ON $k$–GRACEFULNESS OF $r$–CROWNS FOR COMPLETE BIPARTITE GRAPHS

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Abstract: Let $I_r(K_{m,n})$ denote a $r$– crown of a complete bipartite graph $K_{m,n}$ obtained by adding $r$ hanged edges to each vertex of $K_{m,n}$. Ma kejie conjectured that 1-crown of complete bipartite graph $K_{m,n}$ ($m \leq n$) is $k$– graceful graph for $k \geq 2$. The conjecture has been shown true when $m = 1, 2, 3, 4$ for arbitrary $n \geq m$ and $r \geq 2$. In this paper we discuss the $k$–gracefulness of $r$–crown $I_r(K_{m,n})$ ($m \leq n, r \geq 2$) for complete bipartite graph $K_{m,n}$ and prove the conjecture when $m = 5$, for arbitrary $n \geq m$ and $r \geq 2$.

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1. Introduction

Most graph labeling methods trace their origin to one introduced by Rosa [1] in 1966, or one given by Graham and Sloane [2] in 1980. Let $G(V, E)$ be a simple undirected graph, if there exist a single-valued mapping $f : V(G) \rightarrow \{0, 1, \cdots, |E|\}$ such that $f(x) \neq f(y)$ for distinct $x, y \in V(G)$ and an induced mapping is defined as $f^* : E(G) \rightarrow \{1, 2, \cdots, |E|\}$, where $f^*(uv) = |f(u) - f(v)|$ is a bijection for all edges $uv \in E(G)$, then Rosa [1] called the function $f$ the $\beta$–valuation of a graph $G$, Golomb [3] subsequently called such labelings to

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be graceful and the graph is called a graceful graph and $f$ is called a graceful labeling, while $f^*$ is called an induced edge’s graceful labeling.

Although an unpublished result of Erdős says that most graphs are not graceful (cf. [2]), most graphs that have some sort of regularity of structure are graceful. Labeled graphs serve as useful models for a broad range of applications such as: coding theory, $x$-ray crystallography, radar, astronomy, circuit design, communication network addressing and data base management, see [4], [5] and [6] for details.

A natural generalization of graceful graphs is the notion of $k$-graceful graphs introduced independently by Slater [7] in 1982 and by Maheo and Thuillier [8] in 1982. Let $G(V, E)$ be a simple undirected graph, $k$ be an arbitrary natural number larger than 2, if there exists a mapping $f : V(G) \rightarrow \{0, 1, 2, \ldots, |E| + k - 1\}$ such that $f(x) \neq f(y)$ for distinct $x, y \in V(G)$ and an induced mapping is defined as $f^*: E(G) \rightarrow \{k, k + 1, \ldots, |E| + k - 1\}$, where $f^*(uv) = |f(u) - f(v)|$ is a bijection for all edges $uv \in E(G)$, then the graph $G$ is called a $k$-graceful graph, $f$ is called a $k$-graceful labeling, while $f^*$ is called an induced edge’s $k$-graceful labeling.

Obviously, 1-graceful is graceful. Graphs that are $k$-graceful for all $k$ are sometimes called arbitrarily graceful.

Results of Maheo and Thuillier [8] together with those of Slater [7] show that: $C_n$ is $k$-graceful if and only if either $n \equiv 0$ or $1(mod 4)$ with $k$ even and $k \leq (n - 1)/2$, or $n \equiv 3(mod 4)$ with $k$ odd and $k \leq (n^2 - 1)/2$. Maheo and Thuillier [8] also proved that the wheel $W_{2k + 1}$ is $k$-graceful and conjectured that $W_{2k}$ is $k$-graceful when $k \neq 3$ or $k \neq 4$. This conjecture was proved by Liang, Sun, and Xu [9]. Kang [10] proved that $P_m \times C_4$ is $k$-graceful for all $k$. Lee and Wang [11] showed that the graphs obtained from a nontrivial path of even length by joining every other vertex to one isolated vertex (a lotus), the graphs obtained from a nontrivial path of even length by joining every other vertex to two isolated vertices (a diamond), and the graphs obtained by arranging vertices into a finite number of rows with $i$ vertices in the $i$th row and in every row the $j$th vertex in that row is joined to the $j$th vertex and $(j + 1)$st vertex of the next row (a pyramid) are $k$-graceful.

Liang and Liu [12] have shown that $K_{m,n}$ is $k$-graceful. Bu, Gao, and Zhang [13] have proved that $P_n \times P_2$ and $(P_n \times P_2) \cup (P_n \times P_2)$ are $k$-graceful for all $k$. Acharya (see [14]) has shown that a $k$-graceful Eulerian graph with $q$ edges must satisfy one of the following conditions: $q \equiv 0(mod 4)$, $q \equiv 1(mod 4)$ if $k$ is even, or $q \equiv 3(mod 4)$ if $k$ is odd. Bu, Zhang, and He [15] have shown that an even cycle with a fixed number of pendant edges adjoined to each vertex is $k$-graceful. Lu, Pan, and Li [16] have proved that $K_{1,m} \cup K_{p,q}$ is $k$-graceful.
when \( k > 1 \), and \( p \) and \( q \) are at least 2. Seoud and Elsakhawi [17] proved: paths and ladders are arbitrarily graceful; and for \( n > 3 \), \( K_n \) is \( k \)-graceful if and only if \( k = 1 \) and \( n = 3 \) or 4.

**Definition 1.** The \( r \)-crown of the complete bipartite graph \( K_{m,n} \), denoted as \( I_r(K_{m,n}) \), is obtained by adding \( r \) hanged edges to each vertex of \( K_{m,n} \).

Obviously, The 1-- crown \( I_1(K_{m,n}) \) of a complete bipartite graph is a graceful graph. Ma ke-jie [18] presented the following conjecture:

**Conjecture 1.** (see [18]) 1--crown of complete bipartite graph \( K_{m,n} \) \((m \leq n)\) is \( k \)-- gracefull graph for \( k \geq 2 \).

This conjecture has not proved or disproved up to now. Jirimutu [19] has showed that this conjecture is true when \( m = 1 \). Jirimutu, Yu-Lan Bao, Fan-li Kong [20] have proved that this conjecture is true when \( m = 2, 3 \), for arbitrary \( n \geq m \) and \( r \geq 2 \). Siqinqimuge and Jirimutu [21] have proved that this conjecture is true when \( m = 4 \), for arbitrary \( n \geq m \) and \( r \geq 2 \). In this paper we have proved that this conjecture is true when \( m = 5 \), for arbitrary \( n \geq 5 \) and \( r \geq 2 \).

### 2. The Main Result

**Theorem 1.** For \( m = 5 \), \( r \geq 2 \) and \( n \geq 5 \), the \( r \)--crown \( I_r(K_{m,n}) \) of a complete bipartite graph \( K_{m,n} \) is a \( k \)-- graceful graph for \( k \geq 2 \).

**Proof.** We first give some notations used in the following proof. In \( I_r(K_{m,n}) \), let \( X = \{x_1, x_2, \ldots, x_m\} \) and \( Y = \{y_1, y_2, \ldots, y_n\} \), \((X,Y)\)is a bipartition of \( K_{m,n} \), and let the vertices of the \( r \)--hanged edges connected to each vertex \( x_i (i = 1, 2, \ldots, m) \) in \( X \) are denoted by \( x_{it} (t = 1, 2, \ldots, r) \); the vertices of the \( r \)--hanged edges connected to each vertex \( y_j (j = 1, 2, \ldots, n) \) in \( Y \) are denoted by \( y_{jt} (j = 1, 2, \ldots, n, t = 1, 2, \ldots, r) \). Based on above notations we define the vertex label \( f \) of \( I_r(K_{m,n}) (n \geq 5, r \geq 2) \) as follows. Let

\[
\begin{align*}
  f(x_i) &= \begin{cases} 
  k + (6 - i)n + (5 + n)r - 1, & i = 1, 2, \\
  k + (6 - i)n + (5 + n - \frac{(i-1)(i-2)}{2})r - 1, & i = 3, 4, 5,
  \end{cases} \\
  f(x_{it}) &= \begin{cases} 
  t - 1, & i = 1; \ t = 1, 2, \ldots, r \\
  n + (i - 1)r + t - 1, & i = 2, 3, 4, 5; \ t = 1, 2, \ldots, r
  \end{cases} \\
  f(y_j) &= r + j - 1, \quad j = 1, 2, \ldots, n,
\end{align*}
\]
The edge label induced by $20 \text{ Deligen, L. Zhao, Jirimutu}$

$$f(y_{jt}) = \begin{cases} 
  k + (r + 1)j - t, & j = 1, 2, 3, \cdots, n - 6; t = 1, 2, \cdots, r, \\
  k + 2r + j(r + 1) - (t + 1), & j = n - 5, n - 4, n - 3; t = 1, 2, \cdots, r, \\
  k + n + 3r + j(r + 1) - (t + 1), & j = n - 2, n - 1; t = 1, 2, \cdots, r, \\
  k + 3n + (j + 4)r - (t + 1), & j = n; t = 1, 2, \cdots, r.
\end{cases}$$

It is easy to check that $f$ is a single-valued mapping from $V(I_{r}(K_{m,n}))$ to $\{0, 1, 2, \cdots, |E(I_{r}(K_{5,n}))| + k - 1\}$.

Now we prove that the induced mapping $f^* : E(G) \rightarrow \{k, k + 1, \cdots, |E| + k - 1\}$, where $f^*(uv) = |f(u) - f(v)|$, is a bijective mapping for all edges $uv \in E(G)$. Let

$$A_i = \{|f(x_i) - f(x_{it})| : t = 1, 2, \cdots, r\}, i = 1, 2, \cdots, 5,$$

$$B_i = \{|f(x_i) - f(y_j)| : j = 1, 2, \cdots, n\}, i = 1, 2, \cdots, 5,$$

$$C_j = \{|f(y_j) - f(y_{jt})| : t = 1, 2, \cdots, r\}, j = 1, 2 \cdots, n.$$ 

The edge label induced by $f^*$ is as follows.

$$A_1 = \{|f(x_1) - f(x_{1t})| : t = 1, 2, \cdots, r\} = \{k + 5n + (n + 5)r - 1, k + 5n + (n + 5)r - 2, \cdots, k + 5n + (n + 4)r\},$$

$$B_1 = \{|f(x_1) - f(y_j)| : j = 1, 2, \cdots, n\} = \{k + 5n + (n + 4)r - 1, k + 5n + (n + 4)r - 2 \cdots, k + 5n + (n + 4)r - n\},$$

$$B_2 = \{|f(x_2) - f(y_j)| : j = 1, 2, \cdots, n\} = \{k + 4n + (n + 4)r - 1, k + 4n + (n + 4)r - 2 \cdots, k + 3n + (n + 4)r\},$$

$$A_2 = \{|f(x_2) - f(x_{2t})| : t = 1, 2, \cdots, r\} = \{k + 3n + (n + 4)r - 1, k + 3n + (n + 4)r - 2, \cdots, k + 3n + (n + 3)r\},$$

$$B_3 = \{|f(x_3) - f(y_j)| : j = 1, 2, \cdots, n\} = \{k + 3n + (n + 3)r - 1, k + 3n + (n + 3)r - 2 \cdots, k + 2n + (n + 2)r\},$$

$$E(G) = \{k, k + 1, \cdots, |E| + k - 1\},$$

$$f^*(uv) = |f(u) - f(v)|.$$
\[ C_n = \{ |f(y_n) - f(y_{n,t})| : t = 1, 2, \ldots, r \} \]
\[ = \{ k + 2n + (n+3)r - 1, k + 2n + (n+3)r - 2, \ldots, k + 2n + (n+2)r \}, \]

\[ A_3 = \{ |f(x_3) - f(x_{3,t})| : t = 1, 2, \ldots, r \} \]
\[ = \{ k + 2n + (n+2)r - 1, k + 2n + (n+2)r - 2, \ldots, k + 2n + (n+1)r \}, \]

\[ B_4 = \{ |f(x_4) - f(y_j)| : j = 1, 2, \ldots, n \} \]
\[ = \{ k + 2n + (n+1)r - 1, k + 2n + (n+1)r - 2, \ldots, k + n + (n+1)r \}, \]

\[ C_{n-1} = \{ |f(y_{n-1}) - f(y_{(n-1),t})| : t = 1, 2, \ldots, r \} \]
\[ = \{ k + n + (n+1)r - 1, k + n + (n+1)r - 2, \ldots, k + n + nr \}, \]

\[ C_{n-2} = \{ |f(y_{n-2}) - f(y_{(n-2),t})| : t = 1, 2, \ldots, r \} \]
\[ = \{ k + n + n + r - 1, k + n + nr - 2, \ldots, k + n + (n-1)r \}, \]

\[ A_4 = \{ |f(x_4) - f(x_{4,t})| : t = 1, 2, \ldots, r \} \]
\[ = \{ k + n + (n-1)r - 1, k + n + (n-1)r - 2, \ldots, k + n + (n-2)r \}, \]

\[ B_5 = \{ |f(x_5) - f(y_j)| : j = 1, 2, \ldots, n \} \]
\[ = \{ k + n + (n-2)r - 1, k + n + (n-2)r - 2, \ldots, k + (n-2)r \}, \]

\[ C_{n-3} = \{ |f(y_{n-3}) - f(y_{(n-3),t})| : t = 1, 2, \ldots, r \} \]
\[ = \{ k + (n-2)r - 1, k + (n-2)r - 2, \ldots, k + (n-3)r \}, \]

\[ C_{n-4} = \{ |f(y_{n-4}) - f(y_{(n-4),t})| : t = 1, 2, \ldots, r \} \]
\[ = \{ k + (n-3)r - 1, k + (n-3)r - 2, \ldots, k + (n-4)r \}, \]

\[ C_{n-5} = \{ |f(y_{n-5}) - f(y_{(n-5),t})| : t = 1, 2, \ldots, r \} \]
\[ = \{ k + (n-4)r - 1, k + (n-4)r - 2, \ldots, k + (n-5)r \}, \]
\[ A_5 = \{|f(x) - f(x,t)| : t = 1, 2, \cdots, r\} = \{k + (n - 5)r - 1, k + (n - 5)r - 2, \cdots, k + (n - 6)r\}, \]

\[ C_{n-6} = \{|f(y) - f(y_{(n-6),t})| : t = 1, 2, \cdots, r\} = \{k + (n - 6)(r - 1), k + (n - 6)(r - 1) - 1, \]
\[ \cdots, k + (n - 7)(r - 1)\} \]

\[ C_1 = \{|f(y) - f(y_{1,t})| : t = 1, 2, \cdots, r\} = \{k + r - 1, k + r - 2, \cdots, k\}. \]

We tidy up the elements of each set and have an union
\[
\bigcup_{i=1}^{n} C_i \cup \bigcup_{i=1}^{n} B_i \cup \bigcup_{i=1}^{n} A_i = C_1 \cup C_2 \cup \cdots \cup C_{n-m-1} \cup A_5 \cup C_{n-5} \cup C_{n-4} \cup C_{n-3} \cup B_5 \cup A_4
\]
\[
\cup C_{n-2} \cup C_{n-1} \cup B_4 \cup A_3 \cup C_n \cup B_3 \cup A_2 \cup B_2 \cup B_1 \cup A_1
\]
\[ = \{k, k + 1, \cdots, |E(I_r(K_{5,n}))| + k - 1\} \]

So, the induced mapping \( f^* \) is a bijective mapping from \( E(I_r(K_{5,n})) \) onto \( \{k, k + 1, \cdots, |E(I_r(K_{5,n}))| + k - 1\} \).

Thus, the \( r \)-crown graph \( I_r(K_{5,n}) \) of a complete bipartite graph \( k_{5,n} \) is a \( K \)-graceful graph.

\[ \square \]

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