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NEW DEFINITIONS AND THEOREMS VIA GENERALIZED CONVEX DOMINATED MAPPINGS

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Abstract: In this article we define new (g, s, m)-convex and (g, h, m)-convex dominated mappings and obtain new Hermite-Hadamard-like type inequalities on the class of these convex dominated mappings.

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1. Introduction

The following double inequality is well-known in the literature as Hermite-Hadamard's inequality for convex mappings: Let $f : I \subseteq R \to R$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$ with $a < b$. Then the following inequality holds:

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

Both the inequalities hold in reversed direction if f is concave on I.

In [4], Hudzik and Maligranda considers, among others, the class of mappings which are s-convex in the second sense. This class is defined as follows:

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Definition 1.1. A mapping $f : I \subseteq R^+ = [0,\infty) \to R$ is said to be s-convex in the second sense on I if the inequality

$$f(tx+(1-t)y) \le t^s f(x) + (1-t)^s f(y) \tag{1}$$

holds, for all $x, y \in I$ and for some fixed $s \in (0,1]$. We denote the class of s-convex mappings in the second sense by $K_s^2(I)$. The inequality (1) holds in reversed direction if f is s-concave in the second sense.

For the recent results based on the above Definition 1.1, you may see the papers [1, 2, 4, 7, 10, 12] and [13].

In [2], Dragomir and Fitzpatrick established the following inequality:

Theorem 1.1. *Let $f : I \subseteq R^+ \to R$ be an s-convex mapping in the second sense on I, where $s \in (0,1)$ and $a, b \in I$ with $a < b$. If f is in $L([0,1])$, then the following double inequality holds:*

$$2^{s-1} f(\frac{a+b}{2}) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{s+1}. \tag{2}$$

Both the inequalities (2) hold in reversed direction if f is s-concave in the second sense.

In [15], Toader introduced the class of m-convex mappings as the following:

Definition 1.2. A mapping $f : [0,b] \to R$, $b > 0$, is said to be m-convex, where $m \in [0,1]$, if the following inequality

$$f(tx + m(1-t)y) \le tf(x) + m(1-t)f(y)$$

holds for all $x, y \in [0,b]$ and $t \in [0,1]$. We say that f is m-concave if $(-f)$ is m-convex.

In [3], Dragomir and Ionescu introduced the class of (s,m)-convex mappings in the second sense.

Definition 1.3. A mapping $f : [0,b] \to R$, $b > 0$, is said to be (s,m)-convex in the second sense on $[0,b]$ if the following inequality

$$f(tx + m(1-t)y) \le t^s f(x) + m(1-t)^s f(y)$$

holds, for all $x, y \in [0,b]$, $(t,m) \in [0,1]^2$ and for some fixed $s \in (0,1]$.

In [11], Park established the following theorem.

Theorem 1.2. *Let $f : I \subseteq R^+ \to R$ be an (s,m)-convex mapping in the second sense on I, where $(s,m) \in (0,1]^2$, and $a,b \in I$ with $0 \le a < b < \infty$. If f is in $L([0,1])$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)dx \le \min\Big\{\frac{f(a)+mf(\frac{b}{m})}{s+1}, \frac{f(b)+mf(\frac{a}{m})}{s+1}\Big\}. \tag{3}$$

For the recent results based on the above definition, you may see the papers [11, 14].

In [16], Varošanec introduced the following class of mappings.

For the intervals I and J in R with $(0,1) \subseteq J$, let $f : I \to R$ and $h : J \to R$ be nonnegative mappings.

Definition 1.4. Let $h : J \to R$ be a nonnegative mapping with $h \not\equiv 0$. We say that $f : I \to R$ is an h-convex mapping, or that f belongs to the class $SX(h,I)$ if f is nonnegative and the following inequality

$$f(tx+(1-t)y) \le h(t)f(x)+h(1-t)g(y).$$

holds, for all $x,y \in I$ and $t \in (0,1]$.

In [6, 7, 8, 9], Kavurmaci et al. and Özdemir considered the class of (g,m)-convex dominated mappings and proved the following theorems.

Definition 1.5. Let $g : [0,b] \to R$ be a m-convex mapping on the interval $[0,b]$, where $m \in [0,1]$. A real mapping $f : [0,b] \to R$ is said to be (g,m)-convex dominated on $[0,b]$ if the following inequality

$$\begin{aligned}&\Big|tf(x)+m(1-t)f(y)-f(tx+m(1-t)y)\Big|\\ &\le tg(x)+m(1-t)g(y)-g(tx+m(1-t)y)\end{aligned}$$

holds, for all $x,y \in [0,b]$ and $t \in [0,1]$.

Theorem 1.3. *Let $g : [0,\infty) \to R$ be an m-convex mapping on the interval $[0,\infty)$, where $m \in (0,1]$, $f : [0,\infty) \to R$ be a (g,m)-convex dominated mapping on $[0,\infty)$ and $a,b \in [0,\infty)$ with $0 \le a < b < \infty$. If $f \in L([a,b])$, then one has the inequalities:*

$$\begin{aligned}(1)\quad &\Big|\frac{1}{b-a}\int_a^b \Big\{\frac{f(x)+mf(\frac{x}{m})}{2}\Big\}dx - f(\frac{a+b}{2})\Big|\\ &\le \frac{1}{b-a}\int_a^b \Big\{\frac{g(x)+mg(\frac{x}{m})}{2}\Big\}dx - g(\frac{a+b}{2}).\end{aligned}$$

$$(2)\quad \Big|\frac{1}{2}\Big[\Big\{\frac{f(a)+mf(\frac{a}{m})}{2}\Big\}+m\Big\{\frac{f(\frac{b}{m})+mf(\frac{b}{m^2})}{2}\Big\}\Big] - \frac{1}{b-a}\int_a^b\Big\{\frac{f(x)+mf(\frac{x}{m})}{2}\Big\}dx\Big| \le \frac{1}{2}\Big[\Big\{\frac{g(a)+mg(\frac{a}{m})}{2}\Big\}+m\Big\{\frac{g(\frac{b}{m})+mg(\frac{b}{m^2})}{2}\Big\}\Big] - \frac{1}{b-a}\int_a^b\Big\{\frac{g(x)+mg(\frac{x}{m})}{2}\Big\}dx.$$

In [5, 6], Kavurmaci et al. and Hwang considered the class of (g, s)-convex dominated mappings in the second sense and proved the following theorems for them.

Definition 1.6. Let $g : [0, b] \to R$ be an s-convex mapping in the second sense on the interval $[0, b]$. A real mapping $f : [0, b] \to R$ is said to be (g, s)-convex dominated in the second sense on $[0, b]$ if the following inequality

$$\Big|t^s f(x) + (1-t)^s f(y) - f(tx + (1-t)y)\Big| \le t^s g(x) + (1-t)^s g(y) - g(tx + (1-t)y)$$

holds, for all $x, y \in [0, b]$ and $t \in [0, 1]$ and $s \in (0, 1]$.

Theorem 1.4. *Let $g : [0, \infty) \to R$ be an s-convex mapping in the second sense on the interval $[0, \infty)$, where $s \in (0, 1]$, $f : [0, \infty) \to R$ be a (g, s)-convex dominated mapping in the second sense on $[0, \infty)$ and $a, b \in [0, \infty)$ with $0 \le a < b < \infty$. If $f \in L([a, b])$, then the following inequalities hold:*

$$(1)\quad \Big|\frac{1}{b-a}\int_a^b f(x) - 2^{s-1}f(\frac{a+b}{2})\Big| \le \frac{1}{b-a}\int_a^b g(x) - 2^{s-1}g(\frac{a+b}{2}).$$

$$(2)\quad \Big|\frac{f(a)+f(b)}{s+1} - \frac{1}{b-a}\int_a^b f(x)dx\Big| \le \frac{g(a)+g(b)}{s+1} - \frac{1}{b-a}\int_a^b g(x)dx.$$

In [6], Kavurmaci et al. considered the class of (g, h)-convex dominated mappings and proved the following theorems for them.

Definition 1.7. Let $h : J \to R$ be a nonnegative mapping with $h \not\equiv 0$ and $g : I \to R$ be an h-convex mapping. A real mapping $f : I \to R$ is said to be (g, h)-convex dominated on I if the following inequality

$$\Big|h(t)f(x) + h(1-t)f(y) - f(tx + (1-t)y)\Big| \le h(t)g(x) + h(1-t)g(y) - g(tx + (1-t)y)$$

holds, for all $x, y \in I$ and $t \in [0, 1]$.

Theorem 1.5. *Let $h : J \to R$ be a nonnegative mapping with $h \not\equiv 0$ and $g : I \to R$ be an h-convex mapping. Suppose that $f : I \to R$ is a (g, h)-convex dominated on I and $a, b \in [0, \infty)$ with $0 \le a < b < \infty$. If $f \in L[a, b]$, then the following inequalities*

$$(1)\quad \Big|\frac{1}{b-a}\int_a^b f(x)dx - \frac{1}{2h(\frac{1}{2})}f(\frac{a+b}{2})\Big| \le \frac{1}{b-a}\int_a^b g(x)dx - \frac{1}{2h(\frac{1}{2})}g(\frac{a+b}{2}).$$

$$(2)\quad \Big|\Big\{f(a)+f(b)\Big\}\int_0^1 h(t)dt - \frac{1}{b-a}\int_a^b f(x)dx\Big| \le \Big\{g(a)+g(b)\Big\}\int_0^1 h(t)dt - \frac{1}{b-a}\int_a^b g(x)dx$$

hold, for all $x \in I$ and $t \in [0, 1]$.

2. (g, s, m)-Convex Dominated Mappings

Definition 2.1. Let $g : [0, b] \to R$ be an (s, m)-convex mapping on the interval $[0, b]$. A real mapping $f : [0, b] \to R$ is said to be (g, s, m)-convex dominated in the second sense on $[0, b]$ if the following inequality

$$\Big|t^s f(x) + m(1-t)^s f(y) - f(tx + m(1-t)y)\Big| \le t^s g(x) + m(1-t)^s g(y) - g(tx + m(1-t)y) \tag{4}$$

holds, for all $x, y \in [0, b]$ and $(t, m) \in [0, 1]^2$ and for a fixed $s \in (0, 1]$.

The following simple characterization of (g, s, m)-convex dominated mappings holds:

Lemma 1. Let $g : [0, b] \to R$ be an (s, m)-convex mapping in the second sense on the interval $[0, b]$ and $f : [0, b] \to R$ be a real mapping on $[0, b]$. Then the following statements are equivalent:

(a) f is a (g, s, m)-convex mapping in the second sense on $[0, b]$.

(b) The mappings $g - f$ and $g + f$ are (s, m)-convex in the second sense on $[0, b]$.

(c) There exist two (s, m)-convex mappings h and k defined on $[0, b]$ such that

$$f = \frac{1}{2}(h - k) \quad\text{and}\quad g = \frac{1}{2}(h + k).$$

Proof. $(a) \leftrightarrow (b)$: The condition (a):

$$\begin{aligned}&\Big|t^s f(x) + m(1-t)^s f(y) - f(tx + m(1-t)y)\Big| \\ &\le t^s g(x) + m(1-t)^s g(y) - g(tx + m(1-t)y)\end{aligned}$$

is equivalent to

$$\begin{aligned}&g(tx + m(1-t)y) - t^s g(x) - m(1-t)^s g(y) \\ &\le t^s f(x) + m(1-t)^s f(y) - f(tx + m(1-t)y) \\ &\le t^s g(x) + m(1-t)^s g(y) - g(tx + m(1-t)y)\end{aligned}$$

for all $x, y \in I$ and $t \in [0, 1]$. The two inequalities may be rearranged as

$$\begin{aligned}(g+f)(tx + m(1-t)y) &\le t^s (g+f)(x) + m(1-t)^s (g+f)(y) \\ (g-f)(tx + m(1-t)y) &\le t^s (g-f)(x) + m(1-t)^s (g-f)(y),\end{aligned}$$

which are equivalent to the (s, m)-convexity of $g + f$ and $g - f$, respectively.

$(b) \leftrightarrow (c)$: Define the mappings f and g as $f = \frac{1}{2}(h - k)$ and $g = \frac{1}{2}(h + k)$, respectively. Then, if we sum and substract f and g, respectively, we have $g + f = h$ and $g - f = k$. By (b), the mappings $g + f$ and $g - f$ are (s, m)-convex in the second sense on $[0, b]$, so, h and k are (s, m)-convex mappings in the second sense.

Theorem 2.1. *Let $g : [0, \infty) \to R$ be an (s, m)-convex mapping in the second sense on the interval $[0, \infty)$, $f : [0, \infty) \to R$ be a real (g, s, m)-convex dominated mapping in the second sense, where $s \in (0, 1]$, $m \in [0, 1]$, and $a, b \in [0, \infty)$ with $0 \le a < b < \infty$. If $f \in L([a, b])$, then the following inequalities hold:*

$$\begin{aligned}(a)\quad &\Big|\frac{1}{b-a}\int_a^b \Big(\frac{f(x) + mf(\frac{x}{m})}{2}\Big)dx - 2^{s-1} f(\frac{a+b}{2})\Big| \\ &\le \frac{1}{b-a}\int_a^b \Big(\frac{g(x) + mg(\frac{x}{m})}{2}\Big)dx - 2^{s-1} g(\frac{a+b}{2}).\end{aligned}$$

$$\begin{aligned}(b)\quad &\Big|\frac{1}{s+1}\Big\{\Big(\frac{f(a) + mf(\frac{a}{m})}{2}\Big) + m\Big(\frac{f(\frac{b}{m}) + mf(\frac{b}{m^2})}{2}\Big)\Big\} \\ &\qquad - \frac{1}{b-a}\int_a^b \Big(\frac{f(x) + mf(\frac{x}{m})}{2}\Big)dx\Big| \\ &\le \frac{1}{s+1}\Big\{\Big(\frac{g(a) + mg(\frac{a}{m})}{2}\Big) + m\Big(\frac{g(\frac{b}{m}) + mg(\frac{b}{m^2})}{2}\Big)\Big\}\end{aligned}$$

$$-\frac{1}{b-a}\int_a^b\Big(\frac{g(x)+mg(\frac{x}{m})}{2}\Big)dx.$$

Proof. (a) By (3), we have

$$\begin{aligned}&\Big|t^s f(x)+m(1-t)^s f(y)-f(tx+m(1-t)y)\Big|\\&\le t^s g(x)+m(1-t)^s g(y)-g(tx+m(1-t)y)\end{aligned}\tag{5}$$

holds, for all $x,y\in[0,\infty)$, $(t,m)\in[0,1]^2$ and for a fixed $s\in(0,1]$.

If we choose $t=\frac{1}{2}$, then, since f is (g,s,m)-convex dominated mapping in the second sense, we have

$$\Big|\frac{1}{2^s}\Big\{f(x)+mf(y)\Big\}-f(\frac{x+my}{2})\Big|\le\frac{1}{2^s}\Big\{g(x)+mg(y)\Big\}-g(\frac{x+my}{2})\tag{6}$$

holds, for all $x,y\in[0,\infty)$ and $m\in[0,1]^2$ and for some fixed $s\in(0,1]$.

In (6), if we choose $x=ta+(1-t)b$ and $y=(1-t)\frac{a}{m}+t\frac{b}{m}$ for $t\in[0,1]$ and $m\in(0,1]$, we get

$$\begin{aligned}&\Big|\frac{1}{2^s}\Big\{f(ta+(1-t)b)+mf((1-t)\frac{a}{m}+t\frac{b}{m})\Big\}-f(\frac{a+b}{2})\Big|\\&\le\frac{1}{2^s}\Big\{g(ta+(1-t)b)+mg((1-t)\frac{a}{m}+t\frac{b}{m})\Big\}-g(\frac{a+b}{2}).\end{aligned}\tag{7}$$

Integrating the inequality (7) over t on $[0,1]$, we deduce that

$$\begin{aligned}&\Big|\frac{1}{b-a}\int_a^b\Big\{\frac{f(x)+mf(\frac{x}{m})}{2}\Big\}dx-2^{s-1}f(\frac{a+b}{2})\Big|\\&\le\frac{1}{b-a}\int_a^b\Big\{\frac{g(x)+mg(\frac{x}{m})}{2}\Big\}dx-2^{s-1}g(\frac{a+b}{2}).\end{aligned}$$

(b) Since f is a (g,s,m)-convex dominated mapping in the second sense, we have

$$\begin{aligned}&\Big|t^s f(x)+m(1-t)^s f(y)-f(tx+m(1-t)y)\Big|\\&\le t^s g(x)+m(1-t)^s g(y)-g(tx+m(1-t)y)\end{aligned}\tag{8}$$

for any $x,y>0$.

(i) If we let $x=a$ and $y=\frac{b}{m}$ in the inequality (8), then, for $t\in[0,1]$ we have

$$\Big|t^s f(a)+m(1-t)^s f(\frac{b}{m})-f(ta+m(1-t)\frac{b}{m})\Big|$$

$$\leq t^s g(a) + m(1-t)^s g(\frac{b}{m}) - g(ta + m(1-t)\frac{b}{m}). \tag{9}$$

(ii) If we choose $x = \frac{a}{m}$ and $y = \frac{b}{m^2}$ in the inequality (8), then, by multiplying with m, we have

$$\Big| mt^s f(\frac{a}{m}) + m^2(1-t)^s f(\frac{b}{m^2}) - mf(t\frac{a}{m} + (1-t)\frac{b}{m}) \Big|$$
$$\leq mt^s g(\frac{a}{m}) + m^2(1-t)^s g(\frac{b}{m^2}) - mg(t\frac{a}{m} + (1-t)\frac{b}{m}) \tag{10}$$

for all $t \in [0,1]$.

By properties of modulus, if we add the above inequalities (9) and (10), we get

$$\Big| t^s \Big\{ f(a) + mf(\frac{a}{m}) \Big\} + m(1-t)^s \Big\{ f(\frac{b}{m}) + mf(\frac{b}{m^2}) \Big\}$$
$$- \Big\{ f(ta + (1-t)b) + mf(t\frac{a}{m} + (1-t)\frac{b}{m}) \Big\} \Big|$$
$$\leq t^s \Big\{ g(a) + mg(\frac{a}{m}) \Big\} + m(1-t)^s \Big\{ g(\frac{b}{m}) + mg(\frac{b}{m^2}) \Big\}$$
$$- \Big\{ g(ta + (1-t)b) + mg(t\frac{a}{m} + (1-t)\frac{b}{m}) \Big\}. \tag{11}$$

Thus, integrating the inequality (11) over t on $[0,1]$, we get the desired inequality (b).

3. (g,h,m)-Convex Dominated Mappings

Definition 3.1. Let $h : J \to R$ be a nonnegative mapping on the interval J with $h \not\equiv 0$. We say that $f : I \to R$ is an (h,m)-convex mapping on I if f is nonnegative and the following inequality

$$f(tx + m(1-t)y) \leq h(t)f(x) + mh(1-t)f(y) \tag{12}$$

holds, for all $x, y \in I$ and $(t,m) \in (0,1]^2$.

Definition 3.2. Let $h : J \to R$ be a nonnegative mapping on the interval J with $h \not\equiv 0$ and $g : I \to R$ be an (h,m)-convex mapping on I. A real mapping $f : I \to R$ is called (g,h,m)-convex dominated on I if the following inequality

$$\Big| h(t)f(x) + mh(1-t)f(y) - f(tx + m(1-t)y) \Big|$$
$$\leq h(t)g(x) + mh(1-t)g(y) - g(tx + m(1-t)y) \tag{13}$$

holds, for all $x, y \in I$ and $(t,m) \in (0,1]^2$.

The following simple characterization of (g, h, m)-convex dominated mappings holds:

Lemma 2. Let $h : J \to R$ be a nonnegative mapping on the interval J with $h \not\equiv 0$ and $g : I \to R$ be an (h, m)-convex mapping on I. Then, for a real mapping $f : I \to R$, the following statements are equivalent:

(a) f is a (g, h, m)-convex dominated mapping on I.

(b) The mappings $g - f$ and $g + f$ are (h, m)-convex on I.

(c) There exist two (h, m)-convex mappings l and k defined on $[0, b]$ such that

$$f = \frac{1}{2}(l - k) \text{ and } g = \frac{1}{2}(l + k).$$

Proof. By the similar way as in the proof of Lemma 1, this lemma is proved.

Theorem 3.1. *Let $h : J \to R$ be a nonnegative mapping on the interval J with $h \not\equiv 0$ and $g : I \to R$ be an (h, m)-convex mapping on I, where $m \in (0, 1)$. Then for a real (g, h, m)-convex mapping $f : I \to R$ on I, the following inequalities hold:*

$$\begin{aligned}
(a)\quad & \Big|\frac{1}{b-a}\int_a^b \Big\{\frac{f(x) + mf(\frac{x}{m})}{2}\Big\}dx - \frac{1}{2h(\frac{1}{2})}f(\frac{a+b}{2})\Big| \\
& \le \frac{1}{b-a}\int_a^b \Big\{\frac{g(x) + mg(\frac{x}{m})}{2}\Big\}dx - \frac{1}{2h(\frac{1}{2})}g(\frac{a+b}{2}),\\
(b)\quad & \Big|\Big[\Big\{\frac{f(a) + mf(b)}{2}\Big\} + m\Big\{\frac{f(\frac{b}{m}) + mf(\frac{b}{m^2})}{2}\Big\}\Big]\int_0^1 h(t)dt \\
& \qquad - \frac{1}{b-a}\int_a^b \Big\{\frac{f(x) + mf(\frac{x}{m})}{2}\Big\}dx\Big| \\
& \le \Big[\Big\{\frac{g(a) + mg(b)}{2}\Big\} + m\Big\{\frac{g(\frac{b}{m}) + mg(\frac{b}{m^2})}{2}\Big\}\Big]\int_0^1 h(t)dt \\
& \qquad - \frac{1}{b-a}\int_a^b \Big\{\frac{g(x) + mg(\frac{x}{m})}{2}\Big\}dx
\end{aligned}$$

Proof. (a) By (13), we have

$$\Big|h(\frac{1}{2})\Big\{f(x) + mf(y)\Big\} - f(\frac{x+my}{2})\Big| \le h(\frac{1}{2})\Big\{g(x) + mg(y)\Big\} - g(\frac{x+my}{2}). \tag{14}$$

holds, for all $x, y \in I$ and $m \in (0, 1]$.

In (14), if we choose $x = ta + (1-t)b$ and $y = (1-t)\frac{a}{m} + t\frac{b}{m}$, then the following inequality

$$\begin{aligned} &\Big|h(\frac{1}{2})\Big\{f(ta+(1-t)b) + mf((1-t)\frac{a}{m} + t\frac{b}{m})\Big\} - f(\frac{a+b}{2})\Big| \\ &\le h(\frac{1}{2})\Big\{g(ta+(1-t)b) + mg((1-t)\frac{a}{m} + t\frac{b}{m})\Big\} - g(\frac{a+b}{2}) \qquad (15) \end{aligned}$$

holds, for all $x, y \in I$ and $m \in (0,1]$.

Integrating the inequality (15) over t on $[0,1]$, the first inequality is proved.

(b) Since f is a (g,h,m)-convex dominated mapping, we have that the following inequality

$$\begin{aligned} &\Big|h(t)f(x) + mh(1-t)f(y) - f(tx+m(1-t)y)\Big| \\ &\le h(t)g(x) + mh(1-t)g(y) - g(tx+m(1-t)y) \qquad (16) \end{aligned}$$

holds, for any $x, y > 0$.

(i) If we let $x = a$ and $y = \frac{b}{m}$ in the inequality (16), then we have

$$\begin{aligned} &\Big|h(t)f(a) + mh(1-t)f(\frac{b}{m}) - f(ta+(1-t)b)\Big| \\ &\le h(t)g(a) + mh(1-t)g(\frac{b}{m}) - g(ta+(1-t)b). \qquad (17) \end{aligned}$$

(ii) If we choose $x = \frac{a}{m}$ and $y = \frac{b}{m^2}$ in the inequality (16), then, by multiplying with m, we have

$$\begin{aligned} &\Big|mh(t)f(\frac{a}{m}) + m^2h(1-t)f(\frac{b}{m^2}) - mf(t\frac{a}{m} + (1-t)\frac{b}{m})\Big| \\ &\le mh(t)g(\frac{a}{m}) + m^2h(1-t)g(\frac{b}{m^2}) - mg(t\frac{a}{m} + (1-t)\frac{b}{m}) \qquad (18) \end{aligned}$$

for all $t \in [0,1]$.

By properties of modulus, if we add the above inequalities (17) and (18), we get

$$\begin{aligned} &\Big|h(t)\Big\{f(a) + mf(\frac{a}{m})\Big\} + mh(1-t)\Big\{f(\frac{b}{m}) + mf(\frac{b}{m^2})\Big\} \\ &\quad - \Big\{f(ta+(1-t)b) + mf(t\frac{a}{m} + (1-t)\frac{b}{m})\Big\}\Big| \\ &\le h(t)\Big\{g(a) + mg(\frac{a}{m})\Big\} + mh(1-t)\Big\{g(\frac{b}{m}) + mg(\frac{b}{m^2})\Big\} \end{aligned}$$

$$-\left\{g(ta+(1-t)b)+mg(t\frac{a}{m}+(1-t)\frac{b}{m})\right\}. \tag{19}$$

Thus, integrating the inequality (19) over t on $[0,1]$, we get the desired inequality (b).

References

[1] M. Avci, Havva Kavurmaci, M.E. Özdemir, New inequalities of Hermite-Hadamard type via s-convex functions in the second sense with applications, *Appl. Math. and Comput.*, **217** (2011), 5171-5176.

[2] S.S. Dragomir, S. Fitzpatrick, The Hadamard's inequality for s-convex functions in the second sense, *Demonstratio Math.*, **32**, No. 4 (1999), 687-696.

[3] S.S. Dragomir, N.M. Ionescu, On some inequalities for convex dominated functions, *Anal. Num. Theor. Approx.*, **19** (1990), 21-28.

[4] H. Hudzik, L. Maligranda, Some remarks on s-convex functions, *Aequationes Math.*, **48** (1994), 100-111.

[5] Shiow-Ru Hwang, Ming-In Ho, Chung-Shin Wang, Inequalities of Fejér type for G-convex dominated functions, *Tamsui Oxford J. of Math. Sci.*, **25**, No. 1 (2009), 55-69.

[6] Havva Kavurmaci, M.E. Özdemir, M.Z. Sarikaya, New definitions and theorems via different kinds of convex dominated functions, *RGMIA Res. Rep. Coll.*, **15** (2012), Art 9.

[7] U.S. Kirmaci, K. Klaričić, M.E. Özdemir, J. Pečarić, Hadamard-type inequalities for s-convex functions, *Appl. Math. Comput.*, **193**, No. 1 (2007), 26-35.

[8] M.E. Özdemir, Havva Kavurmaci, M. Tunc, Hermite-Hadamard-type inequalities for new different kinds of convex dominated functions, *arXiv:1202.2054v1 [math.CA], 9 Feb 2012.*

[9] M.E. Özdemir, M. Tunc, Havva Kavurmaci, Two new different kinds of convex dominated functions and inequalities via Hermite-Hadamard type, *arXiv:1202.2055v1 [math.CA], 9 Feb 2012.*

[10] J. Park, Generalizations of the Simpson-like type inequalities for co-ordinated s-convex mappings in the second sense, *Inter. J. of Math. and Math. Sci.*, **2012** (2012), Art 715751.

[11] J. Park, Hermite-Hadamard type inequalities for differentiable (s, m)-convex mappings in the second sense, *Far East J. of Math. Sci.*, **52**, No. 1 (2011), 57-73.

[12] J. Park, Generalizations of some Simpson-like type inequalities via differentiable s-convex mappings in the second sense, *Inter. J. of Math. and Math. Sci.*, **2011** (2011), Art 493531.

[13] J. Park, New generalizations of Simpson-type inequalities for differentiable s-convex mappings, *Far East J. of Math. Sci.*, **51(1)** (2011), 41-58.

[14] J. Park, Some inequalities of Hermite-Hadamard type via differentiable (s, m)-convex mappings, *Far East J. of Math. Sci.*, **52**, No. 2 (2011), 209-221.

[15] Gh. Toader, Some generalizations of the convexity, *Proc. of the Coll. on Approx. and Optim.*, Univ. Cluj-Napoca, Cluj-Napoca (1984), 329-338.

[16] S. Varošanec, On h-convexity, *J. Math. Anal. Appl.*, **326** (2007), 303-311.