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STRONG CONVERGENCE THEOREMS OF VISCOSITY APPROXIMATION PROCESS FOR QUASI-NONEXPANSIVE MAPPINGS IN HILBERT SPACES

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Abstract: In this paper, we have discussed the strong convergence of two modifications of the viscosity approximation method in Hilbert spaces, relatively to the computation of fixed points of operators in the class of quasi-nonexpansive mappings.

Our convergence theorems extend a recent result of P.E. Maingé, see [1].

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1. Introduction

Throughout this paper, we assume that H is a real Hilbert space endowed with an inner product and its induced norm denoted by $\langle\cdot,\cdot\rangle$ and $|\cdot|$, respectively. Ω is a closed convex subset of H and $T : \Omega \to H$ is a mapping on Ω with a nonempty fixed point set denoted by $F(T) := \{x \in \Omega \mid Tx = x\} \neq \emptyset$ and $P_\Omega : H \to \Omega$ is the metric projection, $C : \Omega \to \Omega$ is a Banach contraction with modulus $\rho \in [0, 1)$, i.e.,

$$\|Cx - Cy\| \leq \rho\|x - y\|, \quad \forall x, y \in \Omega. \tag{1.1}$$

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Recently, Maingé [1] proposed a new analysis of the viscosity approximation method in Hilbert spaces, relatively to the computation of fixed points of operators in the wide class of quasi-nonexpansive mappings. He considered the following iteration process,

$$x_{n+1} = \alpha_n C x_n + (1-\alpha_n) T_\omega x_n. \tag{1.2}$$

Under appropriate conditions, he proved that $\{x_n\}$ converges to the unique solution of the variational inequality problem VIP($I-C$,$F(T)$): find x^* in $F(T)$ such that

$$\langle (I-C)x^*, q-x^* \rangle \geq 0, \ \forall q \in F(T). \tag{1.3}$$

Motivated and inspired by the Maingé [1] we propose a modification of viscosity approximation method. We define $\{x_n\}$ in the following way:

$$x_{n+1} = \alpha_n C x_n + (1-\alpha_n) P_\Omega T_\omega x_n. \tag{1.4}$$

where $\{\alpha_n\}$ is a slow vanishing sequence, $T_\omega := (1-\omega)I + \omega T, \quad \omega \in (0,1]$, I being the identity mapping on Ω, with two main conditions on T:
(i1) $T \in \varepsilon_Q$ is quasi-nonexpansive, i.e. $\|Tx - Tq\| \leq \|x - q\|$ for any $(x,q) \in \Omega \times F(T)$
(i2) T is demiclosed on Ω, that is $\{z_k\} \subset \Omega, z_k \rightharpoonup z$ weakly, $(I-T)(z_k) \to 0$ strongly $\Rightarrow z \in F(T)$.

In our second modification of viscosity approximation method, our attention will be focused on the following variant of algorithm (1.4):

$$x_{n+1} = \beta_n x_n + (1-\beta_n)[\alpha_n C x_n + (1-\alpha_n) P_\Omega T x_n], \tag{1.5}$$

where $\{\beta_n\}$ is a sequence in (0,1) satisfying $\limsup_{n\to\infty} \beta_n < 1$.

By using simplified method for proofs, this paper establishes the strong convergence of the two sequences given by (1.4) and (1.5) to the unique solution of (1.3). No additional conditions are made on the operator T.

2. Preliminaries

In this section we give a series of preliminary results which are needed for the convergence analysis.

Remark 2.1. Since T is quasi-nonexpansive, it is well-known that $F(T)$ is a closed and convex subset of H, hence $P_{F(T)}$ is well-defined. Note also that, $P_{F(T)}$ is nonexpansive and $C : \Omega \to \Omega$ is a Banach contraction, then

$P_{F(T)}C$ is a Banach contraction, so there exists unique element $x^* \in F(T)$, s.t. $x^* = (P_{F(T)}C)x^*$ which equivalently solves the variational inequality problem (1.3).

Lemma 2.2. *Let T be a quasi-nonexpansive mapping on Ω with $F(T) \neq \emptyset$, and set $T_\omega := (1-\omega)I + \omega T$ for $\omega \in (0,1]$. Then the following statements are reached:*

(i) T is equivalent to

$$\langle x - Tx, x - q\rangle \geq (1/2)\|x - Tx\|^2,\ \forall (x, q) \in \Omega \times F(T);$$

(ii) T_ω with $\omega \in (0,1]$ satisfies

$$\|T_\omega x - q\|^2 \leq \|x - q\|^2 - \omega(1-\omega)\|Tx - x\|^2;$$

(iii)T_ω and $P_\Omega T_\omega$ are quasi-nonexpansive mappings;

(iv) $F(T) = F(T_\omega) = F(P_\Omega T_\omega)$.

Proof. (i) By the following classical equality:

$$\|x - y\|^2 = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2$$

and from T is quasi-nonexpansive, $\forall\ (x, q) \in \Omega \times F(T)$ we have

$$\begin{aligned}
\langle x - Tx, x - q\rangle &= (1/2)\|x - Tx\|^2 + (1/2)\|x - q\|^2 - (1/2)\|x - Tx - x + q\|^2 \\
&= (1/2)\|x - Tx\|^2 + (1/2)\|x - q\|^2 - (1/2)\|Tx - q\|^2 \\
&\geq (1/2)\|x - Tx\|^2 + (1/2)\|x - q\|^2 - (1/2)\|x - q\|^2. \\
&= (1/2)\|x - Tx\|^2.
\end{aligned}$$

(ii) From (i) we know

$$\begin{aligned}
\|T_\omega x - q\|^2 &= \|((1-\omega)I + \omega T)x - q\|^2 \\
&= \|(x - q) - \omega(x - Tx)\|^2 \\
&= \|x - q\|^2 - 2\omega\langle x - q, x - Tx\rangle + \omega^2\|x - Tx\|^2 \\
&\leq \|x - q\|^2 - 2\omega \times (1/2)\|x - Tx\|^2 + \omega^2\|x - Tx\|^2 \\
&= \|x - q\|^2 - \omega(1-\omega)\|x - Tx\|^2.
\end{aligned}$$

(iii) T_ω is quasi-nonexpansive from (ii); Since P_Ω is nonexpansive, and T_ω is quasi-nonexpansive, we have

$$\|(P_\Omega T_\omega)x - q\| \leq \|(T_\omega)x - q\| \leq \|x - q\| \quad \forall (x, q) \in \Omega \times F(T_\omega),$$

so $P_\Omega T_\omega$ is quasi-nonexpansive.
(iv) First we will prove $F(T) = F(T_\omega)$. For any $x \in F(T)$ that is $Tx = x$, then

$$T_\omega x = [(1-\omega)I + \omega T]x = ((1-\omega)Ix + \omega Tx = x - \omega x + \omega x = x$$

Therefore, $x \in F(T_\omega)$. The converse inclusion can be proved similarly. □

Next we shall prove that $F(T_\omega) = F(P_\Omega T_\omega)$. For any $x \in F(T_\omega) \subset \Omega$ and P_Ω is a metric projection on Ω, so $P_\Omega x = x$. Then we have $(P_\Omega T_\omega)x = P_\Omega(x) = x$, i.e. $x \in F(P_\Omega T_\omega)$, so $F(T_\omega) \subset F(P_\Omega T_\omega)$.

On the other hand we will prove the converse inclusion $F(P_\Omega T_\omega) \subset F(T_\omega)$. Assume that $x \in F(P_\Omega T_\omega)$, i.e. $x = (P_\Omega T_\omega)x$, then for any $q \in F(T_\omega)$ we have

$$\begin{aligned} \|T_\omega x - q\|^2 = &\ \|T_\omega x - x + x - q\|^2 \\ = &\ \|T_\omega x - x\|^2 + 2\langle Tx - x, x - q\rangle + \|x - q\|^2 \\ = &\ \|T_\omega x - x\|^2 + 2\langle Tx - (P_\Omega T_\omega)x, (P_\Omega T_\omega)x - q\rangle + \|x - q\|^2 \\ \geq &\ \|T_\omega x - x\|^2 + \|x - q\|^2. \end{aligned}$$

And for T_ω is quasi-nonexpansive, that is

$$\|T_\omega x - q\|^2 \leq \|x - q\|^2.$$

Combining above yields $\|T_\omega x - x\|^2 = 0$. Therefore, $x \in F(T_\omega)$. This completes the proof.

The next result is of fundamental importance for the techniques of analysis used in this paper. It was established in [3] and its proof is given for the sake of completeness.

Lemma 2.3. (see [3], Lemma 1.3) *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}_{j\geq 0}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 0$. Also consider the sequence of integers $\{\tau(n)\}_{n\geq n_0}$ defined by $\tau(n) = max\{k \leq n \mid \Gamma_k < \Gamma_{k+1}\}$. Then $\{\tau(n)\}_{n\geq n_0}$ is a nondecreasing sequence verifying $\lim_{n\to\infty}\tau(n) = \infty$, and for all $n \geq n_0$, it holds that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ and we have*

$$\Gamma_n \leq \Gamma_{\tau(n)+1}. \tag{2.1}$$

Proof. Clearly, we can see that $(\tau(n))$ is a well-defined sequence, and the fact that it is nondecreasing is obvious as well as $\lim_{n\to\infty}\tau(n)$ $= \infty$ and $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$. Let us prove (2.1). It is easily observed that $\tau(n) \leq n$. Consequently, we prove (2.1) by distinguishing the three cases: $(c1)$ $\tau(n) =$

n; ($c2$) $\tau(n) = n-1$; ($c3$) $\tau(n) < n-1$. In the first case (i.e.,$\tau(n) = n$), (2.1) is immediately given by $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$. In the second case (i.e.,$\tau(n) = n-1$), (2.1) becomes obvious. In the third case, by $\tau(n) = max\{k \leq n \mid \Gamma_k < \Gamma_{k+1}\}$ and for any integer $n \geq n_0$, we easily observe that $\Gamma_j \geq \Gamma_{j+1}$ for $\tau(n)+1 \leq j \leq n-1$, namely

$$\Gamma_{\tau(n)+1} \geq \Gamma_{\tau(n)+2} \geq \cdots \geq \Gamma_{n-1} \geq \Gamma_n,$$

which entails the desired result. □

Lemma 2.4. *Suppose $x^* \in F(T)$ is the unique solution of (1.3), $T : \Omega \to H$ is quasi-nonexpanxive and demi-closed, $\{y_n\} \subset \Omega$ is a bounded sequence such that $\|Ty_n - y_n\| \to 0$, then*

$$\liminf_{n\to\infty} \langle (I-C)x^*, y_n - x^* \rangle \geq 0. \tag{2.2}$$

Proof. Clearly, by $\|Ty_n - y_n\| \to 0$ and T demi-closed, we known that any weak cluster-point of $\{y_n\}$ belongs to $F(T)$. It is also a simple matter to see that there exists $\overline{y}$ and a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $\{y_{n_k}\} \rightharpoonup \overline{y}$ weakly as $k \to \infty$(hence $\overline{y} \in F(T)$) and such that

$$\liminf_{n\to\infty} \langle (I-C)x^*, y_n - x^* \rangle = \lim_{k\to\infty} \langle (I-C)x^*, y_{n_k} - x^* \rangle,$$

by (1.3) obviously leads to

$$\liminf_{n\to\infty} \langle (I-C)x^*, y_n - x^* \rangle = \langle (I-C)x^*, \overline{y} - x^* \rangle \geq 0,$$

that is the desired result. □

3. Main Result

Theorem 3.1. *Let $\{x_n\}$ be the sequence given by (1.4) with T being quasi-nonexpansive and demi-closed on Ω, $\omega \in (0,1]$ and $\{\alpha_n\} \subset (0,1)$ such that*

$$\lim_{n\to\infty} \alpha_n = 0; \sum_n \alpha_n = \infty.$$

Then $\{x_n\}$ converges strongly to the unique element x^* in $F(T)$ verifying

$$x^* = (P_{F(T)}C)x^*,$$

which equivalently solves the following variational inequality problem:

$$x^* \in F(T), \text{ and } \langle (I-C)x^*, q-x^*\rangle \geq 0, \quad \forall q \in F(T).$$

Proof. Let x^* be the solution of $x^* = (P_{F(T)}C)x^*$, From (1.4) we have

$$x_{n+1} - x^* = \alpha_n(Cx_n - x^*) + (1-\alpha_n)[P_\Omega T_\omega x_n - x^*]. \tag{3.1}$$

By (3.1) and since $C : \Omega \to \Omega$ is a contraction with modulus $\rho \in [0,1)$, we have

$$\begin{aligned} & \|x_{n+1} - x^*\|^2 \\ = \ & \alpha_n\langle Cx_n - x^*, x_{n+1} - x^*\rangle + (1-\alpha_n)\langle (P_\Omega T_\omega)x_n - x^*, x_{n+1} - x^*\rangle \\ = \ & \alpha_n\langle Cx_n - Cx^* + Cx^* - x^*, x_{n+1} - x^*\rangle \\ & +(1-\alpha_n)\langle (P_\Omega T_\omega)x_n - x^*, x_{n+1} - x^*\rangle \\ \leq \ & \alpha_n\langle Cx_n - Cx^*, x_{n+1} - x^*\rangle + \alpha_n\langle Cx^* - x^*, x_{n+1} - x^*\rangle \\ & +(1-\alpha_n)\|(P_\Omega T_\omega)x_n - x^*\| \, \|x_{n+1} - x^*\| \\ \leq \ & \alpha_n\rho\|x_n - x^*\| \, \|x_{n+1} - x^*\| + \alpha_n\langle Cx^* - x^*, x_{n+1} - x^*\rangle \\ & +(1-\alpha_n)\|(P_\Omega T_\omega)x_n - x^*\| \, \|x_{n+1} - x^*\| \\ \leq \ & (\alpha_n\rho/2)[\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2] + \alpha_n\langle Cx^* - x^*, x_{n+1} - x^*\rangle \\ & +(1-\alpha_n)/2[\|(P_\Omega T_\omega)x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2] \end{aligned} \tag{3.2}$$

(3.2) can be equivalently rewritten as

$$\begin{aligned} & 2\|x_{n+1} - x^*\|^2 \\ \leq \ & \alpha_n\rho[\|x_n - x^*\|^2 + \|x_{n+1} - x^*\|^2] + 2\alpha_n\langle (C-I)x^*, x_{n+1} - x^*\rangle \\ & +(1-\alpha_n)\|(P_\Omega T_\omega)x_n - x^*\|^2 + (1-\alpha_n)\|x_{n+1} - x^*\|^2 \end{aligned} \tag{3.3}$$

by Lemma 2.2, P_Ω is quasi-nonexpansive and $x^* \in F(T_\omega)$, we have

$$\|(P_\Omega T_\omega)x_n - x^*\|^2 \leq \|T_\omega x_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \omega(1-\omega)\|x_n - Tx_n\|^2,$$

and (3.3) equivalently

$$\begin{aligned} & 2\|x_{n+1} - x^*\|^2 \\ \leq \ & \alpha_n\rho\|x_n - x^*\|^2 + \alpha_n\rho\|x_{n+1} - x^*\|^2 + 2\alpha_n\langle (C-I)x^*, x_{n+1} - x^*\rangle \\ & +(1-\alpha_n)\|x_n - x^*\|^2 - (1-\alpha_n)\omega(1-\omega)\|x_n - Tx_n\|^2 \\ & +(1-\alpha_n)\|x_{n+1} - x^*\|^2 \end{aligned}.$$

then we have

$$\begin{aligned} & (1+(1-\rho)\alpha_n)\|x_{n+1} - x^*\|^2 - (1-(1-\rho)\alpha_n)\|x_n - x^*\|^2 \\ & \leq 2\alpha_n\langle (C-I)x^*, x_{n+1} - x^*\rangle - (1-\alpha_n)\omega(1-\omega)\|x_n - Tx_n\|^2, \end{aligned} \tag{3.4}$$

and (3.4) can be rewriten as

$$\begin{aligned} &[\|x_{n+1}-x^*\|^2-\|x_n-x^*\|^2]+(1-\rho)\alpha_n[\|x_{n+1}-x^*\|^2+\|x_n-x^*\|^2] \\ &\leq 2\alpha_n\langle (C-I)x^*, x_{n+1}-x^*\rangle-(1-\alpha_n)\omega(1-\omega)\|x_n-Tx_n\|^2. \end{aligned} \tag{3.5}$$

Then we shall divide the proof into two parts:

Case 1. Suppose that there exists n_0 such that $\Gamma_n := \|x_n - x^*\|^2,\ n \geq n_0$ is nonincreasing. In this situation, $\{\Gamma_n\}$ is then convergent because it is also nonnegative, so that $\lim_{n\to\infty}(\Gamma_{n+1}-\Gamma_n)=0$; By (3.4) we have

$$\begin{aligned} & \|x_{n+1}-x^*\|^2 \\ \leq\ & (1-(1-\rho)\alpha_n)\|x_n-x^*\|^2+2\alpha_n\langle (C-I)x^*, x_{n+1}-x^*\rangle \\ & -(1-\alpha_n)\omega(1-\omega)\|x_n-Tx_n\|^2 \\ \leq\ & (1-(1-\rho)\alpha_n)\|x_n-x^*\|^2+2\alpha_n\langle (C-I)x^*, x_{n+1}-x^*\rangle. \end{aligned} \tag{3.6}$$

Set $\sigma_n = \max\{0, \langle (C-I)x^*, x_{n+1}-x^*\rangle\}$, then (3.6) can be equivalently rewritten as

$$\|x_{n+1}-x^*\|^2 \leq (1-(1-\rho)\alpha_n)\|x_n-x^*\|^2+2\alpha_n\sigma_n. \tag{3.7}$$

It follows from (3.5) that $\|x_n - Tx_n\| \to 0,\ (n\to\infty)$, by *Lemma*2.4 we have $\liminf_{n\to\infty}\langle (I-C)x^*, x_{n+1}-x^*\rangle \ \geq 0$, then $\lim_n \sigma_n = 0$, thus $\sigma_n\alpha_n = \circ(\alpha_n)$. Consequently $\Gamma_n \to 0, (n\to\infty)$, ([4]), i.e. $x_n \to x^*(n\to\infty)$ strongly.

Case 2. Suppose there exists subsequence $\{\Gamma_{n_k}\}_{k\geq 0}$ of $\{\Gamma_n\}_{n\geq 0}$, such that $\{\Gamma_{n_k}\} \leq \{\Gamma_{n_{k+1}}\},\ \forall k \geq 0$. In this situation, we consider the sequence of indices $\{\tau(n)\}$ as defined in *Lemma* 2.3. It follows that $\Gamma_{\tau(n)+1}-\Gamma_{\tau(n)} > 0$, which from (3.5) amounts to

$$0 \leq (1-\rho)[\|x_{\tau(n)+1}-x^*\|^2+\|x_{\tau(n)}-x^*\|^2] \leq 2\langle (C-I)x^*, x_{\tau(n)+1}-x^*\rangle,$$

similar to Case 1 and from (3.5) and invoking Lemma 2.4 we have

$$\liminf_{n\to\infty}\langle (I-C)x^*, x_{\tau(n)+1}-x^*\rangle \geq 0$$

that $\lim_n \|x_{\tau(n)}-x^*\|^2 = 0$. By *Lemma* 2.3 we have $\lim_n\|x_n-x^*\| = 0$, so $x_n \to x^*$ strongly. □

Theorem 3.2. *Let $\{x_n\}$ be the sequence given by* (1.5) *with T being quasi-nonexpansive and demi-closed on Ω, $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in (0,1), such that*

$$\lim_{n\to\infty}\alpha_n = 0; \sum_n \alpha_n = \infty; \limsup_{n\to\infty}\beta_n < 1$$

Then $\{x_n\}$ converges strongly to the unique element x^* in $F(T)$ verifying

$$x^* = (P_{F(T)}C)x^*,$$

which equivalently solves the following variational inequality problem:

$$x^* \in F(T), \text{ and } \langle (I-C)x^*, q - x^* \rangle \geq 0, \quad \forall q \in F(T).$$

Proof. Let x^* be the solution of (1.3), by the definition of x_n we have

$$x_{n+1} - x_n + (1-\beta_n)\alpha_n(x_n - C(x_n)) = (1-\beta_n)(1-\alpha_n)(P_\Omega T x_n - x_n), \quad (3.8)$$

so

$$\begin{aligned} & \langle x_{n+1} - x_n + (1-\beta_n)\alpha_n(x_n - C(x_n)), x_n - x^* \rangle \\ = \ & -(1-\beta_n)(1-\alpha_n)\langle x_n - P_\Omega T x_n, x_n - x^* \rangle. \end{aligned} \quad (3.9)$$

From $\|P_\Omega T x_n - x^*\|^2 \leq \|x_n - x^*\|^2$, we have

$$\begin{aligned} & 2\langle x_n - P_\Omega T x_n, x_n - x^* \rangle \\ = \ & \|P_\Omega T x_n - x_n\|^2 + \|x_n - x^*\|^2 - \|P_\Omega T x_n - x^*\|^2 \\ \geq \ & \|P_\Omega T x_n - x_n\|^2, \end{aligned} \quad (3.10)$$

together with (3.9) and (3.10) we have

$$\begin{aligned} & -\langle x_n - x_{n+1}, x_n - x^* \rangle \\ \leq \ & -(1-\beta_n)\alpha_n\langle (I-C)x_n, x_n - x^* \rangle - \tfrac{1}{2}(1-\beta_n)(1-\alpha_n) \\ & \|P_\Omega T x_n - x_n\|^2, \end{aligned} \quad (3.11)$$

furthermore

$$-\langle x_n - x_{n+1}, x_n - x^* \rangle = \frac{1}{2}\|x_{n+1} - x^*\|^2 - \frac{1}{2}\|x_n - x^*\|^2 - \frac{1}{2}\|x_n - x_{n+1}\|^2. \quad (3.12)$$

Put $\Gamma_n := \frac{1}{2}\|x_n - x^*\|^2$, then we have

$$\begin{aligned} & \Gamma_{n+1} - \Gamma_n - \tfrac{1}{2}\|x_n - x_{n+1}\|^2 \\ \leq \ & -(1-\beta_n)\alpha_n\langle (I-C)x_n, x_n - x^* \rangle - \tfrac{1}{2}(1-\beta_n)(1-\alpha_n) \\ & \|P_c T x_n - x_n\|^2. \end{aligned} \quad (3.13)$$

Frome (3.8), we have

$$\begin{aligned} & \|x_{n+1} - x_n\|^2 \\ = \ & \|(1-\beta_n)\alpha_n(C(x_n) - x_n) + (1-\beta_n)(1-\alpha_n)(P_\Omega T x_n - x_n)\|^2 \\ \leq \ & (1-\beta_n)^2\alpha_n^2\|(C(x_n) - x_n)\|^2 + (1-\beta_n)^2(1-\alpha_n)^2\|P_\Omega T x_n - x_n\|^2. \end{aligned} \quad (3.14)$$

together with (3.13) and (3.14) we have

$$\Gamma_{n+1}-\Gamma_n+\tfrac{1}{2}(1-\beta_n)(1-\alpha_n)[1-(1-\beta_n)(1-\alpha_n)]\|P_\Omega Tx_n-x_n\|^2$$
$$\le(1-\beta_n)\alpha_n[\tfrac{1}{2}(1-\beta_n)\alpha_n\|C(x_n)-x_n\|^2-\langle(I-C)x_n,x_n-x^*\rangle]. \tag{3.15}$$

Then we shall divide the proof into two parts:

Case 1. Suppose that there exists n_0 such that $\Gamma_n:=\|x_n-x^*\|^2,\ n\ge n_0$ is nonincreasing. In this situation, $\{\Gamma_n\}$ is then convergent because it is also nonnegative, so that $\lim_{n\to\infty}(\Gamma_{n+1}-\Gamma_n)=0$, together with (3.15) we obtain

$$\lim_{n\to\infty}\|P_\Omega Tx_n-x_n\|^2=0. \tag{3.16}$$

From (3.15) again, we obtain

$$(1-\beta_n)\alpha_n[\frac{1}{2}(1-\beta_n)\alpha_n\|C(x_n)-x_n\|^2-\langle(I-C)x_n,x_n-x^*\rangle]\ge\Gamma_{n+1}-\Gamma_n. \tag{3.17}$$

Then by $\sum_n\alpha_n=\infty$, we obviously deduce that

$$\liminf_{n\to\infty}(-\frac{1}{2}(1-\beta_n)\alpha_n\|C(x_n)-x_n\|^2+\langle(I-C)x_n,x_n-x^*\rangle)\le 0$$

or equivalently

$$\liminf_{n\to\infty}(\langle(I-C)x_n,x_n-x^*\rangle)\le 0. \tag{3.18}$$

From $\langle(I-C)x-(I-C)y,x-y\rangle\ge(1-\rho)\|x-y\|^2$ we have

$$\langle(I-C)x_n-(I-C)x^*,x_n-x^*\rangle\ge(1-\rho)\|x_n-x^*\|^2=2(1-\rho)\Gamma_n, \tag{3.19}$$

which by (3.18) entails

$$\liminf_{n\to\infty}(2(1-\rho)\Gamma_n+\langle(I-C)x^*,x_n-x^*\rangle)\le 0, \tag{3.20}$$

hence, recalling that $\lim_{n\to\infty}\Gamma_n$ exists, we have

$$2(1-\rho)\lim_{n\to\infty}\Gamma_n\le-\liminf_{n\to\infty}\langle(I-C)x^*,x_n-x^*\rangle\le 0, \tag{3.21}$$

so $\lim_{n\to\infty}\Gamma_n=0$, i.e., $\{x_n\}$ converges strongly to x^*.

Case 2. Suppose there exists subsequence $\{\Gamma_{n_k}\}_{k\ge0}$ of $\{\Gamma_n\}_{n\ge0}$, such that $\Gamma_{n_k}\le\Gamma_{n_{k+1}},\ \forall k\ge 0$. In this situation, we consider the sequence of indices

$\{\tau(n)\}$ as defined in *Lemma* 2.3. It follows that $\Gamma_{\tau(n)+1} - \Gamma_{\tau(n)} > 0$, which by (3.15) amounts to

$$\begin{aligned} &\tfrac{1}{2}(1-\beta_{\tau(n)})(1-\alpha_{\tau(n)})(1-(1-\beta_{\tau(n)})(1-\alpha_{\tau(n)})\|P_\Omega T x_{\tau(n)} - x_{\tau(n)}\|^2 \\ &\leq \Gamma_{\tau(n)} - \Gamma_{\tau(n)+1} + (1-\beta_{\tau(n)})\alpha_{\tau(n)}[\tfrac{1}{2}(1-\beta_{\tau(n)})\alpha_{\tau(n)} \\ &\|C(x_{\tau(n)}) - x_{\tau(n)}\|^2 - \langle (I-C)x_{\tau(n)}, x_{\tau(n)} - x^*\rangle]. \end{aligned} \tag{3.22}$$

Hence, by the boundedness of $\{x_n\}$ and $\lim_{n\to\infty} \alpha_n \to 0$, we immediately obtain

$$\begin{aligned} \liminf_{\tau(n)\to\infty} \|P_\Omega T x_{\tau(n)} - x_{\tau(n)}\|^2 = 0, \\ \liminf_{\tau(n)\to\infty} \|\Gamma_{\tau(n)} - \Gamma_{\tau(n)+1}\|^2 = 0. \end{aligned} \tag{3.23}$$

Similar to (3.17) – (3.20), we obtain

$$2(1-\rho) \lim_{\tau(n)\to\infty} \Gamma_{\tau(n)} \leq -\liminf_{\tau(n)\to\infty} \langle (I-C)x^*, x_{\tau(n))} - x^*\rangle \leq 0, \tag{3.24}$$

so $\lim_{\tau(n)\to\infty} \|x_{\tau(n)} - x^*\|^2 = 0$. By *Lemma* 2.3 we have $\lim_{n\to\infty} \|x_n - x^*\| = 0$, so $x_n \to x^*$ strongly. □

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