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NON-LINEARLY NORMAL SPECIAL CURVES WITH MAXIMAL RANK IN $\mathbb{P}^r$

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Abstract: Fix integers r, g such that $r \geq 4$ and $g \geq r+1$. Let $C \subset \mathbb{P}^r$ be a general smooth curve with genus g, degree $g+r$ and $h^1(C, \mathcal{O}_C(1)) = 1$ (such a curve is not linearly normal). In this paper we prove that C has maximal rank, i.e. $h^0(\mathcal{I}_C(t)) \cdot h^1(\mathcal{I}_C(t)) = 0$ for all $t \in \mathbb{N}$.

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1. Introduction

Let $X \subset \mathbb{P}^r$ be a closed subscheme. We say that X has *maximal rank* if for all integers $t \geq 1$ the restriction map $H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(t)) \to H^0(X, \mathcal{O}_X(t))$ has maximal rank, i.e. it is either injective or surjective. Now assume that X is a reduced and connected curve of degree d and arithmetic genus g, spanning $\mathbb{P}^r$ and with $h^1(X, \mathcal{O}_X(1)) \leq 1$. Riemann-Roch gives $d \geq g+r-1$. Obviously if $d = r$ (and hence X is a rational normal curve) then we have $h^1(X, \mathcal{O}_X(1)) = 0$; in this case we say say that X has *critical value* 1 and that 1 is the critical value of the triple $(r, 0, r)$. In this paper any curve X will have $h^1(X, \mathcal{O}_X(2)) = 0$ and hence $h^0(X, \mathcal{O}_X(t)) = td + 1 - g$ for all $t \geq 2$. Now assume $d > r$. Let k be

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the minimal integer ≥ 2 such that $\binom{r+k}{r} \geq kd + 1 - g$. We say that k is the *critical value* of X and of the triple (d, g, r). X has maximal rank if and only if $h^0(\mathcal{I}_X(t)) = 0$ for all $t < k$ and $h^1(\mathcal{I}_X(t)) = 0$ for all $t \geq k$. If $k \geq 3$, we have $h^1(X, \mathcal{O}_X(k-1)) = 0$. Hence if $k \geq 3$ Castelnuovo-Mumford's lemma says that if $h^1(\mathcal{I}_X(k)) = 0$, then $h^1(\mathcal{I}_X(t)) = 0$ for all $t > k$. Hence X has maximal rank if and only if $h^0(\mathcal{I}_X(k-1)) = 0$ and $h^1(\mathcal{I}_X(k)) = 0$. This statement is true even in the case $k = 2$ and $h^1(X, \mathcal{O}_X(1)) = 1$ and a general such a curve has maximal rank (Lemma 5). Let $B(d, g, r, e)$, $e = 0, 1$ denote the set of all smooth, integral and non-degenerate curves $X \subset \mathbb{P}^r$ such that $\deg(X) = d$, $p_a(X) = g$ and $h^1(X, \mathcal{O}_X(1)) = e$. We have $B(d, g, r, 0) \neq \emptyset$ if and only if $d \geq g + r$. We have $B(d, g, r, 1) \neq \emptyset$ if and only if $d \geq g + r - 1$ and $d \leq 2g - 2$. A general $C \in B(d, g, r, 0)$ has maximal rank (see [3] if $r = 4$, [4] if $r = 3$ and [5] if $r \geq 5$). It is easy to see that $B(g + r, g, r, 1)$ is contained in the closure in $\mathrm{Hilb}(\mathbb{P}^r)$ of $B(g + r, g, r, 0)$ (Proposition 2). A general $C \in B(g + r - 1, g, r, 1)$ has maximal rank (see [6] or [12] if $r = 3$, the case $k = 2$ of [7] if $r \geq 4$). In [6], [7] and [12] only linearly normal special curves are considered. In this paper we prove the following result.

Theorem 1. *Fix integers $r \geq 4$ and $g \geq r + 1$. Then a general $X \in B(r + g, g, r, 1)$ has maximal rank.*

In the case $r = 3$ there are some counterexamples (see [9], Example B) at page 55, [11]).

In the range of quadrics we prove a statement for more pairs (d, g) (see Proposition 1). We also gives other examples of general linear projections of canonically embedded bielliptic curves, which have not maximal rank (see [9], Example B) at page 55, [11]).

2. Preliminaries

Fix a reduced curve $Y \subset \mathbb{P}^r$. We say that a line D is *1-secant* (resp. *2-secant*) to Y if $\sharp(Y \cap D) = 1$ (resp. $\sharp(Y \cap D) = 2$), Y is smooth at each point of $Y \cap D$ and D is not a tangent line of Y at one of the points of $Y \cap D$.

For any nodal curve $Y \subset \mathbb{P}^r$ let N_Y denote the normal bundle of Y in $\mathbb{P}^r$. If $r \geq 4$ and $H \subset \mathbb{P}^r$ is a hyperplane, then we often write $B(d, g, H, e)$ and $B(d, g, H, e)'$ instead of $B(d, g, r - 1, e)$ or $B(d, g, r - 1, e)'$ to emphasize that we are working with curves contained in H.

The algebraic set $B(d, g, r, 0)$ is smooth, irreducible and of dimension $(r + 1)d + (r - 3)(1 - g)$. For all integers d, g, r, e such that $r \geq 3$ and $e \in \{0, 1\}$ let $B(d, g, r, e)'$ denote the closure of $B(d, g, r, e)$ in $\mathrm{Hilb}(\mathbb{P}^r)$.

We have $B(d,g,r,0) \neq \emptyset$ if and only if $r \geq 3$, $g \geq 0$ and $d \geq g+r$. We have $B(d,g,r,1) \neq \emptyset$ if and only if $r \geq 3$, $g \geq r+1$ and $g+r-1 \leq d \leq 2g-2$. Fix integer g,d such that $g \geq 4$ and $g-1 \leq d \leq g$. Fix any $C \in \mathcal{M}_g$. The set, $\mathcal{S}$, of all $L \in \mathrm{Pic}^d(C)$ such that $h^1(C,L)=1$ is irreducible, non-empty and of dimension $2g-2-d$; indeed if $d=2g-2$, then $\mathcal{S}=\{\omega_C\}$, while if $d \neq 2g-2$, then $\mathcal{S}$ is isomorphic to a non-empty open subset of the symmetric product of $2g-2-d$ copies of C. If C is general, then a general $L \in \mathcal{S}$ is very ample; if C is not general, then the set of all very ample $L \in \mathcal{S}$ is open. Hence if $r \geq 3$, $g \geq r+1$ and $g+r-1 \leq d \leq 2g-2$, then $B(d,g,r,1)$ is non-empty, irreducible and of dimension $3g-3+(2g-2-d)+(r+1)(d+1-g-2r)$.

Remark 1. Fix a closed subscheme $W \subset \mathbb{P}^r$ and an effective Cartier divisor F of $\mathbb{P}^r$. Set $a := \deg(F)$. We will take as F either a hyperplane or (if $r=3$) a smooth quadric surface. Let $\mathrm{Res}_F(W)$ be the residual scheme of W with respect to F, i.e. the closed subscheme of $\mathbb{P}^r$ with $\mathcal{I}_W : \mathcal{I}_F$ as its ideal sheaf. If W is reduced, then $\mathrm{Res}_F(W)$ is the union of the irreducible components of W not contained in F. For any $t \in \mathbb{Z}$ we have the following exact sequence of coherent sheaves

$$0 \to \mathcal{I}_{\mathrm{Res}_F(W)}(t-a) \to \mathcal{I}_W(t) \to \mathcal{I}_{W\cap F,F}(t) \to 0 \tag{1}$$

From (1) we get

$$h^i(\mathcal{I}_W(t)) \leq h^i(\mathcal{I}_{\mathrm{Res}_F(W)}(t-a)) + h^i(F, \mathcal{I}_{W\cap F,F}(t))$$

for all $i \geq 0$ and all $t \in \mathbb{Z}$.

Lemma 1. *Assume $B(d,g,r,1) \neq \emptyset$ and $B(d,g,r,1) \neq \emptyset$, i.e. assume $r \geq 3$, $g \geq r+1$ and $g+r \leq d \leq 2g-2$. Then $B(d,g,r,1) \subset B(d,g,r,0)'$.*

Proof. Fix any $X \in B(d,g,r,1)$. Since $\chi(N_X) = (r+1)d+(r-3)(1-g)$, any irreducible component of $\mathrm{Hilb}(\mathbb{P}^r)$ has dimension at least $(r+1)d+(r-3)(1-g)$. Since $d \geq g+r$, we have $\dim(B(d,g,r,1)) < (r+1)d+(r-3)(1-g)$. Hence every $X \in B(d,g,r,1)$ is a flat limit of a family $\{C_\lambda\}$ of non-special subcurves of $\mathbb{P}^r$. Since X is non-degenerate, a general C_λ is non-degenerate. Hence $B(d,g,r,1) \subset B(d,g,r,0)'$. □

Notice that $h^0(X, \mathcal{O}_X(1)) = r+2$ for any $X \in B(r+g,g,r,1)$. Fix $X \in B(d,g,r,1)$ and assume $d \neq 2g-2$. Hence $h^0(X, \omega_X(-1)) = 1$. Let E be the only effective divisor of $|\omega_C(-1)|$. Any point P of the support of E is called a *special linking point* of X. In [1] we proved the following lemma.

Lemma 2. *Fix $X \in B(d,g,r,1)$, $d \neq 2g-2$, and let P be any special linking point of X. Let $D \subset \mathbb{P}^r$ be any 1-secant line of X such that $\{P\} = X \cap D$. Then $X \cup D \in B(d,g,r,1)'$.*

Lemma 3. *Fix $X \in B(d,g,r,1)$. Let D be a 2-secant line of X. Then $h^0(X \cup D, \mathcal{O}_{X\cup D}(1)) = r+2$, $h^1(X \cup D, \mathcal{O}_{X\cup D}(1)) = 1$ and $X \cup D \in B(d+1, g+1, r, 1)'$.*

Proof. Since $h^i(D, \mathcal{O}_D(1)(-D \cap X)) = 0$, $i = 0, 1$, a Mayer-Vietoris exact sequence gives the $h^0(X \cup D, \mathcal{O}_{X\cup D}(1)) = r+2$ and $h^1(X \cup D, \mathcal{O}_{X\cup D}(1)) = 1$.

First assume $d = g+r-1$. In this case the lemma just means that $X \cup D$ is smoothable (use [10], Theorem 4.1, or see [13], Proposition 3.1 for much more). Now assume $d \geq g+r$. Hence X is a isomorphic linear projection of a curve $X' \subset \mathbb{P}r+1$, say from some $O \in \mathbb{P}^{r+1} \setminus X'$. Call $\ell : \mathbb{P}^{r+1} \setminus \{O\} \to \mathbb{P}^r$ this linear projection with $\ell(X') = X$. Let $S \subset X'$ be the set of cardinality two such that $\ell(S) = X \cap D$. Let $D' \subset \mathbb{P}^{r+1}$ be the line spanned by S. D' is a 2-secant line. The case $d = g+(r+1)+r.-1$ just proved gives $X' \cup D' \in B(d+1, g+1, r+1, 1)'$. Project into $\mathbb{P}^{r+1}$ using ℓ a family of curves in $\mathbb{P}^{r+1}$ with $X' \cup D'$ as a special fiber and with a general member in $B(d+1, g+1, r+1, 1)$. □

Lemma 4. *Fix a nodal curve $A \cup D \cup B \subset \mathbb{P}^r$, $r \geq 3$, such that D is a line, $A \in B(u,v,r,1)$, B is smooth (but not necessarily connected; we allow the case $B = \emptyset$), $D \cap B = \emptyset$, either $B = \emptyset$ or $1 \leq \sharp(A \cap B) \leq r$, $A \cap B$ is linearly independent and D is 1-secant to A. Let E be the unique divisor of $\omega_A(-1)$ and assume that the point $P := D \cap A$ is in the support of E. Then $A \cup D \cup B \in B(\deg(A \cup D \cup B), p_a(A \cup D \cup B, r, 1)'$.*

Proof. Assume for the moment $u = v+r-1$, i.e. assume that A is linearly normal. See $\mathbb{P}^r$ as a hyperplane M of $\mathbb{P}^{r+1}$. Let $D' \subset \mathbb{P}^{r+1}$ be any line such that $D' \cap M = \{P\}$. The curve $A \cup D'$ is non-degenerate and smoothable in $\mathbb{P}^{r+1}$ (see [1]). Let $\{Y_t\}_{t\in\Delta}$ be a smoothing of $A \cup D'$ in $\mathbb{P}^{r+1}$ with Δ an integral curve, $o \in \Delta$ and $Y_o = A \cup D'$. Call $\pi : \mathcal{Y} \to \Delta$ this flat family of curves. If $B \neq \emptyset$ we make a ramified covering of Δ to get the existence of $\sharp(A \cap B)$ sections of π, say π_O, $O \in \Delta$. Restricting if necessary Δ we may assume that $s_O(t)$, $O \in A \cap B$, are $\sharp(A \cap B)$ different points of $\mathbb{P}^{r+1}$ in linearly general position. Hence restricting π to a small neighborhood of o in Δ we may find a regular family h_t, $t \in \Delta$ of automorphisms of $\mathbb{P}^{r+1}$ such that h_o is the identity and $h_t(A \cap B) = \{s_O(t)\}_{O \in A\cap B}$. Take the family $Y_t \cup h_t(B)$. Taking a smaller neighborhood of $o \in \Delta$ we may assume that all curves $Y_t \cup h_t(B)$ are nodal. For $t \notin o$, the curve $Y_t \cup h_t(B)$ is smoothable. For degree reasons $Y_t \cup h_t(B) \in B(\deg(A \cup D \cup B), p_a(A \cup D \cup B), r+1, 1)'$.

Taking a linear projection from a general point of $\mathbb{P}^{r+1} \setminus M$ onto M (which is the identity on M and hence on $A \cup D \cup B$) we get a family of smooth curves in $B(\deg(A \cup D \cup B, p_a(A \cup D \cup B, r, 1)$ with $A \cup D \cup B$ as one of its flat limits. Now assume that A is not linearly normal, say it is the linear projection of $A' \subset \mathbb{P}^m$. We see $\mathbb{P}^m$ as a hyperplane of $\mathbb{P}^{m+1}$. Let P' be the point of A' whose projection is P. Let $D'' \subset \mathbb{P}^{m+1}$ be any line such that $D'' \cap M' = \{P'\}$. For each $O \in A \cap B$ let O' be the point of A' with O as its projection. We use M', $A' \cup D''$ and the points O', $O \in A \cap B$, to make the same construction and then project the smoothing in $\mathbb{P}^r$. □

We may iterate lemmas 3 and 4 using the following observation.

Remark 2. Fix a flat family $\{Y_\lambda\}_{\lambda \in \Delta}$ of curves $Y_\lambda \subset \mathbb{P}^r$, where Δ is a connected affine curve. Call $u : \mathcal{Y} \to \Delta$ the corresponding family. Fix $o \in \Delta$ and take a line $D \subset \mathbb{P}^r$ which is 2-secant to Y_o. Taking a finite covering of Δ if necessary, we may assume that u has two disjoint section s_1, s_2 with $\{s_1(o), s_2(o)\} = Y \cap D$. For any $t \in \Delta$ let D_t be the line spanned by $s_1(t)$ and $s_2(t)$. There is an open neighborhood Δ' of o in Δ instead of Δ we reduce to the case in which $\sharp(Y_t \cap D_t) = 2$ for all t and D_t is 2-secant to Y_t for all $t \in \Delta'$.

Lemma 5. *Fix integers r, k such that $r \geq 3$ and $0 \leq k \leq (r^2 - 3r)/2 + 1$. Then a general $C \in B(2r + k, r + 1 + k, r, 1)$ satisfies $h^1(\mathcal{I}_C(2)) = 0$ and $h^0(\mathcal{I}_C(2)) = 1 + (r^2 - 3r)/2 - k$.*

Proof. By Riemann-Roch it is sufficient to prove $h^1(\mathcal{I}_C(2)) = 0$. We use induction on k starting from the case $k = 0$, i.e. a canonically embedded linearly normal curve. Now assume $k > 0$ and take $A \in B(2r + k - 1, r + k, r, 1)$ such that $h^1(\mathcal{I}_A(2)) = 0$, i.e $h^0(\mathcal{I}_A(2)) = (r^2 - 3r)/2 + 2 - k > 0$. Fix any $Q \in |\mathcal{I}_A(2)|$. Fix a general $P \in A$. Since A is non-degenerate and the singular locus of a quadric hypersurface is a linear space, for general P we may assume that Q is not a cone with vertex containing P. Since A is non-degenerate, we get that a general 2-secant line of A through P is not contained in Q. Hence $Q \notin |\mathcal{I}_{A \cup T}(2|$. Hence $h^0(\mathcal{I}_{A \cup T}(2)) < h^0(\mathcal{I}_A(2))$, i.e. (Riemann-Roch) $h^1(\mathcal{I}_{A \cup T}(2)) = 0$. Since $A \cup T \in Z(2r + k, r + 1 + k, r)'$, we may apply semicontinuity. □

Proposition 1. *Fix integers r, k such that $r \geq 4$ and $1 \leq k \leq (r^2 - 3r)/2 - 1$. Then a general $C \in B(2r + k + 1, r + 1 + k, r, 1)$ satisfies $h^1(\mathcal{I}_C(2)) = 0$ and $h^0(\mathcal{I}_C(2)) = (r^2 - 3r)/2 - k - 1$.*

Proof. By Riemann-Roch it is sufficient to prove $h^0(\mathcal{I}_C(2)) = r^2 - 3r)/2 - k - 1$. Take a general $A \in Z(2r + k, r + 1 + k, r)$. Riemann-Roch gives $h^1(A, \mathcal{O}_A(1)) = 1$. Call E the only effective divisor of $|\omega_A(-1)|$. Since $Z(2r +$

$k, r+1+k, r)$ is irreducible and a general element of it has general moduli, we may assume that E is formed by k general points of the abstract curve A (but we may not assume that the pair $(\mathcal{O}_A(1), E)$ is general in $\mathrm{Pic}^{2r+k}(A) \times S^k(A)$). Hence we may assume that E is formed by k distinct points, $P_1, \dots, P_k$. Fix $P \in \{P_1, \dots, P_k\}$. Lemma 5 gives $h^0(\mathcal{I}_A(2)) = 1 + (r^2 - 3r)/2 - k$. Fix a general line line $T \subset \mathbb{P}^r$ passing through P. Since $A \cup T \in B(2r+k+1, r+1+k, r, 1)'$, it is sufficient to prove $h^0(\mathcal{I}_{A\cup T}(2)) \le h^0(\mathcal{I}_A(2)) - 2$. Since T contains a general point, O, of $\mathbb{P}^r$, we have $h^0(\mathcal{I}_{A\cup T}(2)) < h^0(\mathcal{I}_A(2))$. Hence it is sufficient to prove $h^0(\mathcal{I}_{A\cup T}(2)) < h^0(\mathcal{I}_{A\cup\{O\}}(2))$. Fix a general $Q \in |\mathcal{I}_{A\cup\{O\}}(2)|$. Instead of O we take a general $O' \in Q$. Since $O \in Q$, we may consider O' as a general point of $\mathbb{P}^r$ (before fixing Q). Hence $h^0(\mathcal{I}_{A\cup\{O\}}(2)) = h^0(\mathcal{I}_{A\cup\{O'\}}(2))$. Instead of T we take the line T' spanned by P and O'. Since $A \cup T' \in B(2r+k+1, r+1+k, r, 1)'$, it is sufficient to prove $h^0(\mathcal{I}_{A\cup T'}(2)) < h^0(\mathcal{I}_{A\cup\{O'\}}(2))$. For a general $O' \in Q$ we have $Q \notin |\mathcal{I}_{A\cup T'}(2)|$ if and only if Q is nor a cone with vertex containing P. Hence we are done, unless every quadric hypersurface containing A is a cone with vertex containing the linear space $\langle E\rangle$. Since E is general in A and we are in characteristic zero, we may assume $h^0(A, \omega_A(-2E)) = \max\{0, r+1+k-2k\}$. Hence $\dim(\langle|E|) = \min\{r, k-1\}$. Since A is non-degenerate, the singular locus of any quadric hypersurface containing A has codimension at least 2 in $\mathbb{P}^r$. Hence we are done if $k \ge r$. Hence we may assume $1 \le k \le r-1$. Since $\mathcal{O}_A(1)$ is very ample, $\omega_A(1)(-P)$ is spanned and $h^0(A, \mathcal{O}_A(1)(-P)) = 0$. Assume that every element of $|\mathcal{I}_A(2)|$ is a cone with vertex containing $\langle E\rangle$. Let $\psi : A \to \mathbb{P}^{r-1}$ be morphism induced by $|\mathcal{O}_A(1)(-P)|$. Since every element of $|\mathcal{I}_A(2)|$ is a cone with vertex containing $\langle E\rangle$, we get $h^0(\mathbb{P}^{r-1}, \mathcal{I}_{\psi(A)}(2)) = h^0(\mathcal{I}_A(2))$. First assume $r \ge k+3$. For general $A \in \mathcal{M}_{r+2}$ the line bundle $\omega_A(-2P)$ is very ample. Hence for a general $E' \subset A$ with $\sharp(E') = k-1$ the line bundle $\omega_A(-2P-E') = \mathcal{O}_A(1)(-P)$ is very ample (take $E := \{P\} \cup E'$) and $h^1(A, \mathcal{O}_A(1)(-P)) = 1$. Since A is general it has Clifford index $\mathrm{Cliff}(A) = \lfloor (r+k-1)/2 \rfloor$. We have $\deg(\mathcal{O}_A(1)(-P)) = 2r+k-1 \ge 2(r+1+k) - 1 - \lfloor (r+k-1)/2 \rfloor$. By [8], Theorem 1, the line bundle $\mathcal{O}_A(1)(-P)$ is normally generated. Hence $h^0(\mathbb{P}^{r-1}, \mathcal{I}_{\psi(A)}(2)) = \binom{r+1}{2} - 2\deg(A) + 2 - 1 + p_a(A) > h^0(\mathcal{I}_A(2))$, a contradiction. Now assume $k \in \{r-1, r-2\}$; these inequalities exclude the cases $r = 4$ and $r = 5$. The complete linear system $|\mathcal{O}_A(1)(-E)|$ induces a rational map $A \dashrightarrow \mathbb{P}^{r-k}$ whose image is contained in $h^0(\mathcal{I}_A(2))$ linearly independent quadrics hypersurfaces (conics if $k = r-2$, pair of points if $k = r-1$), absurd. □

Proposition 2. *Fix integers g, r such that $3 \le r \le g-3$ and $r(r+1) > 4g$. Let $C \subset \mathbb{P}^{g-1}$ be any canonically embedded smooth bielliptic curve of genus g.*

Let $X \subset \mathbb{P}^r$ be a general linear projection of C. Then:

(i) $h^0(\mathcal{I}_X(2)) \geq (r+1)r/2 - 2(g-1) - 1 > 0$ and $h^1(\mathcal{I}_X(2)) \geq g-2-r$;

(ii) X has not maximal rank.

Proof. Notice that the assumptions imply $r \geq 6$. Since C is bielliptic, there is a 2-dimensional cone $S \subset \mathbb{P}^{g-1}$ with the following properties. S is a cone, say with vertex O, over a degree $g-1$ linearly normal smooth elliptic curve of $\mathbb{P}^{g-2}$, $O \notin C$, C is the scheme-theoretic intersection of S and a quadric hypersurface and the linear projection from O induces the degree 2 morphism $C \to E$ with E an elliptic curve. Take a general codimension $g-2-r$ linear subspace V of $\mathbb{P}^{g-1}$. For general X and V we may assume $X = \ell_V(C)$, where $\ell_V : \mathbb{P}^{g-1} \setminus V \to \mathbb{P}^r$ is the linear projection from V. For general V we have $V \cap S = \emptyset$ and $T := \ell_V(S)$ is a degree $g-1$ cone with vertex $\ell_V(O) \notin X$ and as a basis a smooth elliptic curve $F \subset \mathbb{P}^{r-1}$; here we use that V is a general linear linear subspace of codimension ≥ 5 and hence V does not intersect the secant variety of S (which is a cone with vertex O and as a basis the secant variety of a linearly normal elliptic curve of $\mathbb{P}^{g-2}$) . Since $h^0(F, \mathcal{O}_F(2)) = 2(g-1)$, we have $h^0(\mathbb{P}^{r-1}, \mathcal{I}_F(2)) \geq (r+1)r/2 - 2(g-1)$ (indeed, equality holds, but we don't need it). Since every quadric hypersurface containing T is a cone with vertex O and F as a basis, we get $h^0(\mathcal{I}_T(2)) \geq (r+1)r/2 - 2(g-1)$.

Claim. *We have $h^0(T, \mathcal{I}_{X,T}(2)) \leq 1$.*

Proof of Claim. Assume $h^0(T, \mathcal{I}_{X,T}(2)) \geq 2$. Hence there is a hypersurface $\Sigma \in |\mathcal{I}_{X,T}(2)|$ passing through $\ell_V(O)$. Hence Σ contains 3 points of any general line $L \subset T$, a contradiction.

The Claim gives $h^0(\mathcal{I}_T(2)) \geq (r+1)r/2 - 2(g-1) - 1$. Since $\deg(\mathcal{O}_X(2)) = 3g-3$, we get $h^1(\mathcal{I}_X(2)) \geq r-g-2$.

Part (ii) follows from part (i) and the definition of maximal rank, because $(r+1)r/2 - 2(g-1) - 11 > 0$ and $g-2-r > 0$. □

3. Proof of Theorem 1

Fix a hyperplane $H \subset \mathbb{P}^r$. Set $Z(d,g,H) := B(d,g,r-1,0)$ and $Z(d,g,H)' := B(d,g,r-1,0)'$ to stress that we look at subcurves of H.

For all integers $r \geq 3$ and $t \geq 2$ define the integers $a_{r,t}$ and $b_{r,t}$ by the relations

$$(t-1) \cdot a_{r,t} + 1 + tr + b_{r,t} = \binom{r+t}{r}, \ 0 \leq b_{r,t} \leq t-2 \tag{2}$$

Set $a_{r,0} = a_{r,1} = r$ and $b_{r,0} = b_{r,1} = 0$. The integers $a_{r,t}$ and $b_{r,t}$ were called $g(t,r)$ and $f(t,r)$ in [5]. Taking the difference between the equation in (2) with the same equation for the integers $(r',t') := (r,t)$ we get

$$a_{r,t} + r + (t-1)(a_{r,t} - a_{r,t-1}) + b_{r,t} - b_{r,t-1} = \binom{r+t}{r} \tag{3}$$

For all integers $r \geq 3$ and $t \geq 2$ we define the following assertion $H_{r,t}$:

$H_{r,t}$, $r \geq 3$, $t \geq 2$: There is a triple (A, D, T) with the following properties:

(i) $A \subset B(a_{r,t} + r - 1 - b_{r,t}, a_{r,t} - b_{r,t}, r, 1)$; D is a line 1-secant to A and the point $D \cap A$ is in the support of the unique effective divisor of $|\omega_A(-1)|$; T is a disjoint union of $b_{r,t}$ 1-secant lines of A and $T \cap D = \emptyset$;

(ii) $h^1(\mathcal{I}_{A\cup D\cup T}(t)) = 0$.

Take any (A, D, T) as in part (i) of $H_{r,t}$. We have $h^0(A\cup B\cup T, \mathcal{O}_{A\cup B\cup T}(t)) = \binom{r+t}{r}$. Hence $h^1(\mathcal{I}_{A\cup D\cup T}(t)) = h^0(\mathcal{I}_{A\cup D\cup T}(t))$. Hence (A, D, T) satisfies $H_{r,t}$ if and only if $h^0(\mathcal{I}_{A\cup D\cup T}(t)) = 0$. We only prove $H_{r,t}$ when $r \geq 4$. By [9], Example B) at page 55, or [11] and Lemma 1 $H_{3,3}$ is false. Of course, to see that $H_{r,t}$ makes sense we need to check that $B(a_{r,t} + r - 1 - b_{r,t}, a_{r,t} - b_{r,t}, r, 1)$ is defined, i.e. we need to check that $a_{r,t} - b_{r,t} \geq r + 1$ and that $2(a_{r,t} - b_{r,t}) > a_{r,t} - b_{r,t} + r$. The latter inequality follows from the first one. If $t = 2$ the first inequality is true, because $b_{r,2} = 0$ and $a_{r,2} = \binom{r+2}{2} - 2r - 1 = r(r-1)/2$. For $r \geq 5$ use [5], Lemma 4.3. If $r = 4$ use Remark 3 and Lemma 6.

Remark 3. Since $0 \leq b_{r,t} \leq t - 2$, we have $b_{r,2} = 0$. We have $a_{3,3} = 5$, $b_{3,3} = 0$, $a_{3,4} = 7$, $b_{3,4} = 1$, $a_{3,5} = 10$, $b_{3,5} = 0$ (see [4], III.1, with $d(t) := a_{3,t} + 3$ and $b(t) := b_{3,t}$), $a_{4,2} = 6$, $a_{4,3} = 11$, $b_{4,3} = 0$, $a_{4,4} = 17$, $b_{4,4} = 2$, $a_{4,5} = 26$, $b_{4,5} = 1$, $a_{4,6} = 37$, $b_{4,6} = 0$.

Lemma 6. *For all integers $r \geq 4$ and $t \geq 3$ we have $a_{r,t} - a_{r,t-1} \geq r + t - 2$.*

Proof. For $r \geq 5$, the lemma is [5], Lemma 4.1. Now assume $r = 4$. All cases with $t \leq 6$ are true by Remark 3. We have $a_{4,t} \leq (\binom{t+4}{4} - 4t)/(t-1)$. For $t \geq 6$ use (3) and that $b_{r,t} \leq t - 2$. □

Lemma 7. *Fix integers t, b, a such that $t \geq 2$, $0 \leq b \leq t - 2$ and $3 \leq a \leq a_{3,t} + 1 - b$. Let Z be a general element of $B(a, a-3, 3, 0)$. Let $T \subset \mathbb{P}^3$ be a general disjoint union of b lines, each of them 1-secant to Z. Then $h^1(\mathcal{I}_{Z\cup T}(t)) = 0$.*

Proof. The lemma is true if $b = 0$ by [4]. Hence we may assume $b > 0$. Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface. Until step (c) we assume $a =$. L

(a) In this step we assume $0 < b \le a_{3,t} - a_{3,t-2} - 3$. Let $E \subset \mathbb{P}^3$ be a general element of $B(a_{3,t-2} + 3, a_{3,t-2}, 3, 0)$ with a line $J \subset Q$, say as a line of type $(1, 0)$, as a 2-secant line. Since $\mathrm{Aut}(\mathbb{P}^3)$ is transitive on the set of all lines of $\mathbb{P}^3$, E may be seen as a general element of $B(a_{3,t-2} + 3, a_{3,t-2}, 3, 0)$. By [4] we have $h^1(\mathcal{I}_E(t-2)) = 0$ and $h^0(\mathcal{I}_E(t-2)) = b_{m,t-2} \le \max\{0, t-4\}$. Apply the case $s_1 = 2$ and $s_2 = s_3 = 0$ of [12], Proposition 4.1. We get that $E \cap (Q \setminus J)$ may be seen as a general set of $2 \cdot \deg(E) - 2$ points of Q. Hence no two of these points are on a line of type $(0, 1)$ on Q and for all $(c, d) \in \mathbb{N}^2$ we have $h^0(Q, \mathcal{I}_S(c, d)) = \max\{0, (c+1)(d+1) - \sharp(S)\}$ for any $S \subseteq E \cap (Q \setminus J)$. Let L_i, $1 \le i \le a_{3,t} - a_{3,t-2} - 2$, be lines of type $(0, 1)$ on Q such that $L_i \cap J \cap E = \emptyset$ for all i and and $L_i \cap (E \cap Q) \ne \emptyset$ if and only if $1 \le i \le a_{3,t} - a_{3,t-2} - 2 - b$. Notice that b of the lines L_i are 1-secant to $E \cup J$, while the other noes are 2-secant to $E \cup J$. By semicontinuity to prove the lemma it is sufficient to prove $h^1(\mathcal{I}_{E\cup J\cup(\cup_i L_i)}(t)) = 0$. Set $S := E \cap (Q \setminus (J \cup \cup_i L_i))$. Since $\mathrm{Res}_Q(E \cup J \cup (\cup_i L_i) = E$ and $h^1(\mathcal{I}_E(t-2)) = 0$, it is sufficient to prove $h^1(Q, \mathcal{I}_{(Q\cap E)\cup J\cup(\cup_i L_i)}(t, t)) = 0$, i.e. $h^1(Q, \mathcal{I}_S(t-1, t-a_{3,t} - a_{3,t-2} + 2))$. By the quoted result in [12] it is sufficient to prove $\sharp(S) \le t(t - a_{3,t} - a_{3,t-2} + 3)$. This is true for the following reason. We have $b_{3,t-2} \le (t-1)/3$ (more precisely, by [4], III.1, we have $b_{3,t-2} = 0$ if $t \equiv 1, 2, 4, 5 \pmod 3$, $b_{3,t-2} = (t-6)/3 + 1$ if $t \equiv 0 \pmod 6$ and $b_{3,t-2} = (t-3)/3$ if $t \equiv 3 \pmod 6$). Take any $F \in B(a_{3,t} + 3, a_{3,t}, 3, 0)$ and set $Y := E \cup J \cup (\cup_i L_i)$. We have $h^0(Y, \mathcal{O}_Y(t)) = h^0(F, \mathcal{O}_F(t)) + b - 2t + 2$. Use the difference of the equation (2) for $r = 3$ with the same equation for the pair $(r', t') = (3, t-2)$ (in which the integer $2a_{3,t} = 2 \cdot \deg(E)$ appears).

(b) Now assume $b \ge a_{3,t} - a_{3,t-2} - 2$.

(b1) In this step we handle all cases with $t \le 5$. First assume $t = 3$ and hence $b = 1$ and $a \le 2$. Take any $F_1 \in B(a+3, a, 3, 0)$. Let $H \subset \mathbb{P}^3$ be a general plane. Let $T \subset H$ be a line containing exactly one point, O, of $F_1 \cap H$. Obviously, $h^1(H, \mathcal{I}_{H\cap F_1\setminus\{O\}}(3)) = 0$ for a suitable choice of O. Use that $\mathrm{Res}_H(F_1 \cup T) = F_1$ and $h^1(\mathcal{I}_C(2)) = 0$. Now assume $t = 4$. Hence $b \in \{1, 2\}$ and $a \le 5 - b$. Fix a general $F_2 \in B(6, 3, 3, 0)$. Hence $h^i(\mathcal{I}_{F_2}(2)) = 0$, $i = 0, 1$ (see [2] or [4]). Take a smooth quadric surface Q containing a 2-secant line of F_2 (case $b = 1$) or general (case $b = 2$). Add in Q two disjoint lines meeting $Q \cap F_2$, one of them the 2-secant line if $b = 1$ and apply [12], Proposition 4.1. Now assume $t = 5$. We have $b \in \{1, 2, 3\}$ and $a \le 8 - b$. Fix a general $F_3 \in B(8, 5, 3, 0)$. We have $h^i(\mathcal{I}_{F_3}(3)) = 0$, $i = 0, 1$. Take a smooth quadric surface containing $a - 5$ 2-secant line of F_3 and add in Q b lines in the same system of lines, each of them containing a point of $F_3 \cap Q$. We may apply [12], Proposition 4.1, but here the situation is simpler, because we choose the b points of $F_3 \cap Q$ to which we link the b 1-secant lines.

(b2) Now assume $t \geq 6$; we need $t \geq 6$ to have $b - a_{3,t} - a_{3,t-2} - 2 \leq a_{3,t-2} - a_{3,t-4} - 1$ (see [4], III.1). We make the construction of step (a) with respect to the integer $t' := t-2$ and the integer $b' := b$, i.e. we start with a general $E' \in B(a_{3,t-4}, a_{3,t-4} - 3, 3, 0)$; call E', J', L'_i the corresponding objects and Y' their union. We have $h^1(\mathcal{I}_{Y'}(t-2)) = 0$. Then we deform Y' to a general union $Y'' = E_1 \cup T_1$ of $E_1 \in B(a, a-3, 3, 0)$, E_1 smoothing of $E' \cup J \cup L'_j$, $1 \leq j \leq a-1-a$, and $a_{3,t-2} - 1 - a$ general 1-secant lines of E_1. In the next step we add in Q $b - a_{3,t-2} + 1 - a$ lines of type $(0,1)$, each of them containing a unique point of $E_1 \cap Q$. By [12], Proposition 4.1, the postulation of $E_1 \cap Q$ is good for general E_1, but now, for a fixed E_1, we need to control also $T_1 \cap Q$. It may be done in two ways: either quoting [4], VIII (a very small part of it is needed, because the line in T_1 are 1-secant to E_1, not 2-secant to E_1) or using [2], Lemma 6.2 and Proposition 6.6.

(c) Now we assume $a \leq a_{3,t} + 2 - b$. We repeat the proofs in steps (a) and (b) with the following modifications.

(c1) First take the set-up of step (a). If $a \geq a_{3,t-2} + 1$, then we start with $E \cup J$, but we assume that only $a - a_{3,t-2} - 1$ of the lines L_i are 1-secant to E. If $a = a_{3,t-2}$ we take as J a 1-secant line of E and impose $L_i \cap E = \emptyset$ for all i. Now assume $a < a_{3,t-2}$. We start with a general $E_2 \in B(a, a-3, 3, 0$, take as Q a smooth quadric containing no secant line of E_2. We don't add J, but all the added lines L_i, $1 \leq i \leq b$, are 1-secant. There is no problem with $Q \cap E_2$, because $\sharp(Q \cap E_2)) \leq \sharp(Q \cap E)$.

(c2) Now assume that we are in the set-up of (b). We make the modifications detailed in step (c1) taking the integer $t' := t-2$, i.e. from $H^0(\mathcal{O}_{\mathbb{P}^3}(t-4))$ to $H^0(\mathcal{O}_{\mathbb{P}^3}(t-2))$. Then in the step from $H^0(\mathcal{O}_{\mathbb{P}^3}(t-4))$ to $H^0(\mathcal{O}_{\mathbb{P}^3}(t-2))$ we add the remaining $b - (a_{3,t-2} - a_{3,t-4} -)$ 1-secant lines. □

Lemma 8. *Fix $b \in \{1,2\}$. Let $C \subset \mathbb{P}^3$ be any element of $B(5-b, 2-b, 3, 0)$. Let $T \subset \mathbb{P}^3$ a general union of b lines. Then $h^1(\mathcal{I}_{C \cup T}(3)) = 0$.*

Proof. Fix a general quadric surface Q containing T, say as b lines of type $(1,0)$. For general T and general Q the set $C \cap Q$ is formed by $10 - 2b$ points imposing independent conditions to $|\mathcal{O}_Q(3-b, 3)|$. We have $h^1(\mathcal{I}_C(1)) = 0$ and $\mathrm{Res}_Q(C \cup T) = C$. Apply Remark 1. □

Lemma 9. *For all integers $t \geq 5$ we have $a_{4,t-1} \geq 2t + 4$.*

Proof. If $t = 5, 6$, then use Remark 3. For $t \geq 7$ we use the case $r = 4$ of (3) and that $a_{4,t} \leq (\binom{t+4}{4} - 3t - 3)/(t-1)$. □

Lemma 10. *Fix integers m, t, b, a such that $m \geq 4$, $t \geq 2$, $m \leq a \leq a_{m,t} + m - b - 1$ and $0 \leq b \leq t-2$. Let Z be a general element of $B(a, a-m, m, 0)$. Let $T \subset \mathbb{P}^m$ be a general disjoint union of b lines, each of them 1-secant to Z. Then $h^1(\mathcal{I}_{Z \cup T}(t)) = 0$.*

Proof. The lemma is true if $b = 0$ by [3] and [5]. Hence we may assume $b > 0$. Until step (c) we assume $a = a_{m,t} - b + m - 1$, except in the case $m = 4$ and $t = 3$, in which we analyze all possibilities

(a) We assume $m \geq 5$ and that the lemma is true in $\mathbb{P}^{m-1}$; in step (b) we will do the case $m = 4$. Let $H \subset \mathbb{P}^m$ be a hyperplane. Fix a general $A \in B(a_{m,t-1} + m, a_{m,t-1}, m, 0)$. By [3] and [5] A has maximal rank. Hence $h^1(\mathcal{I}_A(t-1)) = 0$ and $h^0(\mathcal{I}_A(t-1)) = b_{m,t-1} \leq \max\{0, t-3\}$. Moving A we may assume that A intersects transversally H and that the set $A \cap H$ is in linearly general position. Lemma 6 gives $a_{m,t} - a_{m,t-1} \geq m + t - 3$ and in particular $a_{m,t} - a_{m,t-1} \geq m - 1 + b$. Let $E \subset H$ be a general element of $B(a_{m,t} - a_{m,t-1} - b, a_{m,t} - a_{m,t-1} - b + 1 - m, m - 1, H)$ with the only restriction that E contains m points of $A \cap H$. Since $\mathrm{Aut}(H)$ is transitive on the r-ples of m points of H in linearly general position, we may see E as a general element of $B(a_{m,t} - a_{m,t-1} - b, a_{m,t} - a_{m,t-1} - b + 1 - m, m - 1, H)$. Let $T \subset H$ be a general union of b 1-secant lines of H. Since $\deg(A) \geq m + 2t$, we have $a_{m,t} - a_{m,t-1} \leq a_{m-1,t} - 1$ (use (3) for $r = m$). Hence we may apply the inductive assumption and get $h^1(H, \mathcal{I}_{E \cup T}(t)) = 0$. By [5], Lemma 1.6, we obtain $h^1(H, \mathcal{I}_{E \cup T \cup (A \cap H), H}(t)) = 0$. Apply Remark 1.

(b) Now assume $m = 4$. To use the previous proof we need to prove $h^1(H, \mathcal{I}_{E \cup T}(t)) = 0$. To apply Lemma 7 we need $\deg(E \cup T) \leq a_{3,t} - 1$, not just $\deg(E \cup T) \leq a_{m,t} - 2$ as in the case $m \geq 5$. If $t \geq 5$ we have $\deg(E \cup T) \leq a_{3,t} - 2$ by Lemma 9. Now assume $t = 3$. Hence $b = 1$. Hence $a \leq 8$. If $a \leq a_{4,2} = 6$, then we start with a general $E_3 \in B(a + 4, a, 4, 0)$ and add in H a 1-secant line of E_3. If $7 \leq a \leq 8$, we start with a general $E_4 \in B(10, 6, 4, 0)$, take as H a hyperplane spanned by two general secant lines of H and add in H $a - 6$ of these 2-secant lines and another line through a point of $H \cap E_3$; we do not use Lemma 7, because in H we only have 2 or 3 disjoint lines. Now assume $t = 4$. Hence $b \in \{1, 2\}$. Recall that $a_{4,3} = 11$, $b_{4,3} = 0$, $a_{4,4} = 17$ and $b_{4,4} = 2$. Take a general $E_5 \in B(14, 11, 4, 0)$. We add in H a general rational curve $G \in B(5 - b, 2 - b, H, 0)$ containing 4 points of $E_5 \cap H$ and b general lines disjoint from G and each of them containing one of the points of $E_5 \cap H$. Apply Lemma 8.

(c) Now assume $a < a_{m,t}+m-b-1$. First assume $a \le a_{m,t-1}$. Take a general $A' \in B(a, a-m, m, 0)$. Since A' has maximal rank, we have $h^1(\mathcal{I}_{A'}(t-1)) = 0$. We add in H b lines, each of them containing a different point of $A' \cap H$. Now assume $a_{m,t-1} < a \le a_{m,t-1} + m$. In this case we add in H the b 1-secant lines to A and instead of E a smooth rational curve E_1 of degree $a - a_{m,t-1}$ linearly normal in its linear span and containing $\deg(E_1) + 1$ points of $A \cap H$. Now assume $a > a_{m,t-1} + m$. We copy step (b), except that instead of E we take a general element of $B(a_{m,t} - a_{m,t-1} - b, a_{m,t} - a_{m,t-1} - b + 1 - m, m - 1, H)$ containing m points of $A \cap H$. □

Lemma 11. *$H_{r,t}$ is true for all integers $r \ge 4$ and $t \ge 2$.*

Proof. Since $b_{r,2} = 0$, for any $r \ge 3$ the case $k = (r^2-3r)/2-1$ of Lemma 5 implies $H_{r,2}$. Now assume $t \ge 3$ and that $H_{r,t-1}$ is true. Fix (A, D, T) satisfying $H_{r,t-1}$. Deforming if necessary (A, D, T) we may assume that $A \cup D \cup T$ is transversal to H. Hence $P := A \cap D \notin H$.

(a) In this step we assume $b_{r,t} \ge b_{r,t-1}$. Lemma 6 gives $a_{r,t} - a_{r,t-1} \ge r - 1 + b_{r,t} - b_{r,t-1}$. Let $E \subset H$ be a general element of $B(a_{r,t} - a_{r,t} - b_{r,t} + b_{r-1,t}, H)$ containing one point of $A \cap H$. Let $F \subset H$ be a general union of $b_{r,t} - b_{r,t-1}$ lines, each of them 1-secant to E. Since these lines are general 1-secant line, we have $F \cap A = \emptyset$. Set $A' := A \cup E$ and $T' := T \cup F$. Hence $T' \cap D = \emptyset$, T' is a disjoint union of $b_{r,t}$ lines and each line of T' is 1-secant to A'. We have $A' \in B(a_{r,t} + r - 1 - b_{r,t}, a_{r,t} - b_{r,t}, r, 1)'$, the only section of $\omega_{A'}(-1)$ has only finitely many zeroes and P is one of them. By Remark 1 and semicontinuity to prove $H_{r,t}$ it is sufficient to prove $h^1(\mathcal{I}_{A'\cup D\cup T'}(t)) = 0$. Since $\mathrm{Res}_H(A' \cup D \cup T') = A \cup D \cup T$ and $h^1(\mathcal{I}_{A\cup D\cup T}(t)) = 0$, it is sufficient to prove $h^1(H, \mathcal{I}_{H\cap(A'\cup B'\cup T'),H}(t)) = 0$. The scheme $H \cap (A' \cup D \cup T'$ is the disjoint union of $F \cup E$, the point $D \cap H$, the $b_{r,t-1}$ points of $T \cap H$ and $\deg(A)-$ points of $A \cap H$. Hence $h^0(H \cap (A' \cup B' \cup T', \mathcal{O}_{H\cap(A'\cup B\cup T')}(t)) = \binom{r+t-1}{r-1}$. By Lemmas 10 and 9 we have $h^1(H, \mathcal{I}_{E\cup F}(t)) = 0$. Set $S := (A \cap H) \setminus A \cap (E \cup F)$.

Claim 1. *We claim that $h^1(H, \mathcal{I}_{E\cup F\cup(A\cap H),H}(t)) = 0$.*

Proof of Claim 1. It is sufficient to prove that we may apply [5], Lemma 1.6, even if one of the key assumptions there is not satisfied: here A is a general element of $B(a_{r,t}-b_{r,t}-1, a_{r,t}-b_{r,t}, r, 0)$, not of $B(a_{r,t}-b_{r,t}-1, a_{r,t}-b_{r,t}-1, r, 0)$. A is a deformation of a general union of a canonically embedded curve $C \subset \mathbb{P}^r$ and $a_{r,t-1} - b_{r,t-1} + r - 1 - 2r$ general secant lines of C. We take C with $C \cap H =$

$S' \sqcup S''$ with $\sharp(S) = r$ and use S' as a subset of $A \cap H$ containing G. A standard exact sequence gives $h^1(H, \mathcal{I}_{C\cap H,H}(2)) = 1$ and hence $h^1(H, \mathcal{I}_{C\cap H,H}(3)) = 0$. Then we apply [5], Lemma 1.4, to these secant lines. Hence we may use A as the curve Y in [5], Lemma 1.6. To use [5], Lemma 1.6, we need to check the numerical conditions for F stated in [5], Lemma 1.6; alternatively we need to check that we may find F passing through r general points of H; this is for free.

By Claim 1 we have $h^0(H, \mathcal{I}_{E\cup F\cup(A\cap H),H}(t)) = b_{r,t-1} + 1$. Now we move the components of $D \cup T$ keeping fixed their point of intersection with A. Notice that we keep $A \cup E \cup F$ and (hence $tE \cup F \cup (A \cap H)$, but obtain as $(D \cup T) \cap H$ any general subset of H with cardinality $b_{r,t-1} + 1$. Hence for general D and T we get $h^0(H, \mathcal{I}_{E\cup F\cup(A\cap H)\cup(H\cap(D\cup T)),H}(t)) = 0$. Hence $h^1(H, \mathcal{I}_{E\cup F\cup(A\cap H)\cup(H\cap(D\cup T)),H}(t)) = 0$, concluding the proof of $H_{r,t}$ in this case.

(b) In this step we assume $b_{r,t} < b_{r,t-1}$. Set $T = T_1 \cup T_2$ with T_2 any of the $b_{r,t-1} - b_{r,t}$ lines of T.

Claim 2. *A general element of $Z(a_{r,t} - a_{r,t-1}, a_{r,t} - a_{r,t-1} - r + 1, H)$ contains $r + b_{r,t-1} - b_{r,t}$ general points of H.*

Proof of Claim 2. First assume $r = 4$. Hence $H \cong \mathbb{P}^3$. In this case it is sufficient to have $\deg(G) \geq 4 + b_{4,t-1} - b_{4,t}$ (e.g. by [12], Proposition 2.6). This inequality is true by Lemma 6. Now assume $r \geq 5$. In this case we use [5], Lemma 4.2, and apply the case $n := r - 1$ of [5], Lemma 1.5.

Since $\deg(A) \geq r$ and $\mathrm{Aut}(H)$ is transitive on the r-ples of linearly independents point of H, we may assume that A contains a set $S_1 \subset H$ with $\sharp(S_1) = r$. For fixed A (and hence for fixed S_1) we may assume that $T_2 \cap H$ is a general subset of H with cardinality $b_{r,t-1} - b_{r,t}$. Hence we may assume that $S_1 \cup (T_2 \cap H)$ is general in H. Let $G \subset H$ be a general element of $Z(a_{r,t} - a_{r,t-1}, a_{r,t} - a_{r,t-1} - r + 1, H)$ containing $S_1 \cup (H \cap T_2)$. We saw that we may consider G as a general element of $Z(a_{r,t} - a_{r,t-1}, a_{r,t} - a_{r,t-1} - r + 1, H)$. Hence $h^1(H, \mathcal{I}_G(t)) = 0$ (see [4] if $r - 1 = 3$, [3] if $r - 1 = 4$ and [5] if $r - 1 \geq 5$).

Claim 3. *For general A, D, T_2, G we have $h^1(H, \mathcal{I}_{G\cup(A\cup D\cup T_2)\cap H,H}(t)) = 0$.*

Proof of Claim 3. The proof of Claim 1, works, except that now we need to find G containing $r + b_{r,t-1} - b_{r,t}$ general points of H. This is true by Claim 2.

By Claim 2 we have $h^1(H, \mathcal{I}_{G\cup(A\cap H)}(t)) = 0$, i.e. $h^0(H, \mathcal{I}_{G\cup(A\cap H),H}(t)) = 1 + b_{r,t-1} - b_{r,t}$. We keep fixed A, G and T_2, but moves the components of $D \cup T_1$ keeping fixed their point of intersection with A. Moving these lines in this way we get $h^1(H, \mathcal{I}_{G\cup(A\cup D\cup T)\cap H,H}(t)) = h^0(H, \mathcal{I}_{G\cup(A\cup D\cup T)\cap H,H}(t)) = 0$. Hence

Remark 1 gives $h^1(\mathcal{I}_{A\cup D\cup T\cup G}(t)) = 0$. We have $A \cup T_2 \cup G \in B(a_{r,t} - b_{r,t} + r - 1, a_{r,t} - b_{r,t}, r, 1)'$ (Lemma 4 and Remark 2), the only (up to a scalar multiple) non-zero-section of $\omega_{A\cup G}(-1)$ has only finitely many zeroes and P is one of them. Each connected component of T_1 is a 2-secant line of A not intersecting D. By Lemmas 3 and 4 we have $A \cup D \cup T_2 \cup G \in B(a_{r,t} + r, a_{r,t} - b_{r,t}, r, 1)'$, concluding the proof of $H_{r,t}$. □

Lemma 12. *Fix integers $r \geq 4$ and $t \geq 2$. Let X be a general element of $B(a_{r,t} + r, a_{r,t}, r, 1)$. Then $h^1(\mathcal{I}_X(t)) = 0$ and $h^0(\mathcal{I}_X(t)) = b_{r,t}$.*

Proof. For any $X \in B(a_{r,t}+r, a_{r,t}, r, 1)$ we have $h^1(\mathcal{I}_X(t)) = 0$, $h^1(\mathcal{I}_X(t)) = b_{r,t} + h^1(\mathcal{I}_X(t)) = 0$. Hence it is sufficient to prove $h^1(\mathcal{I}_X(t)) = 0$. If $t = 2$, then use Lemma 5. Hence we may assume $t \geq 3$. By semicontinuity it is sufficient to find $X_1 \in B(a_{r,t} + r, a_{r,t}, r, 1)'$ such that $h^1(\mathcal{I}_{X_1}(t)) = 0$. Take a hyperplane $H \subset \mathbb{P}^r$ and (A, D, T) satisfying $H_{r,t-1}$ (Lemma 11). We have $a_{r,t} - a_{r,t-1} \geq r-1$ (Remark 3 and Lemma 6). Fix a general $G \in Z(a_{r,t} - a_{r,t-1}, a_{r,t-1} - a_{r,t-1}, r)$ containing r points of $A \cap H$ and the set $T \cap H$. This is possible by Lemma 6. We have $A \cup D \cup T \cup G \in B(a_{r,t}+r, a_{r,t}, r, 1)'$. Set $X_1 := A \cup D \cup T \cup G$. Remark 1 and step (b) of the proof of Lemma 11 gives $h^1(\mathcal{I}_{X_1}(t)) = 0$. Riemann-Roch gives $h^0(\mathcal{I}_{X_1}(t)) = b_{r,t}$. □

Lemma 13. *Fix integers $r \geq 4$ and $t \geq 2$. Let X be a general element of $B(a_{r,t-1} + r + 1, a_{r,t-1} + 1, r, 1)$. Then $h^0(\mathcal{I}_X(t-1)) = 0$.*

Proof. It $t = 2$, then we use that X is non-degenerate. If $t = 3$, then we use Hence it is sufficient to prove $h^1(\mathcal{I}_X(t)) = 0$. If $t = 3$, then use Lemma 5 (or a curve Y given by Lemma 5 union a 2-secant line of Y). Hence we may assume $t \geq 4$. By semicontinuity it is sufficient to find $X' \in B(a_{r,t-1} + r + 1, a_{r,t-1} + 1, r, 1)'$ such that $h^0(\mathcal{I}_{X'}(t-1)) = 0$. Take a hyperplane $H \subset \mathbb{P}^r$ and (A, D, T) satisfying $H_{r,t-2}$ (Lemma 11). We have $a_{r,t-1} - a_{r,t-2} \geq r-1$ (Remark 3 and Lemma 6). Fix a general $G \in Z(a_{r,t} - a_{r,t-1} + 1, a_{r,t-1} - a_{r,t-1} + 1, r)$ containing r points of $A \cap H$ and the set $T \cap H$. This is possible by Lemma 6. We have $A \cup D \cup T \cup G \in B(a_{r,t} + r, a_{r,t}, r, 1)'$. Set $X_1 := A \cup D \cup T \cup G$. Since $h^0(\mathcal{I}_{A\cup D\cup T}(t-2)) = 0$, Remark 1 and step (b) of the proof of Lemma 11 gives $h^0(\mathcal{I}_{A\cup D\cup T\cup G}(t-1)) = 0$. □

Proof of Theorem 1. Fix integers $r \geq 4$ and $g \geq r+2$ and set $d := g+r$. Let k be the only integer ≥ 2 such that $a_{r,k-1} < g \leq a_{r,k}$. Any $C \in B(g+r, g, r, 1)$ has maximal rank if and only if $h^0(\mathcal{I}_C(k-1)) = 0$ and $h^1(\mathcal{I}_Y(k)) = 0$. Since $B(g + r, g, r, 1)$ is irreducible, it is sufficient to prove the existence of $X_i \in B(g + r, g, r, 1)$, $i = 0, 1$, such that $h^0(\mathcal{I}_{X_0}(k-1)) = 0$ and $h^1(\mathcal{I}_{X_1}(k)) = 0$-.

(a) Here we prove the existence of X_0. By Lemma 13 there is $Y \in B(a_{r,k-1}+r+1, a_{r,k-1}+1, r, 1)$ such that $h^0(\mathcal{I}_Y(k-1)) = 0$. If $g = a_{r,k-1}+1$, then take $X_0 := Y$. If $g > a_{r,k-1}+1$, then take as X_0 a general union of Y and $g-a_{r,k-1}-1$ 2-secant lines of Y (Lemma 3).

(b) Here we prove the existence of X_1. If $g = a_{r,k}$, then use Lemma 11. Hence we may assume $g < a_{r,k}$. Hence $\binom{r+k}{r} - kd - 1 + g \le \binom{r+k}{k} - b_{r,k-1}$. By Lemma 5 we may assume $k \ge 3$. Fix a hyperplane $H \subset \mathbb{P}^r$. Take a general $Y \in B(a_{r,k-1}+r, a_{r,k-1}+1, r, 1)$. Lemma 12 gives $h^1(\mathcal{I}_Y(k-1)) = 0$. For general Y we may assume that Y is transversal to H and that the points of $Y \cap H$ are in linearly general position. First assume $d - a_{r,k-1} - r \ge r-1$. Let $B \subset H$ be a general element of $B(d-a_{r,k-1}-r, d-a_{r,k-1}-r+1)$ containing exactly r points of $Y \cap H$. Since $d \le a_{r,k}$ ++ gives $k(d-a_{r,k-1}-r)+1-(a_{r,k-1}-r+1) \le \binom{r+k-1}{r-1}$. Hence $h^1(H, \mathcal{I}_B(k)))00$ (see [4] if $r = 4$, [3] if $r = 5$ and [5] if $r > 5$). As in the proof of Lemma 11 we see that $h^1(H, \mathcal{I}_{B\cup(A\cap H}(k)) = 0$ for a general A. Remark 1 gives $h^1(\mathcal{I}_{Y\cup B}(k)) = 0$. Now assume $d \le a_{r,k} + 2r$. Recall that $g > a_{r,k-1}$. Instead of B we take a general rational curve $B' \subset B$ of degree $g - a_{r,k-1}$ (i.e. a smooth rational curve of degree $g - a_{r,k-1}$ contained in H and linearly normal in its linear span) containing exactly $\deg(B')+1$ points of $Y \cap H$ (here we use that $Y \cap H$ is in linearly general position). □

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