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# OPTIMAL INEQUALITIES BETWEEN GENERALIZED LOGARITHMIC, IDENTRIC AND POWER MEANS

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**Abstract:** In this paper, we present the best possible power mean bounds for the generalized logarithmic mean, and find the optimal lower generalized logarithmic mean bound for the identric mean.

**AMS Subject Classification:** 26E60
**Key Words:** generalized logarithmic mean, identric mean, power mean

## 1. Introduction

The logarithmic and identric means of two positive real numbers $a$ and $b$ are defined by

$$L(a,b)=\begin{cases}\dfrac{b-a}{\log b-\log a}, & a\neq b,\\ a, & a=b\end{cases}$$

and

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$$I(a,b)=\begin{cases}\frac{1}{e}(\frac{b^b}{a^a})^{1/(b-a)}, & a\neq b,\\ a, & a=b,\end{cases} \tag{1.1}$$

respectively. In the recent past, both mean values have been the subject of intensive research. In particular, many remarkable inequalities and properties for the two variables means can be found in the literature [1-26]. The ratio of identric means leads to the weighted geometric mean

$$\frac{I(a^2,b^2)}{I(a,b)}=(a^ab^b)^{1/(a+b)}.$$

It might be surprising that the logarithmic mean has applications in physics, economics, and even in meteorology [27-29]. In [27] the authors study a variant of Jensen's functional equation involving $L$, which appears in a heat conduction problem. A representation of $L$ as an infinite product and an iterative algorithm for computing the logarithmic mean as the common limit of two sequences of special geometric and arithmetic means are given in [30]. In [31, 32] it is shown that $L$ can be expressed in terms of Gauss's hypergeometric function ${}_2F_1$. And, in [31] the authors prove that the reciprocal of the logarithmic mean is strictly totally positive, that is, every $n\times n$ determinant with elements $1/L(a_i,b_i)$, where $0<a_1<a_2<\cdots<a_n$ and $0<b_1<b_2<\cdots<b_n$, is positive for all $n\geq 1$.

The power mean of order $p$ of two positive real numbers $a$ and $b$ is defined by

$$M_p(a,b)=\begin{cases}(\frac{a^p+b^p}{2})^{1/p}, & p\neq 0,\\ \sqrt{ab}, & p=0.\end{cases} \tag{1.2}$$

The main properties for $M_p(a,b)$ are given in [33]. In particular, the function $p\mapsto M_p(a,b)$ $(a\neq b)$ is continuous and strictly increasing on $R$.

For $t>0$ the generalized logarithmic mean $L_t(a,b)$ of two positive real numbers $a$ and $b$ is defined by

$$L_t(a,b)=L(a^t,b^t)^{1/t}=\begin{cases}\left(\frac{b^t-a^t}{t(\log b-\log a)}\right)^{1/t}, & a\neq b,\\ a, & a=b.\end{cases} \tag{1.3}$$

It is well known that $L_t(a,b)$ is continuous and strictly increasing with respect to $t\in(0,+\infty)$ for fixed $a,b>0$ with $a\neq b$.

For all $a,b>0$ with $a\neq b$, the following inequalities can be found in [34, 35]:

$$L(a,b)=L_1(a,b)<AG(a,b)<L_{3/2}(a,b)<M_{1/2}(a,b),$$

where $AG(a,b)$ is the classical arithmetic-geometric mean of $a$ and $b$ which is defined as the common limit of sequences $\{a_n\}$ and $\{b_n\}$ given by

$$a_0 = a, \quad b_0 = b, \quad a_{n+1} = (a_n + b_n)/2, \quad b_{n+1} = \sqrt{a_n b_n}.$$

Stolarsky [36, 37] proved that the inequalities

$$L(a,b) = L_1(a,b) < I(a,b)$$

and

$$M_{2/3}(a,b) < I(a,b)$$

hold for all $a, b > 0$ with $a \neq b$, and $M_{2/3}(a,b)$ is the best possible lower power mean bound for $I(a,b)$.

In [38], Lin proved that the following results: (1) $p \geq 1/3$ implies that $L(a,b) = L_1(a,b) < M_p(a,b)$ for all $a, b > 0$ with $a \neq b$; (2) $p \leq 0$ implies that $L(a,b) = L_1(a,b) > M_p(a,b)$ for all $a, b > 0$ with $a \neq b$; (3) $p < 1/3$ implies that there exist $a, b > 0$ such that $L(a,b) = L_1(a,b) > M_p(a,b)$; (4) $p > 0$ implies that there exist $a, b > 0$ such that $L(a,b) = L_1(a,b) < M_p(a,b)$. Hence the question was answered: what are the greatest value $p$ and the least value $q$ such that the double inequality $M_p(a,b) < L(a,b) < M_q(a,b)$ holds for all $a, b > 0$ with $a \neq b$?

Alzer and Qiu [39] proved that the inequalities

$$\alpha M_1(a,b) + (1-\alpha)M_0(a,b) < I(a,b) < \beta M_1(a,b) + (1-\beta)M_0(a,b)$$

and

$$M_c(a,b) < \frac{1}{2}(L_1(a,b) + I(a,b))$$

hold for all $a, b > 0$ with $a \neq b$ if and only if $\alpha \leq 2/3$, $\beta \geq 2/e = 0.73575\ldots$ and $c \leq \log 2/(1+\log 2) = 0.40938\ldots$.

It is the aim of this paper to find the best possible parameters $\alpha$, $\beta$ and $\gamma$ such that the inequalities

$$M_\alpha(a,b) < L_t(a,b) < M_\beta(a,b)$$

and

$$L_\gamma(a,b) < I(a,b)$$

hold for any fixed $t > 0$ and all $a, b > 0$ with $a \neq b$.

## 2. Main Results

**Theorem 2.1.** *For any fixed $t > 0$, the double inequality*

$$M_0(a,b) < L_t(a,b) < M_{t/3}(a,b)$$

*holds for all $a, b > 0$ with $a \neq b$, and $M_0(a,b)$ and $M_{t/3}(a,b)$ are the best possible lower and upper power bounds for $L_t(a,b)$, respectively.*

*Proof.* From (1.2) and (1.3) we clearly see that both $M_p(a,b)$ and $L_t(a,b)$ are symmetric and homogenous of degree 1. Without loss of generality, we assume that $b = 1$ and $a = x > 1$.

Firstly, we prove that inequality

$$L_t(x,1) < M_{t/3}(x,1) \tag{2.1}$$

holds for any fixed $t > 0$ and all $x > 1$.

From (1.2) and (1.3) one has

$$\begin{aligned} &\log L_t(x,1) - \log M_{t/3}(x,1) \\ = \; &\frac{1}{t}[\log(x^t - 1) - \log(t \log x)] - \frac{3}{t} \log \frac{x^{t/3}+1}{2}. \end{aligned} \tag{2.2}$$

Let

$$f(x) = \frac{1}{t}[\log(x^t - 1) - \log(t \log x)] - \frac{3}{t} \log \frac{x^{t/3}+1}{2}, \tag{2.3}$$

then simple computations lead to

$$\lim_{x \to 1} f(x) = 0, \tag{2.4}$$

$$f'(x) = \frac{(x^{2t/3}+1)g(x)}{x^{1-t/3}(x^t-1)(x^{t/3}+1)\log x}, \tag{2.5}$$

where

$$g(x) = \log x - \frac{x^{4t/3} + x^t - x^{t/3} - 1}{t x^{t/3}(x^{2t/3}+1)}, \tag{2.6}$$

$$g(1) = 0, \tag{2.7}$$

$$g'(x) = -\frac{x^{t/3}(x^{t/3}-1)^4(x^{2t/3}+x^{t/3}+1)}{3x^{2t/3+1}(x^{2t/3}+1)^2} < 0 \tag{2.8}$$

for all $x > 1$.

Therefore, inequality (2.1) follows from equations (2.2)-(2.7) and inequality (2.8).

Secondly, we prove that inequality

$$M_0(x,1) < L_t(x,1) \tag{2.9}$$

holds any fixed $t > 0$ and all $x > 1$.

From (1.2) and (1.3) we get

$$\log L_t(x,1) - \log M_0(x,1)$$
$$= \frac{1}{t}[\log(x^t - 1) - \log(t\log x)] - \frac{1}{2}\log x. \tag{2.10}$$

Let

$$F(x) = \frac{1}{t}[\log(x^t - 1) - \log(t\log x)] - \frac{1}{2}\log x, \tag{2.11}$$

then simple computations lead to

$$\lim_{x\to 1} F(x) = 0, \tag{2.12}$$

$$F'(x) = \frac{(x^t+1)G(x)}{2x(x^t-1)\log x}, \tag{2.13}$$

where

$$G(x) = \log x - \frac{2(x^t-1)}{t(x^t+1)}, \tag{2.14}$$

$$G(1) = 0, \tag{2.15}$$

$$G'(x) = \frac{(x^t-1)^2}{x(x^t+1)^2} > 0 \tag{2.16}$$

for all $x > 1$.

Therefore, inequality (2.9) follows from equations (2.10)-(2.15) and inequality (2.16).

Next, we prove that $M_{t/3}(a,b)$ is the best possible upper power bound for $L_t(a,b)$.

For any $\epsilon > 0$, $t > 0$ and $x > 0$, from (1.2) and (1.3) one has

$$L_t(1+x,1) - M_{t/3-\epsilon}(1+x,1)$$
$$= \left[\frac{(1+x)^t - 1}{t\log(1+x)}\right]^{1/t} - \left[\frac{(1+x)^{t/3-\epsilon}+1}{2}\right]^{\frac{1}{t/3-\epsilon}}. \tag{2.17}$$

Letting $x \to 0$ and making use of Taylor expansion, we have

$$\begin{aligned}
&\left[\frac{(1+x)^t-1}{t\log(1+x)}\right]^{1/t} - \left[\frac{(1+x)^{t/3-\epsilon}+1}{2}\right]^{\frac{1}{t/3-\epsilon}} \\
= &\left[1+\frac{t}{2}x+\frac{t(2t-3)}{12}x^2+o(x^2)\right]^{1/t} \\
&-\left[1+\frac{t-3\epsilon}{6}x+\frac{(t-3\epsilon)(t-3\epsilon-3)}{36}x^2+o(x^2)\right]^{\frac{1}{t/3-\epsilon}} \\
= &\left[1+\frac{1}{2}x+\frac{t-3}{24}x^2+o(x^2)\right]-\left[1+\frac{1}{2}x+\frac{t-3\epsilon-3}{24}x^2+o(x^2)\right] \\
= &\frac{\epsilon}{8}x^2+o(x^2).
\end{aligned} \tag{2.18}$$

Equations (2.17) and (2.18) imply that for any $\epsilon > 0$ and $t > 0$ there exists $\delta = \delta(\epsilon)$, such that $L_t(1+x,1) > M_{t/3-\epsilon}(1+x,1)$ for any $x \in (0,\delta)$.

Finally, we prove that the $M_0(a,b)$ is the best possible lower power bound for $L_t(a,b)$.

For any $\epsilon > 0$, $t > 0$ and $x > 1$, from (1.2) and (1.3) we have

$$\lim_{x\to+\infty}\frac{L_t(x,1)}{M_\epsilon(x,1)} = \lim_{x\to+\infty}\frac{\left(\frac{x^t-1}{t\log x}\right)^{1/t}}{\left(\frac{x^\epsilon+1}{2}\right)^{1/\epsilon}} = 0. \tag{2.19}$$

Equation (2.19) imply that for any $\epsilon > 0$ and $t > 0$ there exists $T = T(\epsilon,t) > 1$, such that $L_t(x,1) < M_\epsilon(x,1)$ for any $x \in (T,+\infty)$. □

**Theorem 2.2.** *Inequality*

$$L_2(a,b) < I(a,b) \tag{2.20}$$

*holds for all $a,b > 0$ with $a \neq b$, and $L_2(a,b)$ is the best possible lower generalized logarithmic mean bound for the identric mean $I(a,b)$.*

*Proof.* From (1.2) and (1.3) we clearly see that both $I(a,b)$ and $L_t(a,b)$ are symmetric and homogenous of degree 1. Without loss of generality, we assume that $b = 1$ and $a = x > 1$.

From (1.1) and (1.3) one has

$$\begin{aligned}
&\log L_2(x,1) - \log I(x,1) \\
= &\frac{1}{2}[\log(x^2-1)-\log(2\log x)] - \frac{x}{x-1}\log x + 1.
\end{aligned} \tag{2.21}$$

Let

$$f(x) = \frac{1}{2}[\log(x^2-1) - \log(2\log x)] - \frac{x}{x-1}\log x + 1, \tag{2.22}$$

then simple computations lead to

$$\lim_{x\to 1} f(x) = 0, \tag{2.23}$$

$$f'(x) = \frac{g(x)}{2x(x+1)(x-1)^2\log x}, \tag{2.24}$$

where

$$g(x) = 2x(x+1)\log^2 x + 2x(1-x)\log x - x^3 + x^2 + x - 1,$$

$$g(1) = 0, \tag{2.25}$$

$$g'(x) = 2(2x+1)\log^2 x + 6\log x - 3x^2 + 3,$$

$$g'(1) = 0, \tag{2.26}$$

$$g''(x) = 4\log^2 x + (8 + \frac{4}{x})\log x + \frac{6}{x} - 6x,$$

$$g''(1) = 0, \tag{2.27}$$

$$g'''(x) = \frac{2}{x^2}h(x), \tag{2.28}$$

where

$$h(x) = (4x-2)\log x - 3x^2 + 4x - 1,$$

$$h(1) = 0, \tag{2.29}$$

$$h'(x) = 4\log x - \frac{2}{x} - 6x + 8,$$

$$h'(1) = 0, \tag{2.30}$$

$$h''(x) = -\frac{2(3x+1)(x-1)}{x^2} < 0 \tag{2.31}$$

for all $x > 1$.

Therefore, inequality (2.20) follows easily from equations (2.21)-(2.30) and inequality (2.31).

Next, we prove that $L_2(a, b)$ is the best possible lower generalized logarithmic mean bound for identric mean $I(a, b)$.

For any $\epsilon > 0$ and $x > 0$, from (1.1) and (1.3) one has

$$L_{2+\epsilon}(1+x,1)-I(1+x,1) = \left[\frac{(1+x)^{2+\epsilon}-1}{(2+\epsilon)\log(1+x)}\right]^{1/(2+\epsilon)} - \frac{1}{e}(1+x)^{(1+x)/x}. \quad (2.32)$$

Letting $x \to 0$ and making use of Taylor expansion, one has

$$\begin{aligned}
& \left[\frac{(1+x)^{2+\epsilon}-1}{(2+\epsilon)\log(1+x)}\right]^{1/(2+\epsilon)} - \frac{1}{e}(1+x)^{(1+x)/x} \\
= & \left[1+\frac{2+\epsilon}{2}x+\frac{(2+\epsilon)(1+2\epsilon)}{12}x^2+o(x^2)\right]^{1/(2+\epsilon)} \\
& -(1+x)\left[1-\frac{1}{2}x+\frac{11}{24}x^2+o(x^2)\right] \\
= & \left[1+\frac{1}{2}x+\frac{\epsilon-1}{24}x^2+o(x^2)\right]-\left[1+\frac{1}{2}x-\frac{1}{24}x^2+o(x^2)\right] \\
= & \frac{\epsilon}{24}x^2+o(x^2). \qquad (2.33)
\end{aligned}$$

Equations (2.32) and (2.33) imply that for any $\epsilon > 0$ there exists $\delta = \delta(\epsilon)$, such that $L_{2+\epsilon}(1+x,1) > I(1+x,1)$ for any $x \in (0,\delta)$. □

**Remark 2.3.** For any $C > 2$, there exists $T = T(C) > 1$ such that $I(x,1) > L_C(x,1)$ for $x \in (T,+\infty)$. In fact, from (1.1) and (1.3) one has

$$\lim_{x\to+\infty}\frac{I(x,1)}{L_C(x,1)} = \lim_{x\to+\infty}\frac{x^{1/(x-1)}(C\log x)^{1/C}}{e(1-x^{-C})^{1/C}} = +\infty.$$

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