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THE *SEPAK TAKRAW* BALL PUZZLE

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Abstract: This article explains the truncated icosahedron by making a *sepak takraw* ball, which is used in a popular ball game in Thailand. Next, many popular topics for examination questions on spatial diagrams are presented such as: how many regular polyhedra can be made, Euler's polyhedron formula, $F - E + V = 2$, and the number of regular pentagons and hexagons in a soccer ball.

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Key Words: truncated icosahedron, regular polyhedra, Euler's polyhedron formula, fullerene C_{60}

1. Let's Make a Sepak Takraw Ball

Both stress and relaxation are important when studying for an examination. After the challenge of university entrance examinations it makes sense to find some enjoyment in puzzles. The diversion I'd like to introduce was revealed to me by Shigenori Ohsawa, a teacher of high school at Saitama prefecture. In Thailand there is a popular ball game called *sepak takraw*. Apparently, '*sepak*' means 'kick' in Malay, and '*takraw*' means ball in Thai. The ball used in this competitive sport is very durable. It is made of rattan and is 12cm in diameter.

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The game involves kicking the ball, and is similar to the Japanese sport of *kemari* (Figure 1). This has become a medal sport at the Asian Games and uses a plastic ball.

The *sepak takraw* ball is related mathematically to a 32-face semi-regular polyhedron, known as a truncated icosahedron. In fact, it is not merely engaging as a plaything, but is also related to the fullerene C_{60} molecule made famous by the award of a Nobel Prize in chemistry, as well as the construction of soccer balls (Figures 2 and 3).

Figure 1: *Sepak takraw*

What I'd like to introduce here is how we can construct a ball with the same form as the *sepak takraw* ball using simple packing tape. By way of preparation, a 3 - 4m length of packing tape (made from polypropylene, known as PP-band) 15 mm wide will be sufficient. The tape should be cut into 6 pieces 56 cm in length. This length and number of strips will be explained later. It's also useful to have 6 clothes pegs which can be used to hold things in place while working.

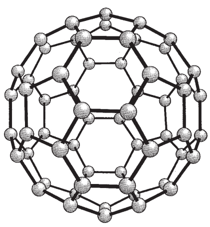


Figure 2: Fullerene

Figure 3: Soccer ball

The following four points regarding the construction are important.

(1) Each strip winds around two 25 cm loops with 6 cm to spare. Two loops are used in order to increase the strength.

(2) Three-way locks are formed where three strips cross (Figure 4). There is another such pattern which is the other way up, but the patterns are equivalent.

(3) When tapes cross each another, they alternate crossing above and below.

(4) Five strips of tape make a regular pentagonal hole (the black part in Figure 5).

It can be seen that in the area indicated with a dotted line, the tape forms a regular hexagon. Keeping the above-mentioned basics in mind, allow me to

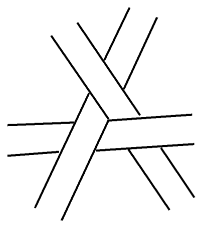


Figure 4: Three-way lock

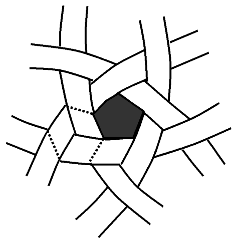


Figure 5: Making a regular pentagon

explain the construction procedure with reference to Figure 6. First there is the assembly of the three-way lock (1). This relationship occurs at every crossing point so as a fundamental, it should be memorized.

Next is the construction of the regular pentagonal hole from five strips of tape (2). When clothes pegs are used to hold the tape in place, the process is easy when the pegs are arranged on radial lines. The number of clothes pegs is five. The point to remember is to pass the sixth strip through them. It doesn't matter where from, but it is best not to forget the three-way lock and pentagonal hole construction. The sixth strip is shown in grey. The second regular pentagon is made (3), then the third (4), then the fourth (5), the fifth (6) and the sixth (7).

At this point the arrangement of the six regular pentagons is such that there are five around the original pentagon. Also, the sixth tape (grey) is a closed loop when the six pentagons are completed, and is wrapped twice.

The whole object is now half complete, and the clothes pegs used we no longer need the clothes pegs to hold it together (7). The sepak takraw ball is completed by passing the five free strips through in order, above and below (8) (12).

2. How Many Regular Polyhedra Can We Make?

The *sepak takraw* ball does not involve solving mathematics with pencil and paper; instead it is constructed by hand and thus gives us a sense of the physical realization of mathematics as well as helping us to understand spatial diagrams. There are many problems that may be presented as examination questions on spatial diagrams and I'd like to introduce a few of them here.

Among polyhedra, there are regular polyhedra and semi-regular polyhedra. When the elements in the construction of a polyhedron, the polygonal faces,

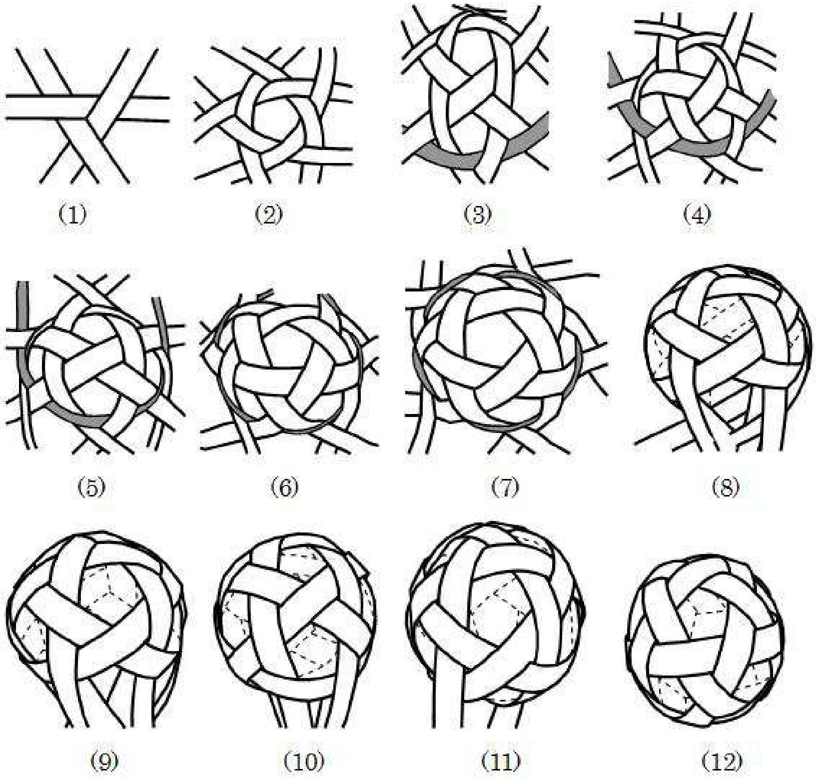


Figure 6: Construction diagram for a sepak takraw ball

are all one type of regular polygon, the polyhedron itself is described as regular. When there are two or more types of regular polygon, the polyhedron is described as semi-regular. It is known that there are five regular polyhedra as shown in Figure 7: the tetrahedron (4 faces), cube (6 faces), octahedron (8 faces), dodecahedron (12 faces), and icosahedron (20 faces). These five regular polyhedra thus exist, and it has been proven that there are no others. Perhaps this is something to learn by heart? I don't think so, because with just a little thought this fact can be easily proved. So let's take a look (Murakami, 1982 or Hitotsumatsu, 1983).[3] [2]

[Problem 1] Prove that there are only five regular polyhedra.

(Proof) Let's try to prove it by thinking of a regular polyhedron as an object in

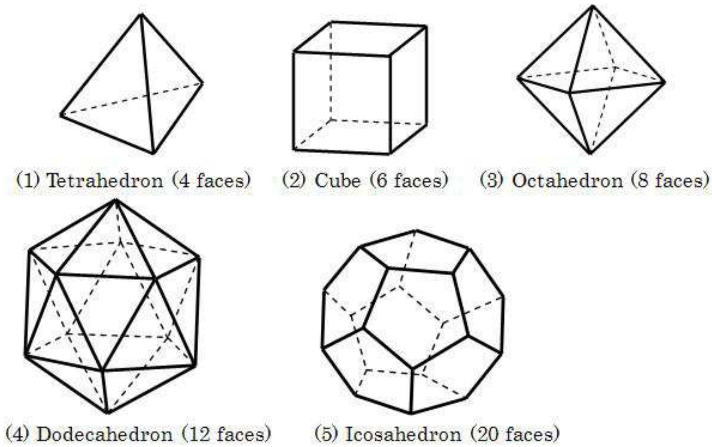


Figure 7: The five regular polyhedra

which m regular n sided polygons meet at a single vertex. The angle at a single vertex of a regular n sided polygon is $180° - \frac{360°}{n}$. Supposing m regular n sided polygons meet at a vertex, then the total angle formed by all the individual vertex angles must be less than 360° so,

$$(180° - \frac{360°}{n})m < 360°.$$

Rearranging this,

$$(m-2)(n-2) < 4 \quad (n, m \geq 3).$$

Solving this yields,

for $n = 3,\ m = 3, 4, 5$ (tetrahedran, octahedron, icosahedron),
for $n = 4,\ m = 3$ (cube),
for $n = 5,\ m = 3$ (dodecahedron). (End of proof)

Check each of the cases above!

3. Euler's Theory of Polyhedra

The tetrahedron is constructed from 4 triangles, the cube from 6 squares, the octahedron from 8 triangles, the dodecahedron from 12 pentagons, and the icosahedron from 20 triangles. There is also the 32-face semi-regular polyhedron known as a truncated icosahedron which is constructed from 12 regular pentagons and 20 regular hexagons (totaling 32 faces).

Writing out the number of faces, edges and vertices of the regular polyhedra and this semi-regular polyhedron yields Table 1.

	Faces (F)	Edges (E)	Vertices (V)	$F - E + V$
Tetrahedron	4	6	4	2
Cube	6	12	8	2
Octahedron	8	12	6	2
Dodecahedron	12	30	20	2
Icosahedron	20	30	12	2
Truncated Icosaderon	32	90	60	2

Table 1. $F - E + V = 2$

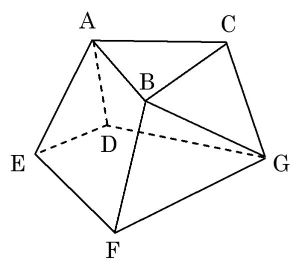
Now, regarding the relationship between these numbers, calculating (Faces - Edges + Vertices) reveals that the following relationship is satisfied

$$F - E + V = 2.$$

This is known as Euler's formula, and proofs may be found in many books. I'll state a method here drawn from a book close to hand, 'The History of Geometry', by Kentaro Yano (see Yano, 1972).[6]

[Problem 2] Describe a method of proving Euler's formula,$F - E + V = 2$, for an arbitrary polyhedron such as that shown in the diagram.

Consider the object that results from removing one face of the polyhedron, such as ABC for example. By so doing, the number of vertices V, and the number of edges E , are unchanged, but the number of faces is reduced by 1, $F' = F - 1$. Thus, although the objective is to prove for the original polyhedron that



$$F - E + V = 2,$$

it is sufficient to prove that

$$F' - E + V = 1$$

for the object that results from removing one face in this way. The proof is made easier by partitioning the polygons into triangles. The faces adjacent to ABC are removed one by one, and it is sufficient to track how the value of $F' - E + V$ develops during this process. In the end, only one face is left (a triangle), and it is made clear that the relationship

$$F' - E + V = 1 - 3 + 3 = 1$$

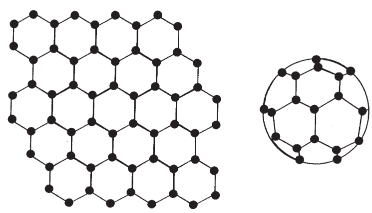
is satisfied. (End of proof)
Refer to the above-mentioned book for the details.

4. Fullerene Molecules and Truncated Icosahedra

Perhaps you know that the discovery of the fullerene C_{60} molecule was connected to Euler's formula (see Baggott, 1996 [1])? At first, for this new molecule with 60 carbon atoms, chemists only considered regular hexagons for the distribution model of the atoms. However, it was understood that it is not possible to cover a spherical surface using such a method. In terms of chemical formulae, a spherical surface formed from six-member rings is not closed, so the existence of five-member rings is theoretically necessary. This issue has been presented as an examination problem.

[Problem 3] Use Euler's theory of polyhedra, $F - E + V = 2$, to prove that graphite sheets formed from 60 atoms cannot be bent and closed in order make a polyhedron.

Having 60 atoms means the number of vertices is 60 ($V = 60$). Since each vertex is shared by 3 polygons, each vertex joins 3 'half-edges'. This is because each edge is shared by two vertices. The total number of edges is therefore 90 ($60 \times 3 \div 2 = 90, E = 90$). Assuming all of the faces are hexagonal, since each edge accounts for one sixth of two faces (one on either side of the edge), the total number of faces is 30 ($90 \times 2 = 180, 180 \div 6 = 30$), ($F = 30$).



Substituting this value into the left hand side yields

$$F - E + V = 30 - 90 + 60 = 0,$$

for which the right hand side is not equal to 2. It is therefore impossible to make a C_{60} polyhedron from a graphite sheet containing only hexagonal surfaces. (End of proof)

I have already mentioned that there are only five regular polyhedra in existence, but there are the semi-regular polyhedra which relax the conditions on the construction of regular polyhedra by permitting two or more different types of regular polygon. A representative example of semi-regular polyhedra is the truncated icosahedron which has 32 faces, and is the closest to a sphere among all of the semi-regular polyhedra, and is also recognizable as a soccer ball!

Figure 8: Truncated icosahedron(32 faces)

The fullerene molecule is written C_{60}, and these 60 carbon atoms are arranged as the vertices of a polyhedron. The molecule is mathematically equivalent to a 32-face truncated icosahedron. '32-face polyhedron' expresses the number of faces, but the number of vertices is 60, and there are 90 edges, so Euler's formula takes the following value.

$$F - E + V = 32 - 90 + 60 = 2.$$

5. The Number of Regular Pentagons and Hexagons

We know that the truncated icosahedron is constructed from 12 regular pentagons and 20 regular hexagons with a total of 32 faces, and I once presented a problem for readers of a math magazine to determine the number of pentagons and hexagons while examining a photograph of a soccer ball (see Nishiyama,1994).[4] This is quite a nice problem, and I think it may be ideal for an entrance exam, so let's take a look at it now.

[Problem 4] A soccer ball is constructed from a total of 32 regular pentagons and hexagons. From the information that can be obtained by looking at the photograph alone, determine the number of pentagons and hexagons individually.

There have been many unique answers, so let's look at the orthodox solution. Denote the number of regular

pentagons by x, and the number of regular hexagons as y.
Since the total is 32,

$$x + y = 32.$$

The problem is how to introduce one more equation. Suitable equations focusing on the number of faces, the number of edges or the number of vertices are all possible.

[Focusing on faces]
Since there are 5 regular hexagons around each regular pentagon, if overlaps are permitted then the total number of hexagons would be $5x$. On the other hand, around each hexagon there are 3 pentagons, so $5x$ counts 3 overlapping layers. Thus, if $5x$ is divided by the degree of overlapping, 3, then it becomes the actual number of regular hexagons.

$$y = \frac{5x}{3}.$$

Solving this yields $x = 12$ and $y = 20$.

[Focusing on edges]
The number of edges around a regular pentagon is 5, so there are $5x$ such edges in total. The number of edges around a regular hexagon is 6, so the total number of such edges is $6y$. Each regular hexagon is only adjacent to a regular pentagon along every other edge, so half the number of hexagon edges is equal to the number of edges around the regular pentagons.

$$5x = \frac{6y}{2}$$

Solving this yields $x = 12$ and $y = 20$.

[Focusing on vertices]
A regular pentagon has 5 vertices, so there are a total $5x$ such vertices. A regular hexagon has 6 vertices, so there are a total of $6y$ such vertices. Paying attention to a single vertex, there is 1 adjacent pentagon, and there are 2 adjacent hexagons. Since the number of hexagon vertices, $6y$, is twice the number of pentagon vertices, $5x$,

$$5x : 6y = 1 : 2,$$

Solving this yields $x = 12$ and $y = 20$. (End of proof)

None of the solution methods above involves complicated equations, but it's clear that having ability for spatial interpretation is particularly important in order to derive these equations. The proportion of entrance examination questions involving mechanical calculation is growing large, but aren't problems that test mathematical thinking also necessary?

6. The Width of the Tape and the Radius of the Ball

After making several *sepak takraw* balls from PP-band tape, you begin to wonder just how big a ball you can make. I therefore had a look into the relationship between the width and length of the tape, and the radius of the ball (see Nishiyama, 2001).[5]

[Problem 5] When the width of tape used to make a *sepak takraw* ball is d, find the length L, of the strip that wraps the around the ball once. Think about how a ball with double the radius would be made.

For the strip to encircle the ball once, it must pass through the 10 hexagons from A to B shown in Figure 9, so the length of the loop is $L = 10\sqrt{3}d$. The value of this coefficient is roughly 17.3. This relational expression may be referred to when estimating how much tape is needed. If the width of the tape is 15 mm, then the length of a loop is approximately 26 cm.

Taking another perspective, by considering that the tape must pass through the course around the truncated icosahedron as shown in Figure 10, the length of the loop L' is

$$L' = \{8 + \frac{4\sqrt{3}}{3}(1 + \sqrt{5 + 2\sqrt{5}})\}d.$$

The value of the coefficient in this expression is 17.4, and is slightly larger than the previous value.

Whichever is used, the length of the loop of tape may be expressed as a proportion of the width. The length of the loop and the radius of the ball are related in the proportion $L = 2\pi r$, so the radius of the ball is proportional to the width of the tape.

This means that if you want to produce a *sepak takraw* ball with twice the radius, you don't need strips of tape that are twice as long, but rather strips that are twice as wide.

It is common in high school mathematics not to deviate from textbooks and reference books. But how about starting afresh and studying mathematics by making things? It may feel like the long way round, but it might in fact be a surprising shortcut.

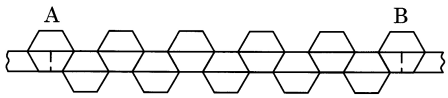


Figure 9

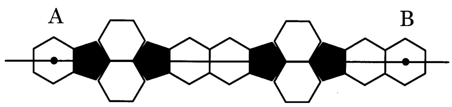


Figure 10

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