FINITE NON-SOLVABLE GROUPS HAVING A UNIQUE IRREDUCIBLE CHARACTER OF A GIVEN DEGREE

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Abstract: It has been conjectured that $\text{PSL}(2, q)$, the projective special linear group of $2 \times 2$ matrices over a field of order $q$, is the only non-solvable group satisfying the property that it has a unique irreducible complex character $\chi$ of degree $m > 1$ and every other irreducible complex character is such that its degree is relatively prime to $m$. (Such a $\chi$ is a particular case of the Steinberg character of finite Chevalley groups.) In this paper, we consider finite non-solvable groups satisfying the above property and show that the derived group $G'$ is a non-abelian simple group and that when $\chi(1) = p$, $p$ an odd prime, $G$ itself is a non-abelian simple group, and is such that its $p$-sylow subgroup $P$ is a cyclic group of order $p$ and equals its centralizer and that all involutions in $G$ are conjugate.

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1. Notations and Preliminary Results

Our notations are quite standard and agree with those in [4]. In what follows, group means finite group and all characters are considered characters over com-
plex numbers. For a character \( \psi \) of a subgroup \( H \) of a group \( G \), \( \psi^G \) denotes the character of \( G \) induced by \( \psi \) (see[4] for a definition). Induction has the following properties:

**Property 1.** If \( H \subset K \subset G \), then \( (\psi^K)^G = \psi^G \).

**Property 2.** Frobenius reciprocity: If \( \varphi \) is a character of \( G \), then \( (\psi^G, \varphi)_G = (\psi, \varphi|_H)_H \).

For \( x \in G \), \( \psi^x \) is defined on \( H^x \) by \( \psi^x(h^x) = \psi(h), h \in H \), and is a character of \( H^x \). If \( H \) is normal in \( G \), the stabilizer of \( \psi = \{ x \in G \mid \psi^x = \psi \} \) is denoted by \( T_{\psi, H} \) or simply by \( T_\psi \) if the domain of \( \psi \) is clear from the context.

If \( \Phi \) is a matrix representation, then \( \det \Phi \) is a linear character. Equivalent representations have the same determinant, so we may put \( \det \Phi = \det \varphi \), where \( \varphi \) is the character of \( \Phi \).

If \( \pi \) is a set of prime numbers, then by a \( \pi \)-number \( b \) we mean that each prime factor of \( b \) is in \( \pi \). If \( |G| \) is a \( \pi \)-number then we say that \( G \) is a \( \pi \)-group.

The following result of Clifford will be used repeatedly.

**Lemma 3** ([6], Theorem 9.10). Let \( H \) be a normal subgroup of \( G \). If \( \eta \) is an irreducible character of \( G \), then there exists an irreducible character \( \theta \) of \( H \) and a positive integer \( a \) such that

\[
\eta|_H = a \sum_{i=1}^{t} \theta^{g_i}, \text{ where } t = [G : T_\theta]
\]

and \( \{g_i\} \) is a complete system of coset representatives of \( T_\theta \) in \( G \).

2. Main Results

**Theorem 4.** Suppose \( G \) is a non-solvable group having exactly one irreducible character \( \chi \) of degree \( m \), and every other irreducible character \( \theta \) of \( G \) is such that \( (\theta(1), m) = 1 \). Further, suppose that \( \chi \) is faithful. Then \( G' \), the derived group of \( G \), is a non-abelian simple group.

**Remark.** It follows from the above hypotheses and the main theorem in [10] that \( G \) has at least one non-linear irreducible character different from \( \chi \).

**Proof.** The proof is carried out in five steps.

**Step 1.** Every abelian, normal subgroup \( A \) of \( G \) is cyclic, and is contained in \( Z(G) \).
By 3, we can write
\[ \chi|_A = a \sum_{i=1}^{t} \psi^{g_i} \]
where \( \psi^{g_i} \) are irreducible characters of \( A \), \( t = [G : T] \), and \( a \) is some positive integer. Since \( A \) is abelian, \( \psi(1) = 1 \) and so \( \chi(1) = a \cdot t \) so that \( a \) and \( t \) are \( \pi \)-numbers, where \( \pi \) is the set of primes dividing \( \chi(1) \). First suppose that \( t > 1 \).

If \( \eta \neq \chi \) is any irreducible character of \( G \), then \( (\eta|_A, \psi)_A = 0 \), for otherwise all conjugates of \( \psi \) appear in \( \eta_A \) (Lemma 3) and so \( t|\eta(1) \), contradicting our hypotheses. This and Frobenius reciprocity (2) imply that \( \psi^G = a \cdot \chi \). Comparing degrees, we obtain
\[ [G : A] \psi(1) = [G : A] = a \chi(1) = a^2 t. \]

Therefore, \( G/A \) is a \( \pi \)-group. But then the hypothesis on the degrees of irreducible characters of \( G \) implies that \( G/A \) cannot have any non-linear irreducible characters. Hence \( G/A \) is abelian and this is a contradiction to our assumption that \( G \) is non-solvable. Therefore, \( t = 1 \) and \( \chi|_A = \chi(1) \cdot \psi \). Then, \( \psi \) is faithful since \( \chi \) is, and so \( A \) is cyclic. Let \( A = \langle x \rangle \) and \( y \) be arbitrary in \( G \). Since \( \psi \) has no conjugates different from \( \psi \), \( \psi^x = \psi \) and this implies \( x^y = x \). Hence \( A \subseteq Z(G) \).

**Step 2.** Let \( N/Z(G) \) be a minimal normal subgroup of \( G/Z(G) \). Then \( G/N \) is abelian.

Let \( \chi|_N = a \sum_{i=1}^{t} \psi^{g_i} \) irreducible characters of \( N \). If \( \psi(1) = 1 \), then \( N' \subseteq \ker \psi^{g_i} \forall i \), and so \( N' \subseteq \ker \chi = 1 \). But \( N \) cannot be abelian by step 1. Therefore, \( \psi(1) \neq 1 \) and clearly \( \psi(1) \) is a \( \pi \)-number. If \( \eta \neq \chi \) is any irreducible character of \( G \), then \( (\eta|_N, \psi)_N = 0 \), for otherwise, \( \psi(1) \) would divide \( \eta(1) \), contrary to our hypotheses. Therefore, \( \psi^G = a \cdot \chi \). Comparing degrees again, we obtain
\[ [G : N] \cdot \psi(1) = a \chi(1) = a^2 t \cdot \psi(1) \]
so that \( G/N \) is a \( \pi \)-group and hence cannot have any non-linear irreducible characters. Thus \( G/N \) is abelian.

**Step 3.** \( |Z(G)| \leq 2 \).

Let \( Z(G) = \langle z \rangle \) and \( \chi|_{Z(G)} = \chi(1) \cdot \varphi \) for some irreducible character \( \varphi \) of \( Z(G) \). Since \( \chi \) is unique of its degree, it is fixed under field automorphisms. Therefore, \( \chi \) is integer valued and hence \( \varphi \) is too. This, together with the fact that \( \varphi(z) \) is a root of unity implies that \( |Z(G)| \leq 2 \).
Step 4. If $|Z(G)| = 2$ then $N' = N$.

If $N' \supseteq Z(G)$, this is obvious. If $N' \nsubseteq Z(G)$, then since $N/Z(G)$ is minimal normal in $G/Z(G)$, we obtain that $N = N' \times Z(G)$. We will show that this cannot happen.

The characters $\{\psi_{g_i}\}$ in the equation $\chi|_N = a \sum_{i=1}^{t} \psi_{g_i}$ are all the non-linear irreducible characters of $N$ whose degrees are $\pi$-numbers, for if $\theta$ is any non-linear irreducible character of $N$ such that $\theta(1)$ is a $\pi$-number, then $(\theta^G, \eta)_G \geq 1$ for some irreducible character $\eta$ of $G$. This implies $(\theta, \eta|_N)_N \geq 1$ and so $\theta(1) | \eta(1)$. Hence, by hypotheses, we have $\eta = \chi$ so that $\theta = \psi_{g_i}$ for some $i$. In particular, all non-linear irreducible characters of $N$ whose degrees are $\pi$-numbers have the same degree and are conjugate. But if $N = N' \times Z(G)$, then the irreducible characters of $N$ are products of irreducible characters of $N'$ and $Z(G)$ (Theorem 3.7.1, p. 100, [7]) and if $\zeta \neq 1_N$ is an irreducible character of $N'$, then $\zeta \times 1_{Z(G)}$ is not conjugate to $\zeta \times \varphi$, where $\varphi$ is as in Step 3. This contradiction proves Step 4.

Step 5. $Z(G) = 1$ and $N$ is non-abelian simple.

Suppose $|Z(G)| = 2$ so that $\varphi(z) = -1$ and $N' = N = G'$. If $\chi(1)$ is odd, then $\det \chi(z) = -1$, a contradiction since $G' \subset \ker \det \chi$. Hence, $2 | \chi(1)$ and so every irreducible character $\theta$ of $N$ different from any $\psi_{g_i}$ has odd degree. (There is at least one such non-linear $\theta$ since $G$ has non-linear characters different from $\chi$.) For any such $\theta$, we must have $Z(G) \subseteq \ker \theta$, for otherwise $\det \theta|_{N'} \neq 1_{N'}$; i.e., $\psi_{g_i}$ for $i = 1, 2, \ldots t$ are all irreducible characters of $N$ which don’t have $Z(G)$ in their kernel. Thus, we have on one hand,

$$|N| = 2 \cdot |N/Z(G)| = 2 \sum \theta(1)^2,$$

where $\theta$ ranges over all irreducible characters of $N/Z(G)$, and on the other hand,

$$|N| = t\psi(1)^2 + \sum \theta(1)^2.$$

Hence, $t\psi(1)^2 = \sum \theta(1)^2 = |N/Z(G)|$. This implies that $\theta(1) | t\psi(1)^2$ since $\theta(1) | |N/Z(G)|$. This is a contradiction since $(\theta(1), \chi(1)) = 1$ and so $Z(G) = 1$.

Since $N$ is non-solvable and minimal normal in $G$, we obtain that $N = N_1 \times N_2 \times \cdots \times N_r$, where the $N_i$’s are non-abelian simple and isomorphic (Theorem 2.1.4, p. 16, [7]) and $r$ is some positive integer. If $r > 1$, then then non-linear irreducible characters of $N$ whose degrees are $\pi$-numbers cannot all have the same degree (Theorem 3.7.1, p. 100, [7]). Hence, $N$ is non-abelian simple and the proof of Theorem 4 is complete.
Theorem 5. Suppose that $G$ and $\chi$ are as in Theorem 4 and that $\chi(1) = p$, $p$ an odd prime. Let $P$ be a $p$-sylow subgroup of $G$. Then the following hold:

(a) $G$ is a non-abelian simple group;
(b) $P = C_G(P)$ and $|P| = p$

(c) All involutions in $G$ are conjugate.

Proof. (a) In step 2 of the proof of Theorem 4, we had $[G : N] = a^2 t$, where $a$ and $t$ are as in the equation

$$\chi|N = a \sum_{i=1}^{t} \psi g_i$$

Also, $\psi(1) \neq 1$ since $N$ is a non-abelian simple group. Thus, if $\chi(1) = p$, then we obtain, by comparing degrees in the above equation, that $a = t = 1$. Therefore, $G = N$. This proves part (a).

(b) Since $G$ is simple, we obtain, by a result of Burnside (Theorem 18.4, p. 94, [6]), that $\chi|P = \mu_P$, where $\mu_P$ is the regular character of $P$, and so $|P| = p$.

Let $P = \langle x \rangle$ and suppose that $y$ is an element of prime order $q$, $q \neq p$ and that $y \in C_G(P)$. Since $\chi|\langle x \rangle = \mu_{\langle x \rangle}$, we obtain that

$$\chi|\langle x \rangle \times \langle y \rangle = \sum_{j=1}^{p} \varphi_j \omega_j,$$

where $\varphi_j$’s are all the linear characters of $\langle x \rangle$, and $\omega_j$’s are among the linear characters of $\langle y \rangle$. Thus, we can write

$$\chi(xy) = \omega_1(y)\lambda + \omega_2(y)\lambda^2 + \cdots + \omega_p(y)\lambda^p,$$

where $\lambda$ is a primitive $p^{th}$ root of unity. Then, we obtain

$$\omega_1(y)\lambda + \omega_2(y)\lambda^2 + \cdots + \omega_{p-1}(y)\lambda^{p-1} = \alpha$$

for some $\alpha \in \mathbb{Q}[\sqrt[p]{1}]$, the field obtained by adjoining $\sqrt[p]{1}$ to the field of rationals. Also, $\lambda + \lambda^2 + \cdots + \lambda^{p-1} = -1$ and so $(-\alpha)\lambda + (-\alpha)\lambda^2 + \cdots + (-\alpha)\lambda^{p-1} = \alpha$ as well. Thus, if $\lambda, \lambda^2, \ldots, \lambda^{p-1}$ are linearly independent over $\mathbb{Q}[\sqrt[p]{1}]$, then all $\omega_i(y)$’s are equal and $\chi(y) = p \cdot \omega_1(y)$, so that $y \in Z(G)$, a
contradiction. Hence, it is enough to show that \( \lambda, \lambda^2, \ldots, \lambda^{p-1} \) are linearly independent over \( \mathbb{Q}[\sqrt[p]{1}] \). Clearly, they span \( \mathbb{Q}[\sqrt[p]{1}] \) as a \( \mathbb{Q}[\sqrt[p]{1}] \)-space. Also \( \mathbb{Q}[\sqrt[p]{1}][\sqrt[p]{1}] = \mathbb{Q}[\sqrt[p]{1}] \), and \( |\mathbb{Q}[\sqrt[p]{1}] : \mathbb{Q}| = (p-1)(q-1) \). The proof of (b) is complete.

(c) A result of Brauer (Theorem 3, [2]), together with parts (a) and (b) above, implies that \( G \) has exactly two \( p \)-blocks, and that the principal block \( B_0(p) \) contains all the irreducible characters of \( G \) except \( \chi \). Part (c), now, follows from a lemma of Richen (p. 299, [9]).

\[ \square \]

**Remark.** The assumption that \( p \) is odd is never used in the proof Theorem 5. But then, using Theorem 5, one can easily show that there are no groups satisfying the hypotheses of Theorem 3.2 when \( p = 2 \) or \( 3 \).

**Remark.** The conjecture that \( \text{PSL}(2,p) \) is the only group satisfying the hypotheses of Theorem 5 is still open. However, Theorem 5 enables us to apply Brauer’s results ([1] and [2]) and obtain the following information about \( p \)-blocks and irreducible characters of \( G \).

Let \( |N_G(P)| = e \cdot p \), where \( e \) is some positive integer dividing \( p-1 \) and let \( t = (p-1)/e \). Let \( \lambda_1, \lambda_2, \ldots, \lambda_t \) denote the irreducible characters of \( N_G(P) \) which are induced from linear characters of \( P \). There are \( 1 + e + t \) irreducible characters of \( G \). There is a unique \( p \)-block \( B(p) \) of defect zero and \( B(p) = \{ \chi \} \). All other irreducible characters belong to the principal block \( B_0(p) \), and so let 

\[ B_0(p) = \{ \zeta = 1_G, \zeta_2, \ldots, \zeta_e, \theta_1, \theta_2, \ldots, \theta_t \}, \]

where the \( \theta_i \)'s are exceptional characters and satisfy \( \lambda_i^G = \lambda_j^G = \varepsilon(\theta_i - \theta_j) \), \( \varepsilon = \pm 1 \), for \( i, j = 1, 2, \ldots, t \). Each \( \zeta_i \) has the property that \( \zeta_i \) has constant value \( a_i \) on all \( p \)-elements and \( a_i = \pm 1 \). As a result, \( \zeta_i(1) = k_i p \pm 1 \) for some positive integer \( k_i \). If \( g \in G \) has order prime to \( p \) then \( \theta_i(g) = \theta_j(g) \forall i, j \). Also, \( \sum_{i=1}^{t} \theta_i \) has constant value \( a \) on the \( p \)-elements and \( a = \pm 1 \). Hence, \( t \cdot \theta_i(1) = \sum_{i=1}^{t} \theta_i(1) = kp \pm 1 \) for some positive integer \( k \). (These \( \theta_i \)'s are algebraically conjugate.)

**References**


