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ON $k-$GRACEFULNESS OF $r-$CROWNS FOR COMPLETE BIPARTITE GRAPHS

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Abstract: Let $I_r(K_{m,n})$ denote a $r-$ crown of a complete bipartite graph $K_{m,n}$ obtained by adding r hanged edges to each vertex of $K_{m,n}$. Ma kejie conjectured that 1-crown of complete bipartite graph $K_{m,n}$ $(m \leq n)$ is $k-$ graceful graph for $k \geq 2$. The conjecture has been shown true when $m = 1, 2, 3, 4$ for arbitrary $n \geq m$ and $r \geq 2$. In this paper we discuss the $k-$gracefulness of $r-$crown $I_r(K_{m,n})$ $(m \leq n, r \geq 2)$ for complete bipartite graph $K_{m,n}$ and prove the conjecture when $m = 5$, for arbitrary $n \geq m$ and $r \geq 2$.

AMS Subject Classification: 05C65
Key Words: complete bipartite graph, graceful graph, $k-$graceful graph

1. Introduction

Most graph labeling methods trace their origin to one introduced by Rosa [1] in 1966, or one given by Graham and Sloane [2] in 1980. Let $G(V,E)$ be a simple undirected graph, if there exist a single-valued mapping $f : V(G) \to \{0, 1, \cdots, |E|\}$ such that $f(x) \neq f(y)$ for distinct $x, y \in V(G)$ and an induced mapping is defined as $f^{\square} : E(G) \to \{1, 2, \cdots, |E|\}$, where $f^{\square}(uv) = |f(u)-f(v)|$ is a bijection for all edges $uv \in E(G)$, then Rosa [1] called the function f the $\beta - valuation$ of a graph G, Golomb [3] subsequently called such labelings to

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be graceful and the graph is called a graceful graph and f is called a graceful labeling, while $f^{\square}$ is called an induced edge's graceful labeling.

Although an unpublished result of Erdös says that most graphs are not graceful (cf. [2]), most graphs that have some sort of regularity of structure are graceful.Labeled graphs serve as useful models for a broad range of applications such as: coding theory, $x-$ray crystallography, radar, astronomy, circuit design, communication network addressing and data base management, see [4], [5] and [6]for details.

A natural generalization of graceful graphs is the notion of k-graceful graphs introduced independently by Slater [7] in 1982 and by Maheo and Thuillier [8] in 1982. Let $G(V,E)$ be a simple undirected graph, k be an arbitrary natural number larger than 2, if there exists a mapping $f : V(G) \to \{0, 1, 2, \cdots, |E| + k-1\}$ such that $f(x) \neq f(y)$ for distinct $x, y \in V(G)$ and an induced mapping is defined as $f^{\square} : E(G) \to \{k, k+1, \cdots, |E|+k-1\}$, where $f^{\square}(uv) = |f(u)-f(v)|$ is a bijection for all edges $uv \in E(G)$, then the graph G is called a $k-$graceful graph, f is called a $k-$graceful labeling, while $f^{\square}$ is called an induced edge's $k-$graceful labeling.

Obviously, 1-graceful is graceful. Graphs that are k-graceful for all k are sometimes called arbitrarily graceful.

Results of Maheo and Thuillier [8] together with those of Slater [7] show that: C_n is k-graceful if and only if either $n \equiv 0$ or $1(mod4)$ with k even and $k \leq (n-1)/2$, or $n \equiv 3(mod4)$ with k odd and $k \leq (n^2-1)/2$. Maheo and Thuillier [8] also proved that the wheel W_{2k+1} is k-graceful and conjectured that W_{2k} is k-graceful when $k \neq 3$ or $k \neq 4$. This conjecture was proved by Liang, Sun, and Xu [9]. Kang [10] proved that $P_m \times C_{4n}$ is k-graceful for all k. Lee and Wang [11] showed that the graphs obtained from a nontrivial path of even length by joining every other vertex to one isolated vertex (a lotus), the graphs obtained from a nontrivial path of even length by joining every other vertex to two isolated vertices (a diamond), and the graphs obtained by arranging vertices into a finite number of rows with i vertices in the ith row and in every row the jth vertex in that row is joined to the jth vertex and $(j+1)st$ vertex of the next row (a pyramid) are k-graceful.

Liang and Liu [12] have shown that $K_{m,n}$ is k-graceful. Bu, Gao, and Zhang [13] have proved that $P_n \times P_2$ and $(P_n \times P_2) \cup (P_n \times P_2)$ are k-graceful for all k. Acharya (see [14]) has shown that a k-graceful Eulerian graph with q edges must satisfy one of the following conditions: $q \equiv 0(mod4)$, $q \equiv 1(mod4)$ if k is even, or $q \equiv 3(mod4)$ if k is odd. Bu, Zhang, and He [15] have shown that an even cycle with a fixed number of pendant edges adjoined to each vertex is k-graceful. Lu, Pan, and Li [16] have proved that $K_{1,m} \cup K_{p,q}$ is k-graceful when $k > 1$, and p and q are at least 2. Seoud and Elsakhawi [17] proved: paths and ladders are arbitrarily graceful; and for $n > 3$, K_n is k-graceful if and only if $k = 1$ and $n = 3$ or 4.

Definition 1. The $r-$ crown of the complete bipartite graph $K_{m,n}$, denoted as $I_r(K_{m,n})$, is obtained by adding r hanged edges to each vertex of $K_{m,n}$.

Obviously, The $1-$ crown $I_1(K_{m,n})$ of a complete bipartite graph is a graceful graph. Ma ke-jie [18] presented the following conjecture:

Conjecture 1. (see [18]) $1-$crown of complete bipartite graph $K_{m,n}$ $(m \leq n)$ is $k-$ graceful graph for $k \geq 2$.

This conjecture has not proved or disproved up to now. Jirimutu [19] has showed that this conjecture is true when $m = 1$. Jirimutu, Yu-Lan Bao, Fan-li Kong [20] have proved that this conjecture is true when $m = 2, 3$, for arbitrary $n \geq m$ and $r \geq 2$. Siqinqimuge and Jirimutu [21] have proved that this conjecture is true when $m = 4$, for arbitrary $n \geq m$ and $r \geq 2$. In this paper we have proved that this conjecture is true when $m = 5$, for arbitrary $n \geq 5$ and $r \geq 2$.

2. The Main Result

Theorem 1. *For $m = 5$, $r \geq 2$ and $n \geq 5$, the $r-$crown $I_r(K_{m,n})$ of a complete bipartite graph $K_{m,n}$ is a $k-$ graceful graph for $k \geq 2$.*

Proof. We first give some notations used in the following proof. In $I_r(K_{m,n})$, let $X = \{x_1, x_2, \cdots, x_m\}$ and $Y = \{y_1, y_2, \cdots, y_n\}$, (X,Y)is a bipartition of $K_{m,n}$, and let the vertices of the $r-$hanged edges connected to each $x_i (i = 1, 2, \cdots m)$ in X are denoted by $x_{it} (t = 1, 2, \cdots, r)$; the vertices of the $r-$ hanged edges connected to each vertex $y_j (j = 1, 2, \cdots, n)$ in Y are denoted by $y_{jt} (j = 1, 2, \cdots, n,\ t = 1, 2, \cdots, r)$. Based on above notations we define the vertex label f of $I_r(K_{m,n})(n \geq 5, r \geq 2)$ as follows. Let

$$f(x_i) = \begin{cases} k + (6-i)n + (5+n)r - 1, & i = 1, 2, \\ k + (6-i)n + (5 + n - \frac{(i-1)(i-2)}{2})r - 1, & i = 3, 4, 5, \end{cases}$$

$$f(x_{it}) = \begin{cases} t - 1, & i = 1;\ t = 1, 2, \cdots, r \\ n + (i-1)r + t - 1, & i = 2, 3, 4, 5;\ t = 1, 2, \cdots, r \end{cases}$$

$$f(y_j) = r + j - 1, \quad j = 1, 2, \cdots, n,$$

$f(y_{jt}) =$

$$\begin{cases} k+(r+1)j-t, & j=1,2,3,\cdots,n-6; t=1,2,\cdots,r, \\ k+2r+j(r+1)-(t+1), & j=n-5,n-4,n-3; t=1,2,\cdots,r, \\ k+n+3r+j(r+1)-(t+1), & j=n-2,n-1; t=1,2,\cdots,r, \\ k+3n+(j+4)r-(t+1), & j=n; t=1,2,\cdots,r. \end{cases}$$

It is easy to check that f is a single-valued mapping from $V(I_r(K_{m,n}))$ to $\{0,1,2,\cdots,|E(I_r(K_{5,n}))|+k-1\}$.

Now we prove that the induced mapping $f^{*}: E(G) \to \{k,k+1,\cdots,|E|+k-1\}$, where $f^{*}(uv) = |f(u)-f(v)|$, is a bijective mapping for all edges $uv \in E(G)$. Let

$$A_i = \{|f(x_i)-f(x_{it})| : t=1,2,\cdots,r\}, i=1,2,\cdots,5,$$

$$B_i = \{|f(x_i)-f(y_j)| : j=1,2,\cdots,n\}, i=1,2,\cdots,5,$$

$$C_j = \{|f(y_j)-f(y_{jt})| : t=1,2,\cdots,r\}, j=1,2\cdots,n.$$

The edge label induced by f^{*} is as follows.

$$A_1 = \{|f(x_1)-f(x_{1t})| : t=1,2,\cdots,r\}$$
$$= \{k+5n+(n+5)r-1, k+5n+(n+5)r-2,\cdots,k+5n+(n+4)r\},$$

$$B_1 = \{|f(x_1)-f(y_j)| : j=1,2,\cdots,n\}$$
$$= \{k+5n+(n+4)r-1, k+5n+(n+4)r-2\cdots,k+5n+(n+4)r-n\},$$

$$B_2 = \{|f(x_2)-f(y_j)| : j=1,2,\cdots,n\}$$
$$= \{k+4n+(n+4)r-1, k+4n+(n+4)r-2\cdots,k+3n+(n+4)r\},$$

$$A_2 = \{|f(x_2)-f(x_{2,t})| : t=1,2,\cdots,r\}$$
$$= \{k+3n+(n+4)r-1, k+3n+(n+4)r-2,\cdots,k+3n+(n+3)r\},$$

$$B_3 = \{|f(x_3)-f(y_j)| : j=1,2,\cdots,n\}$$
$$= \{k+3n+(n+3)r-1, k+3n+(n+3)r-2\cdots,k+2n+(n+2)r\},$$

$$\begin{aligned} C_n &= \{|f(y_n) - f(y_{n,t})| : t = 1, 2, \cdots, r\} \\ &= \{k + 2n + (n+3)r - 1, k + 2n + (n+3)r - 2, \cdots, k + 2n + (n+2)r\}, \end{aligned}$$

$$\begin{aligned} A_3 &= \{|f(x_3) - f(x_{3,t})| : t = 1, 2, \cdots, r\} \\ &= \{k + 2n + (n+2)r - 1, k + 2n + (n+2)r - 2, \cdots, k + 2n + (n+1)r\}, \end{aligned}$$

$$\begin{aligned} B_4 &= \{|f(x_4) - f(y_j)| : j = 1, 2, \cdots, n\} \\ &= \{k + 2n + (n+1)r - 1, k + 2n + (n+1)r - 2, \cdots, k + n + (n+1)r\}, \end{aligned}$$

$$\begin{aligned} C_{n-1} &= \{|f(y_{n-1}) - f(y_{(n-1),t})| : t = 1, 2, \cdots, r\} \\ &= \{k + n + (n+1)r - 1, k + n + (n+1)r - 2, \cdots, k + n + nr\}, \end{aligned}$$

$$\begin{aligned} C_{n-2} &= \{|f(y_{n-2}) - f(y_{(n-2),t})| : t = 1, 2, \cdots, r\} \\ &= \{k + n + n + r - 1, k + n + nr - 2, \cdots, k + n + (n-1)r\}, \end{aligned}$$

$$\begin{aligned} A_4 &= \{|f(x_4) - f(x_{4,t})| : t = 1, 2, \cdots, r\} \\ &= \{k + n + (n-1)r - 1, k + n + (n-1)r - 2, \cdots, k + n + (n-2)r\}, \end{aligned}$$

$$\begin{aligned} B_5 &= \{|f(x_5) - f(y_j)| : j = 1, 2, \cdots, n\} \\ &= \{k + n + (n-2)r - 1, k + n + (n-2)r - 2, \cdots, k + (n-2)r\}, \end{aligned}$$

$$\begin{aligned} C_{n-3} &= \{|f(y_{n-3}) - f(y_{(n-3),t})| : t = 1, 2, \cdots, r\} \\ &= \{k + (n-2)r - 1, k + (n-2)r - 2, \cdots, k + (n-3)r\}, \end{aligned}$$

$$\begin{aligned} C_{n-4} &= \{|f(y_{n-4}) - f(y_{(n-4),t})| : t = 1, 2, \cdots, r\} \\ &= \{k + (n-3)r - 1, k + (n-3)r - 2, \cdots, k + (n-4)r\}, \end{aligned}$$

$$\begin{aligned} C_{n-5} &= \{|f(y_{n-5}) - f(y_{(n-5),t})| : t = 1, 2, \cdots, r\} \\ &= \{k + (n-4)r - 1, k + (n-4)r - 2, \cdots, k + (n-5)r\}, \end{aligned}$$

$$A_5 = \{|f(x_5) - f(x_{5,t})| : t = 1, 2, \cdots, r\}$$
$$= \{k + (n-5)r - 1, k + (n-5)r - 2, \cdots, k + (n-6)r\},$$

$$C_{n-6} = \{|f(y_{n-6}) - f(y_{(n-6),t})| : t = 1, 2, \cdots, r\}$$
$$= \{k + (n-6)(r-1), k + (n-6)(r-1) - 1,$$
$$\cdots, k + (n-7)(r-1)\}$$

$$\vdots$$

$$C_1 = \{|f(y_1) - f(y_{1,t})| : t = 1, 2, \cdots, r\} = \{k + r - 1, k + r - 2, \cdots, k\}.$$

We tidy up the elements of each set and have an union

$$(\bigcup_{i=1}^{n} C_i)\bigcup(\bigcup_{i=1}^{n} B_i)\bigcup(\bigcup_{i=1}^{n} A_i)$$
$$= C_1\bigcup C_2\bigcup\cdots\bigcup C_{n-m-1}\bigcup A_5\bigcup C_{n-5}\bigcup C_{n-4}\bigcup C_{n-3}\bigcup B_5\bigcup A_4$$
$$\bigcup C_{n-2}\bigcup C_{n-1}\bigcup B_4\bigcup A_3\bigcup C_n\bigcup B_3\bigcup A_2\bigcup B_2\bigcup B_1\bigcup A_1$$
$$= \{k, k+1, \cdots, |E(I_r(K_{5,n}))| + k - 1\}$$

So, the induced mapping $f^{\square}$ is a bijective mapping from $E(I_r(K_{5,n}))$ onto $\{k, k+1, \cdots, |E(I_r(K_{5,n}))| + k - 1\}$.

Thus, the $r-$crown graph $I_r(K_{5,n})$ of a complete bipartite graph $k_{5,n}$ is a $K-$ graceful graph. $\square$

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