International Journal of Pure and Applied Mathematics
Volume 80 No. 1 2012, 1-15
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu
ijpam.eu

RELATION BETWEEN PROPERTIES OF TERNARY SEMIRINGS AND PROJECTIVE PLANES

Flamure Ibraimi[1 §], Alit Ibraimi[2]

Department of Mathematics
State University of Tetovo
Tetovo, MACEDONIA

Abstract: The purpose of the present paper is to study the concept of semiring in projective plane. We introduce the properties of ternary semirings and we show that projective plane with these properties is a desarguesian projective plane.

AMS Subject Classification: 51E1
Key Words: projective plane, theorem of desargues, ternary semiring

1. Introduction

Ternary semirings are one of the generalized structures of semirings introduced by Dutta and Kar [1]. D.H. Lehmer [3] introduced the notion of ternary algebraic system called triplexes which turn out to be commutative ternary groups. Some works on ternary semiring may be found in [2], [4], [5], [6], [7], [8], [9] and [10]. Main purpose in this paper is to give the relation between properties of ternary semirings and projective plane.

By coordinatization of projective plane we define addition and multiplication from the set of all points $P \in \mathcal{P} : PIl,\ P \neq J$ to the set of points on l. Then by introducing the properties of ternary semirings, we show that projective plane with these properties is a desarguesian projective plane.

Received: April 22, 2011

§Correspondence author

2. Preliminaries

Definition 1. Incidence structure or incidence geometry is called a tripple $\Sigma = (\mathcal{P}, \mathrm{B}, \mathrm{I})$ of sets $\mathcal{P}$, B, I which satisfy conditions:

$$\mathcal{P} \cap \mathrm{B} = \varphi, \quad I \subseteq \mathcal{P} \times \mathrm{B}$$

($\mathcal{P}$, B are disjunct set and I an binary relation between the elements of $\mathcal{P}$ and B).

Definition 2. The incidence structure $\mathcal{P} = (P, B, I)$ is called the projective plane if and only if:

1. Any two distinct points are incident with a unique line.
2. Any two distinct lines are incident with a unique point.
3. There exist four points such that no three are incident with one line.

Any set of points which satisfies conditions (i) and (ii) is called a *closed configuration*, and any closed configuration which does not satisfy condition (iii) is called a *degenerate plane*.

Lemma 1. *Any projective plane contains a quadrilateral.*

Theorem 1. *Let $\mathcal{P}$ be a finite projective plane. There exists a positive integer $n \geq 2$ such that:*

1. *Each line contains exactly n+1 points;*
2. *Each point is on exactly n+1 lines;*
3. *Plane $\mathcal{P}$ contains $n^2 + n$+1 points and $n^2 + n$+1 lines.*

Definition 3. Ordered triple $(A, +\bullet)$ is said to be a ternary semiring if the following condition are satisfied:

1. $(A, +)$ is commutative semigroup,
2. $(abc)de = a(bcd)e = ab(cde)$,
3. $(a + b)cd = acd + bcd$,
4. $a(b + c)d = abd + acd$,
5. $ab(c + d) = abc + abd$,

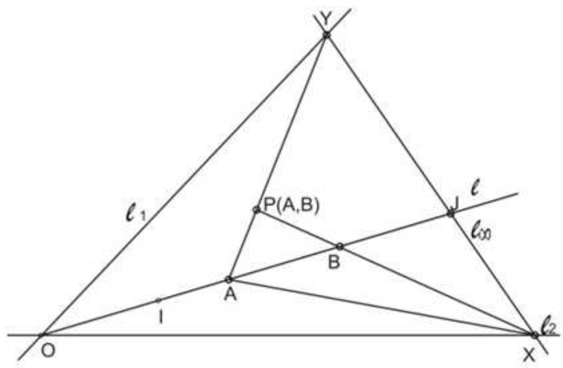

Figure 1

for all a, b, c, d, $e \in A$.

Definition 4. Let $(A, +\bullet)$be a ternary semiring. If there exists an element $0 \in A$such that $0{+}x = x$and $0xy = x0y = xy0 = 0$ for all x, $y \in A$, then “0” is called the zero element or simply the zero of the ternary semiring $(A, +\bullet)$. In this case we say that $S$is a ternary semiring with zero.

Coordinatization. An ordered quadruple $(OXYI)$ of distinct point such that no three are incident whith one line will be called a coordinate system. In this coordinate system $(OXYI)$ we define the following elements $l = OI, l_1 = OY, l_2 = OX, l_\infty = XY$ (see Figure 1).

Now define set $\mathcal{A} = \{P \in \mathcal{P} | PIl,\ P \neq J$.

For any point P not incident with the line l_∞ we can define $A = PY \cap lB = PX \cap l$ and the ordered pair (AB) to be the coordinate of the point $P(AB)$.

Also we should mention that the incidence structure $(\mathcal{P},\ \mathcal{B},\ I)$ is to be Desarguesian projective plane.

Proposition 1. *There exists a one to one correspondence between* $\mathcal{P}^{\prime} = \{P \in \mathcal{P} | P$ *and* $A \times A$.

In addition, we define a function λ_{A+B} from the set $\mathcal{A}$ to the set of points on l. Let (O, X, Y, I) be a coordinate system with associated lines l_1, l_2, l_3, l_∞ and the set $\mathcal{A}$. For any two points $AB \in \mathcal{A}$ we define $a_B = (XB \cap l_1) \cup J$. Now $\lambda_{A+B} = [(YA \cap a_B) \cup X] \cap l$, $\forall AB \in A$ (see Figure 2). So, $\lambda_{A+B} = A + B$, $\forall AB \in A$.

Corollary 1. *Let* $B \in A$, $a_B = (XB \cap l_1) \cup J$ *and* $C(X^{'}Y^{'})$, $C \notin l_\infty$. *Then* CIa_B *if and only if* $Y = X^{'} + B$.

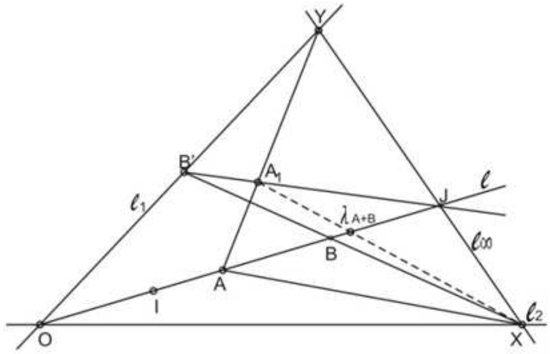

Figure 2

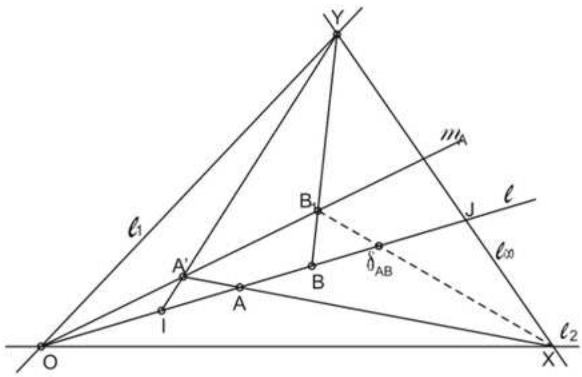

Figure 3

Now, also in $(OXYI)$ we define another function δ_{AB} from the set A to the set of points on l by $\delta_{AB} = [(YB \cap m_A) \cup J] \cap l$, $m_A = [(XA \cap YI) \cup O]$, $\forall AB \in A$ (see Figure 3). So, $\delta_{AB} = A \bullet B, \forall AB \in A$.

Corollary 2. *Let* $A \in A$, $m_A = [(XA \cap YI) \cup O]$ *and* $C(X^{'}Y^{'})$, $C \notin l_{\infty}$. *Then* CIm_A *if and only if* $Y = AX^{'}$.

3. Relation between Properties of Ternary Semirings and Projective Planes

Theorem 1.

1. $(A + B) + C = A + (B + C)$, $\forall ABC \in A$.
2. $A + B = B + A$, $\forall AB \in A$.

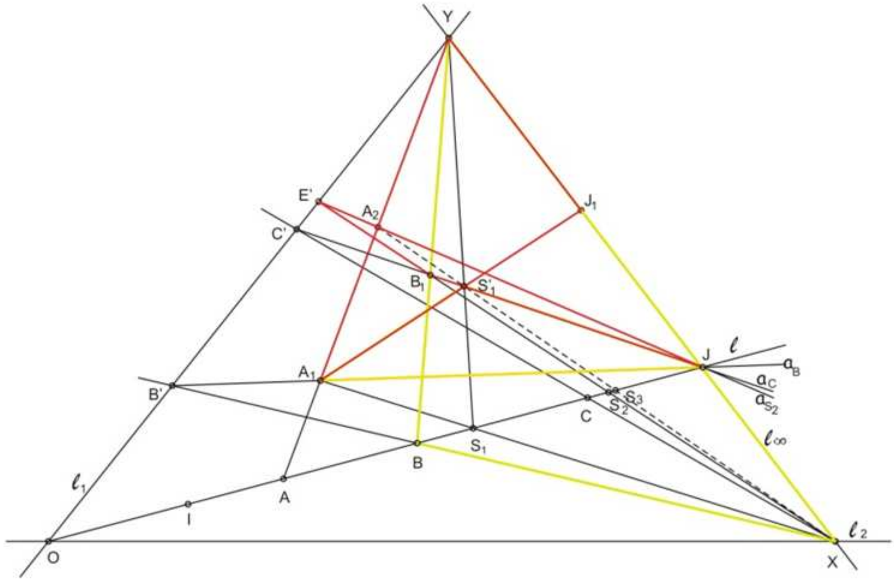

Figure 4

3. $A + O = O + A = A$, $\forall A \in A$.

4. $(A \bullet B) \bullet C = A \bullet (B \bullet C)$, $\forall ABC \in A$.

Proof.

1. To proof that $(A + B) + C = A + (B + C), \forall ABC \in \mathcal{A}$, let $ABC \neq O$ and $A \neq C$. In the coordinate system $(OXYI)$ let $ABC \in l$. To define $S_1 = A + B$ we determine $XB \cap l_1 = \left\{B'\right\}, XC \cap l_1 = \left\{C'\right\}$ then from the points B', C' and Jwe determine the lines $a_B = B'J, a_C = C'J$, $YA \cap a_B = \{A_1\}, XA_1 \cap l = \{S_1$ (see Figure 4).

   Now, $YB \cap a_C = \{B_1\}$, $B_1X \cap l = \{S_2$ define the point $S_2 = B + C$ and $a_{S_2} = (XS_2 \cap l_1) \cup J$ Since $S_1 = A + B$ we define $S_3 = (A + B) + C$ with $YS_1 \cap a_C = \left\{S_1'\right\}, XS_1' \cap l = \{S_3,\ S_1'(A + B\,(A + B) + C)$ also we define $A_1S_1' \cap l_{\boldsymbol{\infty}} = \{J_1$. Let $S_4 = A + (B + C)$ then we define $S_2 = B + C$, $YA \cap a_{S_2} = \{A_2\}\ XA_2 \cap l = \{S_4$. Point $S_4 \equiv S_3$ and $A_2(AA + (B + C))$. We need to prove the collinearity of the points A_2, S_1' and X. By the conclusion of the Theorem of Desargues, we might look for such triangles perspective from a point to obtain the necessary collinearity. We have triangles $YBX$and $S_1'JA_1$perspective from the point

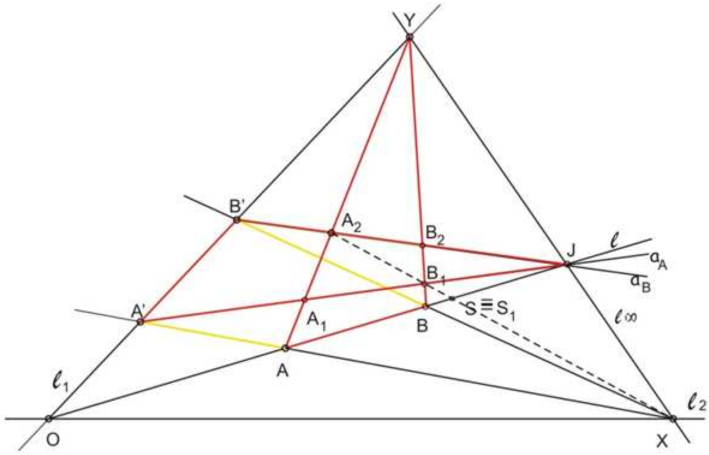

Figure 5

S_1. Thus, $YB \cap S_1'J = \{B_1\}$, $BX \cap JA_1 = \{B'$ and $YX \cap S_1'A_1 = \{J_1\}$, from the theorem of Desargues are collinear. Now we consider triangles $E'JB_1$ and YA_1J_1 perspective from B'. So, the points $E'J \cap YA_1 = \{A_2$, $JB_1 \cap A_1J_1 = \{S_1'$ and $E'B_1 \cap YJ_1 = \{X$ are collinear, based on the Theorem of Desargues. This completes the proof of part (i).

2. For $A + B = B + A, \forall AB \in \mathcal{A}$, let $AB \neq O$ and $A \neq B$ while for $A = B = O$ the result is immediate. Let ABC incident with the line$l$. To define $S = A + B$, we have to determine the point $XB \cap l_1 = \{B'$, $B'J = a_B$, $YA \cap a_B = \{A_2\}$ then $XA_2 \cap l = \{S$ and $A_2 = (AA + B)$ (see Figure 5).

   For $S_1 = B + A$ we define the line $a_A = (XA \cap OY) \cup J$ and point $YB \cap a_A = \{B_1\}$, than $XB_1 \cap l = \{S_1$. Since $S \equiv S_1$ we notice $A + B = B + A$. Now we can prove collinearity of the point A_2, B_1, Xusing the theorem of Desargues. Let us consider triangles $A'AA_1$and $B'BB_2$ perspective from the point O for which we have $A'A \cap B'B = \{X$, $A'A_1 \cap B'B_2 = \{J$, $AA_1 \cap BB_2 = \{Y$. Thus, XJY are collinear point. Now, we choose triangles $A'B'J$ and ABY perspective from O, so that $A'B' \cap AB = \{O$, $A'J \cap AY = \{A_1$, $B'J \cap BY = \{B_2\}$ are collinear. Finally, since the points $A_2(AA + B)$, $B_1(BB + A)$ and X are collinear then $A + B = B + A$.

3. $A + O = O + A = A$, $\forall A \in \mathcal{A}$. If $XA \cap OY = \{A'$, $A'J = a_A$ and $OY \cap a_A = \left\{A'\right\}$ then $A'X \cap l = \{A\}$ (see Figure 6).

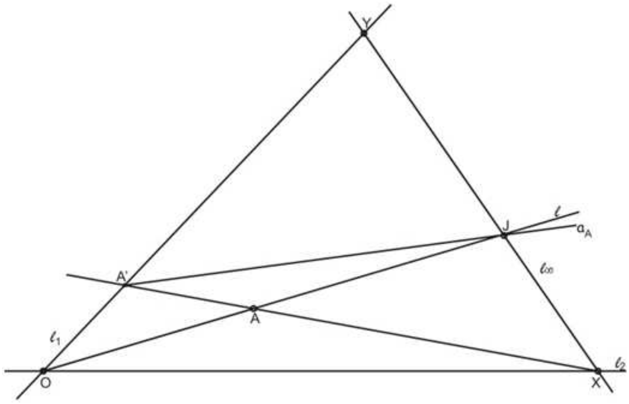

Figure 6

Therefore, $O + A = A \forall A \in \mathcal{A}$. In similar way we can see that $A + O = A, \forall A \in \mathcal{A}$.

4. Let $ABC \in A$ and $ABC \neq O$ to prove that $(A \bullet B) \bullet C = A \bullet (B \bullet C)$. Initially, to determine $P_1 = A \bullet B$ we introduce points $XA \cap YI = \{A^{'}$, $OA^{'} = m_A$, $YB \cap m_A = \{B_1$ the point$B_1 (BA \bullet B)$ then $XB_1 \cap l = \{P_1$. (fig.7).

   If $P_2 = (A \bullet B) \bullet C$, we define $P_1 \cap YI = \left\{ P_1^{'} \right\} OP_1^{'} = m_{P_1}$, $YC \cap m_{P_1} = \{C_2\}$, $C_2 \left(C, (A \bullet B) \bullet C \right)$, then $C_2X \cap l = \{P_2\}$. Now, if $P_3 = B \bullet C$ we have $XB \cap YI = \left\{ B^{'} \right\}$, $OB^{'} = m_B$, $YC \cap m_B = \{C_1\}$, $B_1 \left(C, B \bullet C \right)$ then, $XC_1 \cap l = \{P_3\}$. Next, let define the point $P_4 = A \bullet (B \bullet C)$. We introduce $YP_3 \cap m_A = \{P\}$, $P(BCA \bullet (B \bullet C))$ then, $PX \cap l = \{P_4$. Also from the fig. 7, we can see that $P_4 \equiv P_2$. We need to prove the collinearity of the points C_2, P, X. Now, it's necessary to look for triangles to be used in the theorem of Desargues to establish this collinearity. We have triangles $B' P_1' C_1$and BB_1P_3 perspective from X. Thus, $B' P_1' \cap BB_1 = \{Y$, $P_1' C_1 \cap B_1P_3 = \{F$ and $B' C_1 \cap BP_3 = \{O$ by the theorem of Desargues are collinear points. Then, by the triangles $OP_1' B_1$ and YC_1P_3 perspective from F we have collinearity of the points $OP_1' \cap YC_1 = \{C_2$, $OB_1 \cap YP_3 = \{P$ and $P_1' B_1 \cap C_1P_3 = \{X$. This completes the proof of *(iii).*

Theorem 2.

1. $(A \bullet B \bullet C) \bullet D \bullet E = A \bullet (B \bullet C \bullet D) \bullet E = A \bullet B \bullet (C \bullet D \bullet E), \forall ABC, DE \in \mathcal{A}$.

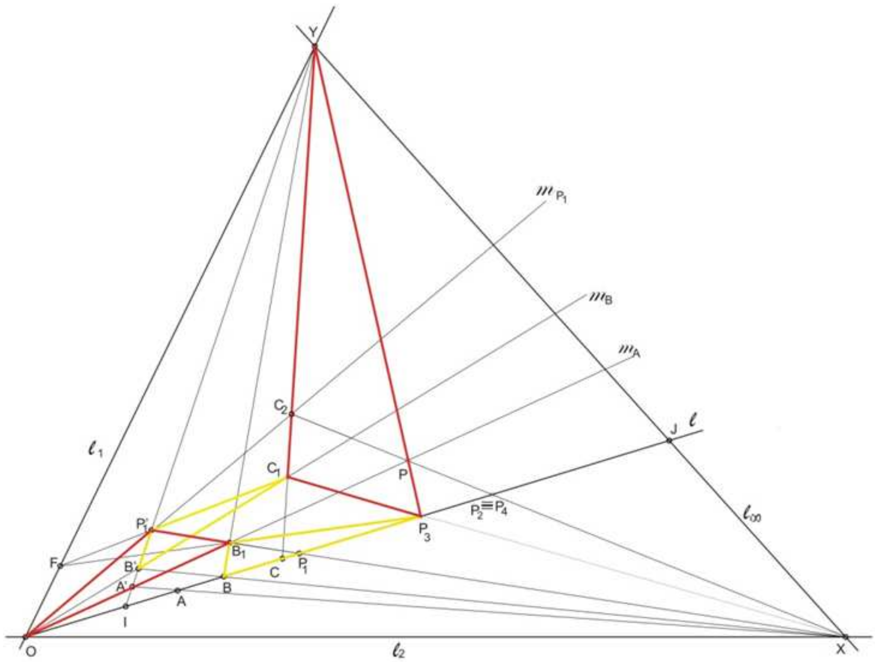

Figure 7

2. $A \bullet B \bullet O = A \bullet O \bullet B = O \bullet A \bullet B = O, \forall AB \in \mathcal{A}$*.*

3. $(A + B) \bullet C \bullet D = A \bullet C \bullet D + B \bullet C \bullet D, \forall ABC, D \in \mathcal{A}$*.*

4. $A \bullet (B + C) \bullet D = A \bullet B \bullet D + A \bullet C \bullet D, \forall ABC, D \in \mathcal{A}$*.*

5. $A \bullet B \bullet (C + D) = A \bullet B \bullet C + A \bullet B \bullet D, \forall ABC, D \in \mathcal{A}$*.*

Proof.

1. For $(A \bullet B \bullet C) \bullet D \bullet E = A \bullet B \bullet (C \bullet D \bullet E)$, $\forall ABC, DE \in \mathcal{A}$, let $ABC, DE \neq OI$. To define $P_1 = A \bullet B$ we determine $XA \cap YI = \left\{A'\right\}$, $OA' = m_A$, $YB \cap m_A = \{B_1\}$ then $XB_1 \cap l = \{P_1\}$. (fig.8). Now, if we determine $P_2 = (A \bullet B) \bullet C$ we have$XP_1 \cap YI = \{P_1'\}$, $P_1' = m_{P_1}$, $YC \cap m_{P_1} = \{C_1\}$,where $XC_1 \cap l = \{P_2\}$.

   Now, for the point $P_3 = D \bullet E$ we introduce $XD \cap YI = \left\{D'\right\}$, $OD' = m_D$, $YE \cap m_D = \{E_1\}$ and $XE_1 \cap l = \{P_3\}$. Finally, to define $P = (A \bullet B \bullet C) \bullet D \bullet E$ we have $XP_2 \cap YI = \{P_2'\}$, $OP_2' = m_{P_2}$, $YP_3 \cap m_{P_2} =$

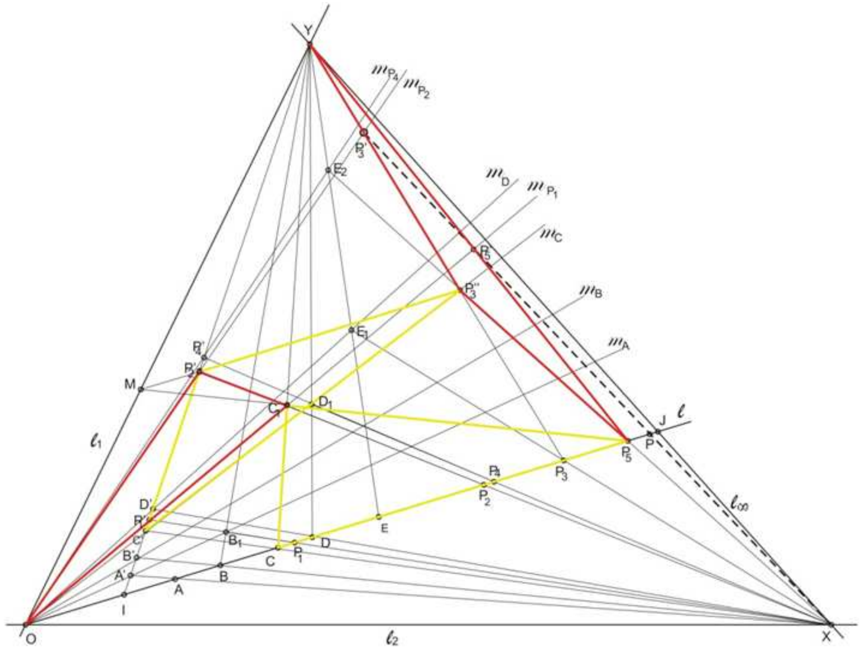

Figure 8

$\{P_3^{\prime}\}$and then $XP_3^{'} \cap l = \{P,\ P_3^{'}(D \bullet E\,(A \bullet B \bullet C) \bullet D \bullet E)$. To proof (i) we have to define also the point $P^{'} = A \bullet B \bullet (C \bullet D \bullet E)$. Using the point P_1 we define $P_4 = C \bullet D$ with $XC \cap YI = \{C^{\prime}\}$, $OC^{'} = m_C$, $YD \cap m_C = \{D_1\}$ than, $D_1X \cap l = \{P_4$. Also we denote $P_3" = m_C \cap YP_3$. Now, if $P_5 = (C \bullet D) \bullet E$ we have, $XP_4 \cap YI = \{P_4^{\prime}\}$, $OP_4^{\prime} = m_{P_4}$, $YE \cap m_{P_4} = \{E_2\}$ than, $E_2X \cap l = \{P_5$. Finally, for $P^{'} = A \bullet B \bullet (C \bullet D \bullet E)$, since $OP_1^{\prime} = m_{P_1}$, $YP_5 \cap m_{P_1} = \{P_5^{\prime}\}$ than, $XP_5^{'} \cap l = \left\{P^{'}\right\}$ where $P_5^{'}\,(C \bullet D \bullet E, A \bullet B \bullet (C \bullet D \bullet E))$ We can see that $P \equiv P^{\prime}$. For this part we have to prove the collinearity of the point $P_3^{'}, P_5^{'}, X$. Let consider triangles CC_1P_5 and $C^{\prime}P_2^{\prime}P_3^{\prime\prime}$ perspective from J. Hence, from the theorem of Desargues the point $CC_1 \cap C^{\prime}P_2^{\prime} = \{Y, C_1P_5 \cap P_2^{'}P_3^{''} = \{M$ and $CP_5 \cap C^{\prime}P_3^{\prime\prime} = \{O$ are collinear. Now, we have triangles $OC_1P_2^{\prime}$ and $YP_5P_3^{\prime\prime}$ perspective from M. Thus, $OC_1 \cap YP_5 = \{P_5^{'}, OP_2^{'} \cap YP_3^{''} = \{P_3^{\prime}\}$ and $C_1P_2^{'} \cap P_5P_3^{''} = \{X$ are collinear point. This, completes the first part of (i). The proof of the second part we will continuous based on case (iii) of Theorem 1. So, let $A \bullet (B \bullet C \bullet D) \bullet E, \forall ABC, DE \in \mathcal{A}$ denote in the form $(A \bullet G) \bullet E = A \bullet (G \bullet E)$ where $G = B \bullet C \bullet D$. To define

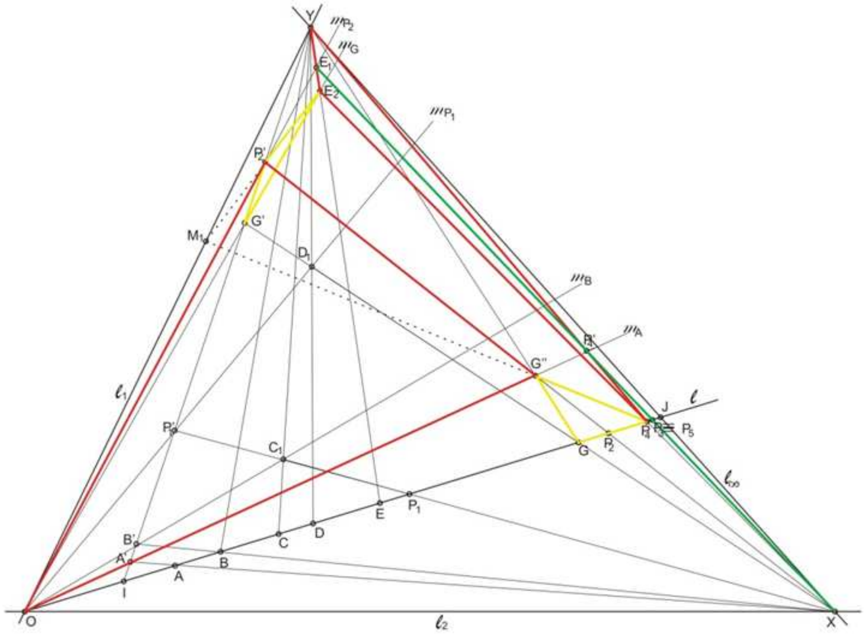

Figure 9

$P_1 = B \bullet C$ we determine $XB \cap YI = \{B'\}$, $OB' = m_B$, $YC \cap m_B = \{C_1\}$, $C_1X \cap l = \{P_1$ (see Figure 9).

Now, we define $\mathrm{G} = \mathrm{B} \bullet \mathrm{C} \bullet \mathrm{D}$ with $\mathrm{XP_1 \cap YI} = \{\mathrm{P_1}'\}$, $\mathrm{OP}_1'=\mathrm{m_{P_1}}$, $\mathrm{YD \cap m_{P_1}} = \{\mathrm{D_1}$ and $\mathrm{D_1X \cap l} = \{\mathrm{G}\}$. Next, for $\mathrm{P_2}= \mathrm{A} \bullet \mathrm{G}$ we have $\mathrm{XA \cap YI} = \{\mathrm{A}'\}$, $\mathrm{OA}' =\mathrm{m_A}$, $\mathrm{YG \cap m_A}= \{\mathrm{G}"\}$ then $\mathrm{XG" \cap l} = \{\mathrm{P_2}$. If we define $\mathrm{P_3}= (\mathrm{A} \bullet \mathrm{G}) \bullet E$, then $\mathrm{XP_2 \cap YI} = \{\mathrm{P_2}'\}$, $\mathrm{OP}_2'=\mathrm{m_{P_2}}$, $\mathrm{YE \cap m_{P_2}}= \{\mathrm{E_1}$ and $\mathrm{XE_1 \cap l} = \{\mathrm{P_3}$. Next, if $\mathrm{XG \cap YI} = \{\mathrm{G}'\}$, $\mathrm{OG}' =\mathrm{m_G}$, $\mathrm{YE \cap m_G}= \{\mathrm{E_2}$, then $\mathrm{XE_2 \cap l} = \{\mathrm{P_4}$ where $\mathrm{P_4}= \mathrm{G} \bullet \mathrm{E}$. Now, finally to define $\mathrm{P_5}= \mathrm{A(G \bullet E)}$ we have $\mathrm{YP_4 \cap m_A} = \{\mathrm{P}_4'$, $\mathrm{XP}_4' \cap \mathrm{l} = \{\mathrm{P_5}$ and $\mathrm{P_5 \equiv P_3}$. Finally, based on the Theorem of Desargues we have to proof that the point $\mathrm{E_1,P_4',X}$ are collinear. From triangles $\mathrm{GG"P_4}$ and $\mathrm{G'P_2'E_2}$ perspective from X we have $\mathrm{GG" \cap G'P_2'}=\{\mathrm{Y}\}$, $\mathrm{G"P_4 \cap P_2'E_2}= \{\mathrm{M_1}$ and $\mathrm{GP_4 \cap G'E_2}= \{\mathrm{O}\}$ collinear. To determine collinearity of the point $\mathrm{E_1,P_4'}$ and X we have triangles $\mathrm{OG"P_2'}$ and $\mathrm{YP_4E_2}$ perspective from $\mathrm{M_1}$ where $\mathrm{OG" \cap YP_4}= \{\mathrm{P_4'}$, $\mathrm{G"P_2' \cap P_4E_2}= \{\mathrm{X}\}$ and $\mathrm{OP_2' \cap YE_2} = \{\mathrm{E_1}$. This completes the proof of (i).

2. $A \bullet B \bullet O = A \bullet O \bullet B = O \bullet A \bullet B = O, \forall AB \in \mathcal{A}$. If we define $P = A \bullet B$ then from figure 10, we can see that hold $A \bullet B \bullet O = O$.

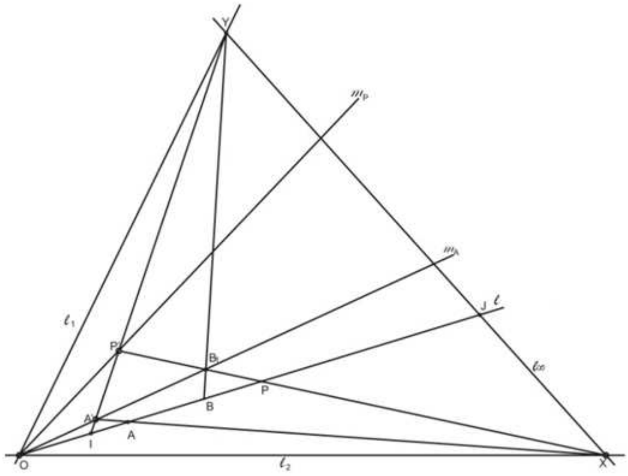

Figure 10

Similarly, $A \bullet O \bullet B = O \bullet A \bullet B = O, \forall AB \in \mathcal{A}$.

3. Let $(A + B) \bullet C \bullet D = A \bullet C \bullet D + B \bullet C \bullet D$, $\forall ABC, D \in \mathcal{A}$, $AB \neq O$ and $CD \neq OI$. First to define $S_1 = A + B$ we have $XB \cap OY = \{B"\}$, $B"J = a_B$, $YA \cap a_B = \{A_1$ and $XA_1 \cap l = \{S_1$ (see Figure 11).

   Now we can determine $P_1 = C \bullet D$ with $XC \cap YI = \{C'$, $OC' = m_C$, $YD \cap m_C = \{D_1$ then, $XD_1 \cap l = \{P_1$. Let note $M = (A+B) \bullet C \bullet D$. To determine this point in l we should define $XS_1 \cap YI = \{S_1'$, $OS_1' = m_{S_1}$, $YP_1 \cap m_{S_1} = \left\{P_1'\right\}$and then $XP_1' \cap l = \{M$. So, the point $P_1'(CD, (A+B) \bullet C \bullet D)$. Also we have to define the sum $S = A \bullet C \bullet D + B \bullet C \bullet D$, $\forall ABC, D \in \mathcal{A}$. Therefore, first we define point $P_2 = A \bullet C \bullet D$ as follows, $XA \cap YI = \{A'$, $OA' = m_A$, $YP_1 \cap m_A = \left\{P_1''\right\}$ then, $XP_1'' \cap l = \{P_2$. Now, if $P_3 = B \bullet C \bullet D$ and $XB \cap YI = \{B'$, $OB' = m_B$, $YP_1 \cap m_B = \left\{P_1'''\right\}$ then $XP_1''' \cap l = \{P_3$. Finally, we can define sum $S = A \bullet C \bullet D + B \bullet C \bullet D$ by $XP_3 \cap YO = \{P_3'$, $P_3'J = a_{P_3}$ $YP_2 \cap a_{P_3} = \{P_2'$ then, $XP_2' \cap l = \{S$ where we have $S \equiv M$. Point $P_2'(A \bullet C \bullet D, A \bullet C \bullet D + B \bullet C \bullet D)$. Thus, based on theorem of Desargues, we have to proof that the points O, A_1 and P_2' are collinear. From triangles $B"A_2A$ and $P_3'P_2''P_2$ perspective from O we have $B"A_2 \cap P_3'P_2'' = \{X$, $A_2A \cap P_2''P_2 = \{Y$ and $B"A \cap P_3'P_2 = \{K$. Points X, Y and K are collinear by the Desargues theorem. Now, from triangles $B"JP_3'$ and AYP_2 perspective from K we have $B"P_3' \cap AP_2 = \{O$,

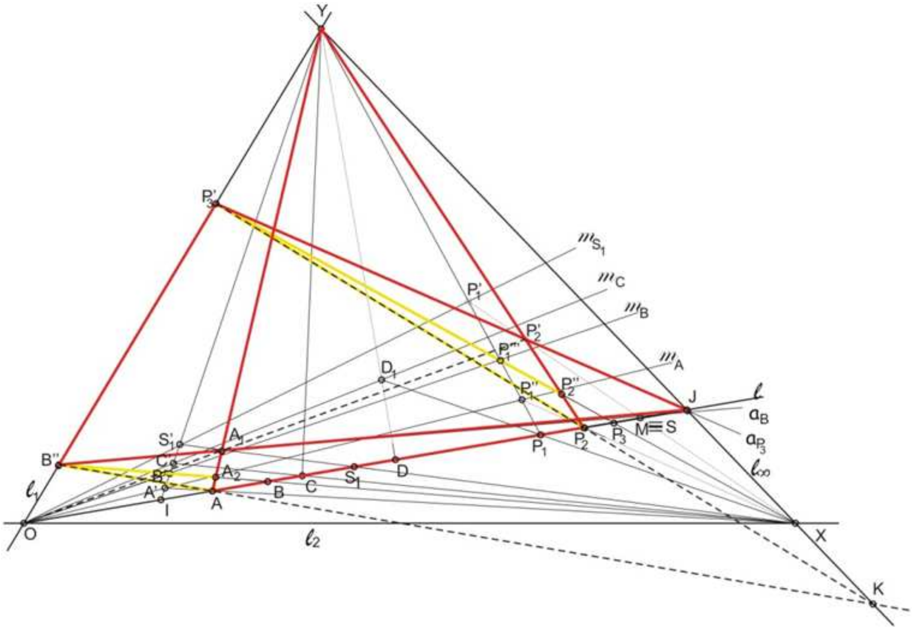

Figure 11

$B"J \cap AY = \{A_1$ and $JP_3' \cap YP_2 = \{P_2'$, so that the points O, A_1 and P_2' are collinear.

4. $A \bullet (B + C) \bullet D = A \bullet B \bullet D + A \bullet C \bullet D, \forall ABC, D \in \mathcal{A}$. First we define $S_1 = B + C$ and we have $XC \cap OY = \{C'$, $C'J = a_C$, $YB \cap a_C = \{B_1$ then, $XB_1 \cap l = \{S_1$. (fig.12).

   Now, we determine $P_1 = A \bullet (B + C)$ then $XA \cap YI = \{A'$, $OA' = m_A$, $YS_1 \cap m_A = \{S_1'$ and $XS_1' \cap l = \{P_1$. Next for $P_2 = A \bullet (B + C) \bullet D$ we have $XP_1 \cap YI = \{P_1'$, $OP_1' = m_{P_1} YD \cap m_{P_1} = \{D_1$, $XD_1 \cap l = \{P_2$. In coordinate system we have to define also the sum $S = A \bullet B \bullet D + A \bullet C \bullet D$. Therefore, if $P_3 = A \bullet B \bullet D$ first we define $P_3' = A \bullet B$ with $YB \cap m_A = \{B_2$, $XB_2 \cap l = \{P_3'$. Now, $XP_3' \cap YI = \{P_3''$, $OP_3'' = m_{P_3'}$, $YD \cap m_{P_3'} = \{D_2\}$, $XD_2 \cap l = \{P_3$. To define $P_4 = A \bullet C \bullet D$, we determine first $P_4' = A \bullet C$ and we have $YC \cap m_A = \{C_1$, $XC_1 \cap l = \{P_4'$. Now, $XP_4' \cap YI = \{P_4''$, $OP_4'' = m_{P_4'}$, $YD \cap m_{P_4'} = \{D_3$, $XD_3 \cap l = \{P_4$. Finally, we define sum $S = A \bullet B \bullet D + A \bullet C \bullet D$ and we have $XP_4 \cap YO = \{G$, $GJ = a_{P_4}$, $YP_3 \cap a_{P_4} = \{F\}$, $FX \cap l = \{S$ where $S \equiv P_2$ We need to prove the collinearity of the points F, B_1 and O. From triangles BB_3C' and P_3F_1G perspective from O we have $BB_3 \cap P_3F_1 = \{Y$, $B_3C' \cap F_1G = \{X$,

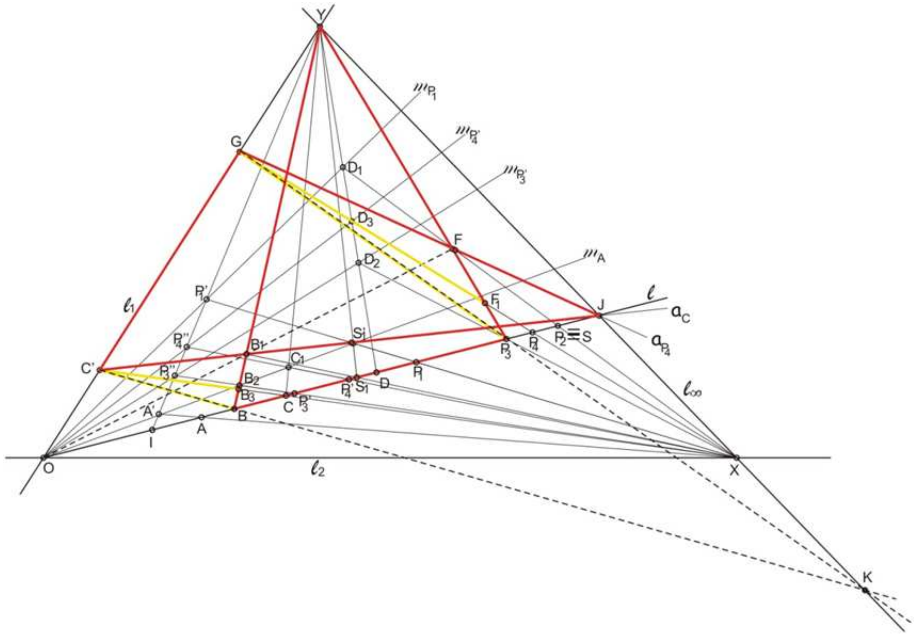

Figure 12

$BC' \cap P_3G = \{K$. We can see that the point Y, X, K are collinear. Let consider triangles YP_3B and JGC' perspective from K. Since, $YP_3 \cap JG = \{F,\ P_3B \cap GC' = \{O,\ YB \cap JC' = \{B_1$ we can see that these points are collinear. This completes the proof of *(iii).*

5. Let $A \bullet B \bullet (C + D) = A \bullet B \bullet C + A \bullet B \bullet D$, $\forall ABC, D \in \mathcal{A}$ and $A, B \neq O, I, CD \neq O$. Let define $P_1 = A \bullet B$, then $XA \cap YI = \{A'$, $OA' = m_A$, $YB \cap m_A = \{B_1$ and $XB_1 \cap l = \{P_1\}$, $B_1(BA \bullet B)$. (fig.13). Now to determine $S_1 = C + D$ we have $XD \cap YO = \{D'$, $D'J = a_D$, $YC \cap a_D = \{C_1$ and then $XC_1 \cap l = \{S_1$, $C_1(CC + D)$.

   Next, we define $P_2 = A \bullet B \bullet (C + D)$and we have $XP_1 \cap YI = \{P_1'$, $OP_1' = m_{P_1}$, $YS_1 \cap m_{P_1} = \{S_1'$, $XS_1' \cap l = \{P_2$. On the other side if $S_2 = A \bullet B \bullet C + A \bullet B \bullet D$ then we define $P_3 = A \bullet B \bullet C$ and $P_4 = A \bullet B \bullet D$. For P_3 we determine $YC \cap m_{P_1} = \{C_2$, $XC_2 \cap l = \{P_3$ so, $C_2(C, A \bullet B \bullet C)$. For P_4we determine $YD \cap m_{P_1} = \{D_1$, $XD_1 \cap l = \{P_4$ so, $D_1(D, A \bullet B \bullet D)$. Now, for $S_2 = A \bullet B \bullet C + A \bullet B \bullet D$ we have $XP_4 \cap YO = \{P_4'$, $P_4'J = a_{P_4}$, $YP_3 \cap a_{P_4} = \{P_3'$, $XP_3' \cap l = \{S_2$ and $P_3'(A \bullet B \bullet C A \bullet B \bullet C + A \bullet B \bullet D)$. Finally, we can see that $S_2 \equiv P_2$. Also we determine the points $C_2 \cap l_{\boldsymbol{\infty}} = \{H\}$, $P_4'H \cap YC = \{Q\}$, $D'H \cap YC = \{C_3$,

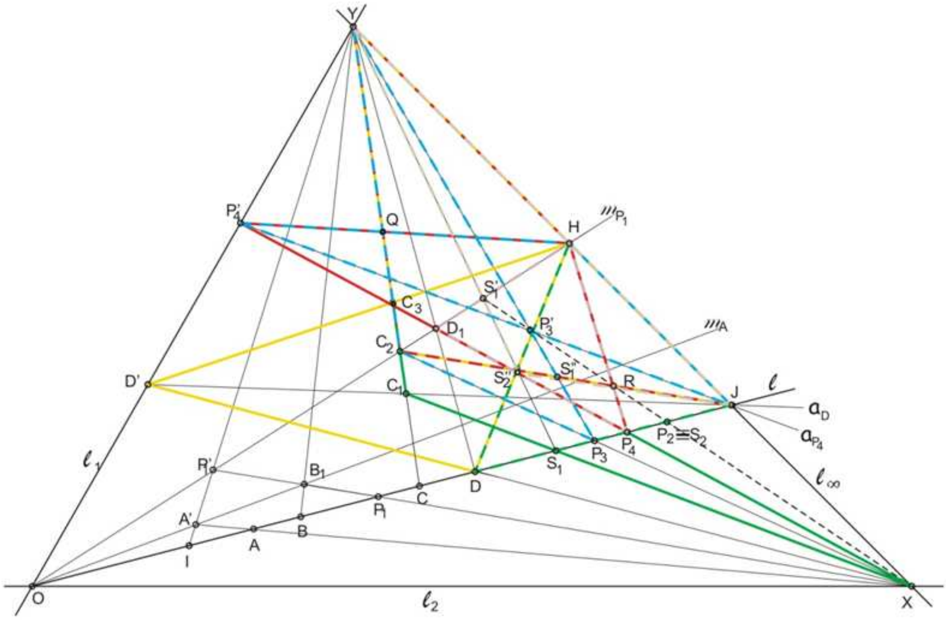

Figure 13

$C_2J \cap P_4H = \{R$ and $C_2J \cap YS_1 = \{S_1^{''}$.

We have to proof collinearity of the points $S_1^{'}, R$ and X. From triangles YJC_2 and $D'DH$ perspective from O we have $YJ \cap D^{'}D = \{X,\ JC_2 \cap DH = \{S_2^{''}$ and $YC_2 \cap D^{'}H = \{C_3$. Therefore, $X, S_2^{''}C_3$ are collinear. From triangles C_3XC_1 and HDJ perspective from $D^{'}$ we have $C_3X \cap HD = \{S_2^{''},\ XC_1 \cap DJ = \{S_1$ and $C_3C_1 \cap HJ = Y$. Points $Y, S_2^{''}, S_1$are collinear. Now with triangles YJC_2 and $P_4^{'}P_4H$ perspective from O we have collinearity of the points $YJ \cap P_4^{'}P_4 = \{X,\ JC_2 \cap P_4H = \{R$ and $YC_2 \cap P_4^{'}H = \{Q$. From triangles YP_3C_2 and $P_4^{'}JH$ perspective from O we have $YP_3 \cap P_4^{'}J = \{P_3^{'},\ P_3C_2 \cap JH = \{X$ and $YC_2 \cap P_4^{'}H = \{Q$. Hence $X, P_3^{'}, Q$ are collinear. And finally, from triangles D_1HP_4 and $YS_1^{''}J$ perspective from D we have collinearity of the points $D_1H \cap YS_1^{''} = \{S_1^{'}$, $HP_4 \cap S_1^{''}J = \{R$ and $D_1P_4 \cap YJ = \{X\}$ This completes the proof of *(iv).*

From Theorem 1, (i), (iii), (iv) and Theorem 2 (i), (iii), (iv), (v), we obtain the following proposition.

Proposition 2. *Projective plane with elements from ternary semiring is a desarguesian plane.*

References

[1] T.K. Dutta, S. Kar, On regular ternary semirings, *Advances in Algebra*, Proceedings of the ICM Satellite Conference in Algebra and Related Topics, World Scientific (2003), 343-355.

[2] T.K. Dutta, S. Kar, A note on regular ternary semirings, *Kyungpook Math. J.*, **46** (2006), 357-365.

[3] D.H. Lehmer, A ternary analogue of Abelian groups, *American Journal of Mathematics*, **59** (1932), 329-338.

[4] Kavikumar, Azme Khamis, Young Bae Jun, Fuzzy bi-ideals in ternary semirings, *International Journal of Computational and Mathematical Sciences*, **3**, No. 4 (2009), 164-168.

[5] T.K. Dutta, S. Kar, On the Jacobson radical of a ternary semiring, *Southeast Asian Bulletin of Mathematics*, **28**, No. 1 (2004), 1-13.

[6] T.K. Dutta, S. Kar, A note on the Jacobson radicals of ternary semirings, *Southeast Asian Bulletin of Mathematics*, **29**, No. 2 (2005), 321-331.

[7] T.K. Dutta, S. Kar, Two types of Jacobson radicals of ternary semirings, *Southeast Asian Bulletin of Mathematics*, **29**, No. 4 (2005), 677-687.

[8] T.K. Dutta, S. Kar, On matrix ternary semirings, *International Journal of Mathematics and Analysis*, **1**, No. 1 (2006), 97-111.

[9] S. Kar, On quasi-ideals and bi-ideals of ternary semirings, *International Journal of Mathematics and Mathematical Sciences*, **2005**, No. 18 (2005), 3015-3023.

[10] S. Kar, On structure space of ternary semirings, *Southeast Asian Bulletin of Mathematics*, **31** (2007), 547-555.

[11] D. Hughes, F. Piper, *Projective Planes Sringer Verlag*, New York (1973).

[12] S. Bilaniuk, *A Problem Course on Projective Planes*, Trent University, Ontario, Canada (2003).