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ANALYTICAL APPROACH TO THE PROBLEM OF CLOSE APPROACH OF TWO SMALL MASSES IN THE FIELD OF MASSIVE OBLATE PRIMARY

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Abstract: The present work deals with the encounter problem of two small masses m_2 and m_3 describing initially elliptical orbits around a massive oblate primary. The equations of motion of the center of mass of m_2 and m_3 are developed in the most general form without any restrictions on the orbital elements. To avoid the appearance of mixed secular terms, we applied the method of multiple variables expansion. To facilitate the application of this method we first transform to the Struble variables.

AMS Subject Classification: 58D30
Key Words: encounter problem, multiple variables expansion, Struble variables

1. Introduction

In the literature the encounter problem is usually defined as follows (e.g. Petit and Henon, 1986) : consider two light bodies m_2 and m_3 describing initially coplanar and circular orbits, with slightly different radii, around a heavy spherical primary m_1. As the distance between m_2 and m_3 gets smaller their mutual attraction is no longer negligible and becomes comparable to that of each to the primary and we say that we have an encounter problem. The measure of the

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difference in semi-major axes at the encounter time is called the impact parameter. The problem may be more generalized such that the orbits be elliptical (e.g. Moons et.al., 1988) and not coplanar (e.g. Brumberg and Ivanova, 1990).

The encounter problem occurs in astronomy in planetary rings, coorbital satellites, the accretion of particles by a protoplanet, the temporary capture of comets, the distribution of particles around the earth, the crowdy region occupied by communication satellites, and in interplanetary missions.

Spirig and Waldvogel (1985) and Waldvogel and Spirig (1988) developed a singular perturbation solution to the three body problem with two small masses as represented by Saturn's coorbiting satellites. Petit and Henon (1986) performed analytic developments for the encounter type solutions of the Hill's problem described as the limiting case of the planar circular three body problem when two of the masses are very small, the behavior of the solutions is examined before, during and after the encounter, both for small and large impact parameter. In another paper considering the same problem, Petit and Henon (1986) performed a detailed numerical study of the family of relative orbits obtained from the variation of the dimensionless impact parameter resulting when the equations of motion are set in the Hill's form, they compared the limiting cases of small and large impact parameters with previous works.

Moons et. al. (1988) formulated the planar encounter problem in canonical form accounting for the eccentricities of the orbits of the small masses. Then using Lie transform method they averaged the Hamiltonian, in the cases of small and large impact parameters, to obtain formulas for the changes in the orbital elements.

Brumberg and Ivanova (1990) extended the results of Henon and Petit (1986) to take into account the eccentricity of the Hill's problem as well as the eccentricity and inclination of the relative motion of the particles. Profoeg (1995) presented an algorithm to assess the importance of distant encounters on planetary systems of arbitrary eccentricity.

Namouni, et. al (1996), derived mapping equations of Hill's problem in the case of encounter type solutions. They reproduce a natural divergence during the encounter when the bodies come close to each other.

Valsecchi, et. al. (2000), deduced an analytic expression for the distribution of energy perturbations at close encounters between small bodies and planets. The analytic formulation reproduces well the results of the numerical integrations that pointed out the asymmetries in the distribution of energy perturbations.

The present work is concerned with two small masses m_2 and m_3 describing initially elliptical orbits around a massive oblate primary. The orbits have slightly different sizes so that m_2 and m_3 approaches each other and interaction occurs between them. During this encounter phase the attraction between m_2 and m_3 is assumed comparable to that of each with m_1.

2. Formulation

In this section the equations of motion for the case of encounter in a field resulting from an oblate planet will be derived. The potentials at m_1, m_2 and m_3, resulting from their own gravities, are

$$V_1 = -\frac{k^2 m_1 m_2}{S_{12}} - \frac{k^2 m_1 m_3}{S_{13}} \tag{1}$$

$$V_2 = -\frac{k^2 m_1 m_2}{S_{12}} - \frac{k^2 m_2 m_3}{S_{23}} - \frac{k^2 m_1 m_2 R_0^2}{2}\frac{J_2}{S_{12}^3}\left(1 - \frac{3z_{12}^2}{S_{12}^2}\right) \tag{2}$$

$$V_2 = -\frac{k^2 m_1 m_3}{S_{13}} - \frac{k^2 m_2 m_3}{S_{23}} - \frac{k^2 m_1 m_3 R_0^2}{2}\frac{J_2}{S_{13}^3}\left(1 - \frac{3z_{13}^2}{S_{13}^2}\right) \tag{3}$$

where s_{ij} is the distance between m_i and m_j. k and R_0 are the Gauassian constant and the mean equatorial radius of the oblate planet, and z_{ij} is the angular distance of m_2 and m_3 from the mean equator of the massive primary, respectively.

From equations (1) - (3) the force components can be written as;

$$m_1\ddot{\bar{S}}_1 = k^2 m_1 m_2 \frac{\bar{S}_{12}}{S_{12}^3} + k^2 m_1 m_3 \frac{\bar{S}_{13}}{S_{13}^3} \tag{4}$$

$$\begin{aligned} m_2\ddot{\bar{S}}_2 = & -k^2 m_1 m_2 \frac{\bar{S}_{12}}{S_{12}^3} + k^2 m_2 m_3 \frac{\bar{S}_{23}}{S_{23}^3} - \frac{3}{2}k^2 m_1 m_2 R_0^2 J_2 \frac{\bar{S}_{12}}{S_{12}^5} \\ & -3k^2 m_1 m_2 R_0^2 J_2 \frac{z_{12}}{S_{12}^5}\hat{z}_{12} + \frac{15}{2}k^2 m_1 m_2 R_0^2 J_2 \frac{\bar{S}_{12}}{S_{12}^7} z_{12}^2 \end{aligned} \tag{5}$$

$$\begin{aligned} m_3\ddot{\bar{S}}_3 = & -k^2 m_1 m_3 \frac{\bar{S}_{13}}{S_{13}^3} - k^2 m_2 m_3 \frac{\bar{S}_{23}}{S_{23}^3} - \frac{3}{2}k^2 m_1 m_3 R_0^2 J_2 \frac{\bar{S}_{13}}{S_{13}^5} \\ & -3k^2 m_1 m_3 R_0^2 J_2 \frac{z_{13}}{S_{13}^5}\hat{z}_{13} + \frac{15}{2}k^2 m_1 m_3 R_0^2 J_2 \frac{\bar{S}_{13}}{S_{13}^7} z_{13}^2 \end{aligned} \tag{6}$$

Where $\hat{z}_{ij}$ are a unit vectors in the specified directions. $\bar{S}_1$, $\bar{S}_2$ and $\bar{S}_3$ are the position vectors of m_1 , m_2 and m_3 with respect to the inertial reference frame. Setting

$$\bar{S}_{12} = \bar{r}_2 \quad , \quad \bar{S}_{13} = \bar{r}_3 \quad , \quad \bar{S}_{23} = \bar{r}_{23} \, , \quad k = 2, 3$$

$$m_k = \varepsilon\mu_k \quad , \quad \varepsilon = \frac{m_2 + m_3}{m_1} \ll 1 \quad , \quad \mu_k = \frac{m_k}{m_2 + m_3}$$

$$\ddot{\bar{r}}_2 = \ddot{\bar{S}}_2 - \ddot{\bar{S}}_1 \quad , \quad \ddot{\bar{r}}_3 = \ddot{\bar{S}}_3 - \ddot{\bar{S}}_1$$

And choosing the units such that $k = m_1 = 1$, we obtain

$$\ddot{\bar{r}}_2 = -\left(1 + \varepsilon\mu_2\right)\frac{\bar{r}_2}{r_2^3} - \varepsilon\mu_3\frac{\bar{r}_3}{r_3^3} + \varepsilon\mu_3\frac{\bar{r}_{23}}{r_{23}^3} - \frac{3}{2}R_0^2 J_2\frac{\bar{r}_2}{r_2^5}$$
$$-3R_0^2 J_2\frac{z_2}{r_2^5}\hat{z}_2 + \frac{15}{2}R_0^2 J_2\frac{\bar{r}_2}{r_2^7}z_2^2 \tag{7}$$

$$\ddot{\bar{r}}_2 = -\left(1 + \varepsilon\mu_3\right)\frac{\bar{r}_3}{r_3^3} - \varepsilon\mu_2\frac{\bar{r}_2}{r_2^3} - \varepsilon\mu_2\frac{\bar{r}_{23}}{r_{23}^3} - \frac{3}{2}R_0^2 J_2\frac{\bar{r}_3}{r_3^5}$$
$$-3R_0^2 J_2\frac{z_3}{r_3^5}\hat{z}_3 + \frac{15}{2}R_0^2 J_2\frac{\bar{r}_3}{r_3^7}z_3^2 \tag{8}$$

Where $\hat{z}_j$ are a unit vectors along the directions $m_1 m_j$, $j = 2, 3$. Substituting for r_2^p and r_3^p the approximate expansions

$$r_2^p = R^p\left\{1 - p\,\mu_3\varepsilon^{1/3}\,\frac{\bar{r}.\,\bar{R}}{R^2} + \frac{1}{2}p\left(p-1\right)\mu_3^2\varepsilon^{2/3}\left(\frac{\bar{r}.\,\bar{R}}{R^2}\right)^2\right\}$$

$$r_3^p = R^p\left\{1 + p\,\mu_2\varepsilon^{1/3}\,\frac{\bar{r}.\,\bar{R}}{R^2} + \frac{1}{2}p\left(p-1\right)\mu_2^2\varepsilon^{2/3}\left(\frac{\bar{r}.\,\bar{R}}{R^2}\right)^2\right\}$$

also using equations (7) and (8), then the equation of motion of the centre of mass $\ddot{\bar{R}} = \mu_2\ddot{\bar{r}}_2 + \mu_3\ddot{\bar{r}}_3$, yields

$$\ddot{\bar{R}} = -\frac{\bar{R}}{R^3} + R_0^2 J_2\left\{-\frac{3\bar{R}}{R^5} + \frac{15}{2}\left(\mu_2 z_2^2 + \mu_3 z_3^2\right)\frac{\bar{R}}{R^7} - \frac{3\left(\mu_2 z_2\,\hat{z}_2 + \mu_3 z_3\,\hat{z}_3\right)}{R^5}\right\} \tag{9}$$

Equations (9) represent the motion of the centre of mass of m_2 and m_3. The first term describes the Keplerian part of the motion arising from the central attraction of the primary. The second term introduced by the oblateness of the primary. Using the straight-forward expansion to solve equations (9) a small divisor $\left(1 - \gamma^2\right)$, where $\gamma = 1 - \frac{1}{2}\left(m_2 + m_3\right)$, will arise in the denominator of the solution. This will approach exact commensurability as $\left(m_2 + m_3\right) \rightarrow$. The source of this divisor is the presence of different time scales which are not taken into account in the expansion procedure. To remove these small divisors we now proceed to express equations (9) in terms of the Struble variables (Kevorkian, 1966).

3. Formulation in Terms of Struble Variables

As is clear, the bracketed terms in equations (9) are factored by $J_2 \equiv \varepsilon^{1/3}$, so that we can replace each of $\bar{r}_2$ and $\bar{r}_3$ by $\bar{R}$ (X, Y, Z), and hence we can replace each of z_2 and z_3 by Z in the last two terms. Equations (9) can be written as

$$\ddot{X} = \frac{\cos\alpha\sin\theta}{R^2} - R_0^2 J_2 \left[\frac{3\cos\alpha\sin\theta}{2R^4} - \frac{15\cos\alpha\sin\theta cos^2\theta}{2R^4}\right]$$

$$\ddot{Y} = -\frac{\sin\alpha\sin\theta}{R^2} - R_0^2 J_2 \left[\frac{3\sin\alpha\sin\theta}{2R^4} - \frac{15\sin\alpha\sin\theta cos^2\theta}{2R^4}\right] \quad (10)$$

$$\ddot{X} = -\frac{\cos\theta}{R^2} - R_0^2 J_2 \left[\frac{9\cos\theta}{2R^4} - \frac{15 cos^3\theta}{2R^4}\right]$$

Where (R, θ, α) are the spherical polar coordinates

In order to transform equation (10) to the variables and geometry proposed by Struble (1960), to facilitate the application of the multiple variables expansion (Kevorkian and Cole, 1981), utilizing the equations for $p^{'}$, $i^{'}$, $\Omega^{'}$, and $u^{''}$ cited in Kevorkian (1966), we have

$$p^{'} = 0 \quad (11)$$

$$\Omega^{'} = J_2 \frac{3R_0^2 u\, cos^3 i\, cos^2\theta}{p^2 sin^2 i} - J_2^2 \frac{9R_0^4 u^2\, cos^7 i\, cos^4\theta}{p^4 sin^4 i} \quad (12)$$

$$i^{'} = -J_2 \frac{3R_0^2 u\, cos^3 i\, \cos\theta}{p^2}\cos\varphi + J_2^2 \frac{9R_0^4 u^2\, cos^7 i\, cos^4\theta}{p^4 sin^4 i}\cos\varphi \quad (13)$$

$$u^{''} = -u + \frac{cos^2 i}{p^2} - R_0^2\, J_2 \{\frac{uu^{'}}{p^2}\cos\varphi[-3\sin i cos^2 i\, cos\theta +$$

$$+\frac{6cos^4 i\, cos\theta}{\sin i}] + \frac{3u^{'2} cos^4 i\, cos^2\theta}{p^2 sin^2 i} + \frac{u^2}{p^2} - \frac{3}{2}cos^2 i + \frac{9}{2}cos^2 i\, cos^2\theta -$$

$$-\frac{6cos^4 i\, cos^2\theta}{\sin^2 i} + \frac{u}{p^4}\left[\frac{6cos^6 i\, cos^2\theta}{\sin^2 i}\right] - R_0^4\, J_2 \frac{u^2 u^{'}}{p^4} [\frac{9cos^6 i\, cos^3\theta\, cos\varphi}{\sin i}$$

$$-\frac{18cos^8 i\, cos^3\theta\, cos\varphi}{\sin^3 i} + \frac{36cos^6 i\, cos^3\theta}{\sin i}] - \frac{9uu^{'2}}{p^4 sin^4 i}cos^8 i\, cos^4\theta$$

$$+\frac{27u^3}{p^4 sin^4 i}cos^8 i\, cos^4\theta - \frac{27u^2}{p^2 sin^4 i}cos^{10} i\, cos^4\theta +$$

$$+\frac{u^3}{p^4}\frac{cos^6 i}{sin^2 i}[9cos^2\theta - 27cos^4\theta]\}. \tag{14}$$

where p, φ, Ω, ω and $u = \frac{1}{R}$ are the component of the angular momentum along the x-axis , the instantaneous value of the angle between the node and the satellite, the longitude of the ascending node, argument of perigee, and the reciprocal radial coordinate, respectively. The prime in equations (11)-(14) denotes differentiation with respect to φ.

4. The Two Variable Expansion Procedure

In this section, we apply the two variable expansion procedure to solve the equations of motion of the centre of mass near to the encounter phase. No restrictions are imposed on the inclination i , but the eccentricity e is assumed of order J_2 To solve equations (11) – (14) let us introduce the two time scales φ , $\varphi_1 = J_2\varphi$, and assume the involved quantities expandable as

$$\begin{aligned} u &= u_0(\varphi,\varphi_1) + J_2 u_1(\varphi,\varphi_1) + \ldots \\ p &= p_0(\varphi,\varphi_1) + J_2 p_1(\varphi,\varphi_1) + \ldots \\ i &= i_0(\varphi,\varphi_1) + J_2 i_1(\varphi,\varphi_1) + \ldots \\ \Omega &= \Omega_0(\varphi,\varphi_1) + J_2 \Omega_1(\varphi,\varphi_1) + \ldots \\ \mathrm{e} &= \mathrm{e}_0(\varphi,\varphi_1) + J_2 \mathrm{e}_1(\varphi,\varphi_1) + \ldots \\ \omega &= \omega_0(\varphi,\varphi_1) + J_2 \omega_1(\varphi,\varphi_1) + \ldots \end{aligned} \tag{15}$$

From which $\frac{di}{d\varphi} = \frac{\partial i_0}{\partial\phi} + J_2\left(\frac{\partial i_0}{\partial\varphi_1} + \frac{\partial i_1}{\partial\phi}\right) + \ldots$

$$\frac{d^2 i}{d\varphi^2} = \frac{\partial^2 i_0}{\partial\varphi^2} + J_2\left(2\frac{\partial^2 i_0}{\partial\varphi\partial\varphi_1} + \frac{\partial^2 i_1}{\partial\phi^2}\right) + \ldots$$

With similar expressions for the remaining variables. From the above developments we readily have

$$\frac{uu'}{p^2} = \frac{u_0}{p_0^2}\frac{\partial u_0}{\partial\phi} + J_2\frac{u_0}{p_0^2}\frac{\partial u_0}{\partial\phi_1} + J_2\frac{u_0}{p_0^2}\frac{\partial u_1}{\partial\varphi} + J_2\frac{u_1}{p_0^2}\frac{\partial u_0}{\partial\varphi} - 2\,J_2\frac{u_0 p_1}{p_0^3}\frac{\partial u_0}{\partial\varphi} + \ldots$$

$$\frac{u^2 u'}{p^4} = \frac{u_0^4}{p_0^4}\frac{\partial u_0}{\partial\phi} \quad , \quad \frac{uu'^2}{p^4} = \frac{u_0}{p_0^4}\left(\frac{\partial u_0}{\partial\phi}\right)^2 \quad , \quad \frac{u^3}{p^4} = \frac{u_0^3}{p_0^4} \quad ,$$

$$\frac{u'^2}{p^2} = \frac{1}{p^2}\left(\frac{\partial u_0}{\partial \phi}\right)^2 + J_2\frac{2}{p_0^2}\frac{\partial u_0}{\partial \phi}\left(\frac{\partial u_0}{\partial \phi_1} + \frac{\partial u_1}{\partial \varphi}\right) - J_2\frac{2p_1}{p_0^3}\left(\frac{\partial u_0}{\partial \phi}\right)^2 + \dots \quad (16)$$

We can similarly calculate the rest of the quantities involved in equations (11)-(14). Now using expressions (16) into equations (11)-(14) and equating the like powers of J_2 on each side, then the coefficients of J_2^0 yield

$$\frac{\partial^2 u_0}{\partial \varphi^2} + u_0 = \frac{cos^2 i}{p^2} \quad (17)$$

$$\frac{\partial p_0}{\partial \varphi} = 0 \quad , \quad \frac{\partial \Omega_0}{\partial \varphi} = 0 \quad , \quad \frac{\partial i_0}{\partial \varphi} = 0 \quad (18)$$

The coefficients of J_2 yield

$$\frac{\partial^2 u_1}{\partial \varphi^2} + u_1 = -2\left[\frac{\partial^2 u_0}{\partial \varphi \partial \varphi_1}\right]_0 - R_0^2\frac{u_0}{p_0^2}\frac{\partial u_0}{\partial \phi}(-3\sin i cos^2 i\, cos\theta \cos\varphi +$$

$$+\frac{6cos^4 i\, cos\theta \cos\varphi}{\sin i}) + \frac{1}{p_0^2}\left(\frac{\partial u_0}{\partial \phi}\right)^2\left(\frac{3cos^4 i\, cos^2\theta}{\sin^2 i}\right) + \frac{u_0^2}{p_0^2}(-\frac{3}{2}cos^2 i + \frac{9}{2}cos^2 i\, cos^2\theta$$

$$-\frac{6cos^4 i\, cos^2\theta}{\sin^2 i}) + \frac{u_0}{p_0^4}\left(\frac{6cos^6 i\, cos^2\theta}{\sin^2 i}\right) \quad (19)$$

$$\frac{\partial p_1}{\partial \varphi} = -\frac{\partial p_0}{\partial \varphi_1} \quad , \quad (20)$$

$$\frac{\partial \Omega_1}{\partial \varphi} = -\frac{\partial \Omega_0}{\partial \varphi_1} - \frac{u_0}{p_0^2}\left(\frac{3R_0^2 cos^3 i_0\, cos^2\theta}{\sin^2 i_0}\right) \quad (21)$$

$$\frac{\partial i_1}{\partial \varphi} = -\frac{\partial i_0}{\partial \varphi_1} - \frac{u_0}{p_0^2}\left(3R_0^2 cos^3 i\ \cos\theta\ \cos\varphi\right). \quad (22)$$

Where we mean by $\left[\frac{\partial^2 u_0}{\partial \varphi \partial \varphi_1}\right]_0$ evaluation of zero orders of *ipe*, and ω

Now we proceed to evaluate the solution of the equations at different orders. A complete solution of order zero will be deferred until the solutions of some involved quantities are obtained from higher orders We will similarly treat the first order one

4.1. Zero Order Solution

Recalling the zero order equations (17) and (18), these equations has solutions

$$i_0 = \sum_{n=0} J^n \, i_{0n} \left(\varphi_1, J_2\right) \tag{23}$$

$$\Omega_0 = \sum_{n=0} J^n \, \Omega_{0n} \left(\varphi_1, J_2\right) \tag{24}$$

$$p_0 = \sum_{n=0} J^n \, p_{0n} \left(\varphi_1, J_2\right) \tag{25}$$

$$u_0 = \frac{cos^2 i_0}{p_0^2} \left[1 + e \cos\left(\phi - \omega\right)\right] \tag{26}$$

$$e = \sum_{n=0} J^n e_n \left(\varphi_1\right) \tag{27}$$

$$\omega = \sum_{n=0} J^n \omega_n \left(\varphi_1\right) \tag{28}$$

Regarding equation (20) to avoid secular terms in p_1 we choose

$$\frac{\partial p_{00}}{\partial \varphi_1} = 0, \quad \text{or} \quad p_{00} = const,$$

and

$$p_1 = p_1 \left(\varphi_1, J_2\right) \tag{29}$$

and hence p_1 is to be absorbed in p_{01}, so that in equation (15) we set p_1 to zero.

4.2. First Order Solution

Consider equation (21), noting that $\cos\theta = \sin i \sin\varphi$, we then have

$$\frac{\partial \Omega_1}{\partial \varphi} = -\frac{\partial \Omega_{00}}{\partial \varphi_1} - \left(\frac{3R_0^2 \, u_0 \, cos^3 i_{00} \, sin^2 \varphi}{p_{00}^2}\right)$$

$$= -\frac{\partial \Omega_{00}}{\partial \varphi_1} + \frac{3R_0^2 \, cos^5 i_{00}}{2p_{00}^4} \left(-1 + \cos 2\varphi\right) - \frac{3R_0^2 \, e_0 \, cos^5 i_{00}}{2p_{00}^4} \cos\left(\varphi - \omega_0\right)$$

$$+ \frac{3R_0^2 \, e_0 \, cos^5 i_{00}}{2p_{00}^4} \cos\left(3\varphi - \omega_0\right) + \frac{3R_0^2 \, e_0 \, cos^5 i_{00}}{2p_{00}^4} \cos\left(\varphi + \omega_0\right)$$

It is required that Ω_1 be bounded function of ϕ so that we choose

$$-\frac{\partial \Omega_{00}}{\partial \varphi_1} - \frac{3R_0^2 \, cos^5 i_{00}}{2p_{00}^4} = 0 \tag{30}$$

Whose solution is deferred until the solution to i_{00} is obtained. And

$$\Omega_1 = -\frac{3R_0^2 \, cos^5 i_{00}}{4p_{00}^4} \sin 2\varphi + \frac{3R_0^2 e_0 \, cos^5 i_{00}}{4p_{00}^4} \cos(\varphi + \omega_0) -$$

$$-\frac{R_0^2 e_0 \, cos^5 i_{00}}{2p_{00}^4} \cos(3\varphi - \omega_0) + \frac{3R_0^2 e_0 \, cos^5 i_{00}}{2p_{00}^4} \sin(\varphi + \omega_0)$$

Now let us consider equation (22), we have

$$\frac{\partial i_1}{\partial \varphi} = -\frac{\partial i_{00}}{\partial \varphi_1} - \frac{3R_0^2 \, sin_{00} \, cos^5 i_{00}}{2p_{00}^4} \left\{ \sin 2\varphi + \frac{e_0}{2} \sin(3\varphi - \omega_0) + \frac{e_0}{2} \sin(\varphi + \omega_0) \right\}$$

It is required that i_1 be bounded function of ϕ, so that we choose

$$\frac{\partial i_{00}}{\partial \varphi_1} = 0 \tag{31}$$

So that

$$i_{00} = const \tag{32}$$

Since all quantities on the right hand side of equation (22) are independent of φ (except φ it self)it is readily integrates to yield

$$i_1 = \frac{3R_0^2 \, sin_{00} \, cos^5 i_{00}}{4p_{00}^4} \left\{ \cos 2\varphi + \frac{e_0}{3} \cos(3\varphi - \omega_0) + e_0 \sin(\varphi + \omega_0) \right\} \tag{33}$$

We can proceed similarly to evaluate the remaining quantities. The second order solution needs lengthily algebra to be obtained.

5. Conclusion

The equations of motion of the center of mass of m_2 and m_3 are developed in the most general form without any restrictions on the orbital elements. To overcome the expected singularity the equations are transformed to be expressed in terms of the Struble variables and solved in terms of two time scales, up to first order. The second order solution is very complicated, such solution will be presented in a forthcoming work.

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