}(\phi_i^* + \psi_e^*)$. This representation may also be introduced directly.

(ii) If $\psi_e^* = -\phi_i^*$ over ∂B, then equation (12) reduces to

$$\int_{\partial B} \left[\mathbf{g}(\mathbf{p},\mathbf{q})_i^* (\phi(\mathbf{q}) - \psi(\mathbf{q}))\right] dq = 4\pi\phi(\mathbf{p}) \quad ; \quad \mathbf{p} \in B_i \tag{14}$$

showing that ϕ may always be generated as a vector double-layer potential defined by the source density $\mu = (4\pi)^{-1}(\phi - \psi)$. This representation may also be introduced directly.

Possibility (i) hinges upon the existence of a unique regular ψ in B_e subject to ψ having prescribed continuous values over ∂B. Similarly, possibility (ii) hinges upon the existence of a unique regular ψ in B_e subject to ψ_e^* having prescribed continuous values over ∂B. These are fundamental existence-uniqueness theorems which are discussed elsewhere [3].

4. Inclusion Problems

Equation (12) has an immediate application to elastic inclusion problems. The inclusion is a definite region B_i of an infinite homogeneous elastic continuum, which undergoes a prescribed irreversible deformation ${}^o\phi$.

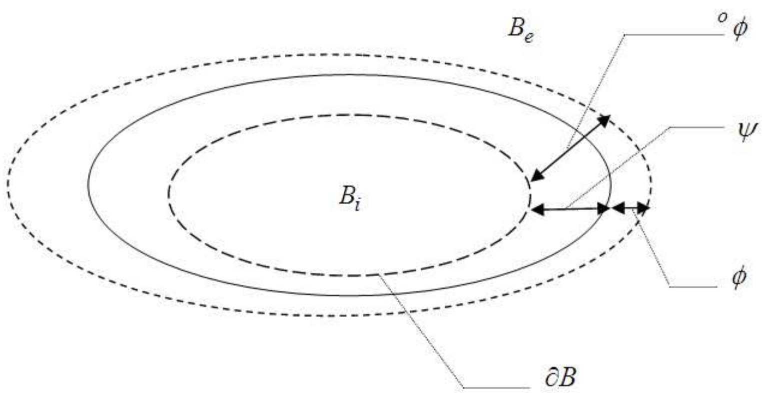

Figure 1: The boundary scenario: Initial boundary ∂B (inner oval), equilibrium boundary (middle oval), and the free boundary (outer oval)

This takes place against the elastic constraints of the surrounding matrix, and as a consequence, the boundary ∂B of B_i suffers an elastic displacement ϕ relative to its irreversible displacement ${}^o\phi$. Hence the net displacement of ∂B being $\phi + {}^o\phi$. Now ∂B forms the internal boundary of B_e as well as the (external) boundary of B_i, and from this point of view it suffers as purely elastic displacement $\psi = \phi + {}^o\phi$, in order to maintain continuity of contact between B_i and B_e. Neither ϕ nor ψ are known separately, but $\phi - \psi = -{}^o\phi$. Under equilibrium conditions $\phi_i^* + \psi_e^* = 0$ over ∂B, in which case (14) becomes

$$-\int_{\partial B} \left[\mathbf{g}(\mathbf{p},\mathbf{q})_i^* \; {}^o\phi(\mathbf{q})\right] dq = 4\pi\phi(\mathbf{p}) \quad ; \quad \mathbf{p} \in B_i \tag{15}$$

Therefore $4\pi\phi$ in B_i may be generated as a vector double-layer potential defined by a known source density $\mu = -\,{}^o\phi$ over ∂B. As $\mathbf{p}$ passes from B_i into B_e, the integral jumps by a total amount $4\pi \, {}^o\phi(\mathbf{p})$ so yielding the value $4\pi(\phi(\mathbf{p}) + {}^o\phi(\mathbf{p})) = 4\pi\psi$ just inside B_e. We infer that

$$-\int_{\partial B} \left[\mathbf{g}(\mathbf{p},\mathbf{q})_i^* \, {}^o\phi(\mathbf{q})\right] dq = 4\pi\psi(\mathbf{p}) \quad ; \quad \mathbf{p} \in B_e \tag{16}$$

Which shows that $4\pi\psi$ in B_e may be generated as the exterior counterpart of $4\pi\phi$ in B_i.

These formulae provide the mathematical justification for Eshelby's intuitive approach, and they also allow effective numerical solutions by exploiting BEM methodology.

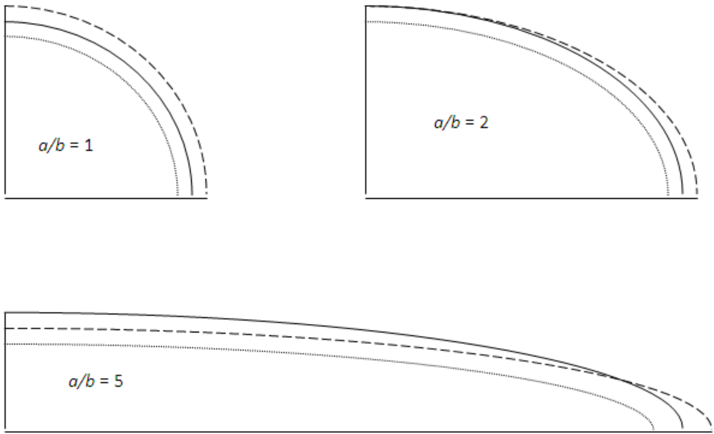

Figure 2: Schematic illustration of relation between initial ellipse (dotted line), free ellipse (broken line) and equilibrium ellipse (full line), at different aspect ratio

5. Elliptic Inclusion and Numerical Findings

In the simplest case we consider an ellipse of semi-axes a, b which undergoes a free homogeneous deformation sufficiently defined by

$$a \to a(1 + E^o_{11})$$
$$b \to b(1 + E^o_{22})$$

To the first order, the equilibrium boundary becomes an ellipse similar and similarly situated to the original ellipse, and to the free ellipse, with equilibrium semi-axes

$$a(1 + E^c_{11})$$
$$b(1 + E^c_{22})$$

Interesting features emerge on examining the ratios $E^c_{11}/E^o_{11}, E^c_{22}/E^o_{22}$ in various particular cases. Taking $E^o_{11} = E^o_{11}$ and $\nu = 1/3$ (Poissons ratio), they have been calculated via the boundary equations (BE) for a series of axial ratios as shown in the following table and as depicted in Fig. (2) above. The analytical results [4][5] are also presented for comparison. We see that the major axis of the ellipse suffers a steadily increasing relative compression when a/b increases until eventually it is forced completely back to the initial length. The minor axis at first suffers compression, but this becomes entirely eliminated at $a/b = 2$; thereafter, the minor axis actually extends beyond its free length, though not exceeding a definite limiting value. Intuitively this surprising effect appears as a direct result of the intense compression suffered by the major axis as the ellipse becomes more elongated.

a/b	E^c_{11}/E^o_{11}		E^c_{22}/E^o_{22}	
	Analytical	BEM	Analytical	BEM
1	$\frac{3}{4}$	0.74899	$\frac{3}{4}$	0.747999
2	$\frac{1}{2}$	0.4889	1	0.98999
3	$\frac{3}{8}$	0.37244	$\frac{9}{8}$	1.124449
10	$\frac{3}{22}$	0.135899	$\frac{15}{11}$	1.35789
∞	0	0.000032	$\frac{3}{2}$	1.44998

Table 1: Comparison between BEM and analytical results

Acknowledgments

I am indebted to Professor M. A. Jaswon, City University, London, UK, for all his help, advice and valuable suggestions. Thanks to Professor Dilcher Dalhousie University, Canada for arranging the research visit, and to Sultan Qaboos University, Oman for granting the research leave.

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