International Journal of Pure and Applied Mathematics
Volume 80 No. 2 2012, 147-160
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

COMPUTATION OF SOME RESULTS INVOLVING MAXIMUM TERMS OF COMPOSITION OF ENTIRE FUNCTIONS

Sanjib Kumar Datta[1 §], Tanmay Biswas[2], Soumen Kanti Deb[3]

[1]Department of Mathematics
University of Kalyani
Kalyani, Dist-Nadia, PIN-741235, West Bengal, INDIA
(Formerly: Department of Mathematics
University of North Bengal
P.O. North Bengal University, Raja Rammohunpur
Dist-Darjeeling, PIN-734013, West Bengal, INDIA)
[2]Rajbari, Rabindrapalli
R.N. Tagore Road, P.O. Krishnagar
Dist. Nadia, PIN-741101, West Bengal, INDIA
[3]Bahin High School
P.O. Bahin, Dist. Uttar Dinajpur, PIN-733157, West Bengal, INDIA

Abstract: In this paper we compare the maximum term of composition of two entire functions with their corresponding left or right factors with order zero.

AMS Subject Classification: 30D30, 30D35
Key Words: order, lower order, entire function, composition, growth rate, maximum term

1. Introduction, Definitions and Notations

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum

Received: February 26, 2012

[§]Correspondence author

term $\mu(r,f)$ of $f=\sum_{n=0}^{\infty}a_n z^n$ on $|z|=r$ is defined by $\mu(r,f)=\max(|a_n|r^n)$. To start our paper we just recall the following definitions.

Definition 1. The order ρ_f and lower order λ_f of an entire function f is defined as

$$\rho_f=\limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log r} \text{ and } \lambda_f=\liminf_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log r}.$$

Since for $0\le r<R$,

$$\mu(r,f)\le M(r,f)\le\frac{R}{R-r}\mu(R,f) \quad \{cf.\ [\mathbf{6}]\ \}$$

it is easy to see that

$$\rho_f=\limsup_{r\to\infty}\frac{\log^{[2]}\mu(r,f)}{\log r} \text{ and } \lambda_f=\liminf_{r\to\infty}\frac{\log^{[2]}\mu(r,f)}{\log r}.$$

If $\rho_f<\infty$ then f is of finite order. Also $\rho_f=0$ means that f is of order zero. In this connection Liao and Yang [4] gave the following definition:

Definition 2. [4]Let f be an entire function of order zero. Then the quantities ρ_f^* and λ_f^* of an entire function f are defined as :

$$\rho_f^*=\limsup_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log^{[2]}r} \text{ and } \lambda_f^*=\liminf_{r\to\infty}\frac{\log^{[2]}M(r,f)}{\log^{[2]}r}.$$

Datta and Biswas [1] gave an alternative definition of zero order and zero lower order of an entire function which is the following:

Definition 3. [1]*Let* f be *an entire function of order zero. Then the quantities* ρ_f^{**} and λ_f^{**} of f *are defined by:*

$$\rho_f^{**}=\limsup_{r\to\infty}\frac{\log M(r,f)}{\log r} \text{ and } \lambda_f^{**}=\liminf_{r\to\infty}\frac{\log M(r,f)}{\log r}.$$

The *growth indicators* ρ_f^{**} and λ_f^{**} for entire f in Definition 3 can be written in terms of its maximum term as

$$\rho_f^{**}=\limsup_{r\to\infty}\frac{\log\mu(r,f)}{\log r} \text{ and } \lambda_f^{**}=\liminf_{r\to\infty}\frac{\log\mu(r,f)}{\log r},$$

which can be proved in the following way :

$$\rho_f^{**} = \limsup_{r\to\infty} \frac{\log M(r,f)}{\log r} \geq \limsup_{r\to\infty} \frac{\log \mu(r,f)}{\log r}$$

and

$$\lambda_f^{**} = \liminf_{r\to\infty} \frac{\log M(r,f)}{\log r} \geq \liminf_{r\to\infty} \frac{\log \mu(r,f)}{\log r}.$$

Now taking $R = 2r$ in the inequality

$$\mu(r,f) \leq M(r,f) \leq \frac{R}{R-r}\mu(R,f) \quad \{cf.\ [\mathbf{6}]\ \}$$

we again get that

$$\begin{aligned}\rho_f^{**} &= \limsup_{r\to\infty} \frac{\log M(r,f)}{\log r} \\ &\leq \limsup_{r\to\infty} \frac{\log 2\mu(2r,f)}{\log 2r} \\ &= \limsup_{r\to\infty} \frac{\log \mu(2r,f) + O(1)}{\log 2r + O(1)} \\ \text{i.e., } \rho_f^{**} &\leq \limsup_{r\to\infty} \frac{\log \mu(r,f)}{\log r}\end{aligned}$$

and

$$\begin{aligned}\lambda_f^{**} &= \liminf_{r\to\infty} \frac{\log M(r,f)}{\log r} \\ &\leq \liminf_{r\to\infty} \frac{\log 2\mu(2r,f)}{\log 2r} \\ &= \liminf_{r\to\infty} \frac{\log \mu(2r,f) + O(1)}{\log 2r + O(1)} \\ \text{i.e., } \lambda_f^{**} &\leq \liminf_{r\to\infty} \frac{\log \mu(r,f)}{\log r}.\end{aligned}$$

Combining the above inequalities we obtain that

$$\rho_f^{**} = \limsup_{r\to\infty} \frac{\log \mu(r,f)}{\log r} \text{ and } \lambda_f^{**} = \liminf_{r\to\infty} \frac{\log \mu(r,f)}{\log r}.$$

In the paper we investigate the comparative growths of maximum term of two entire functions with their corresponding left and right factors with order zero. We do not explain the standard notations and definitions in the theory of entire and meromorphic functions because those are available in [3] and [7].

2. Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1. *[5] Let f and g be two entire functions with $g(0) = 0$. Then for all sufficiently large values of r,*

$$\mu(r, f \circ g) \geq \frac{1}{4}\mu\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right) - |g(0)|, f\right).$$

Lemma 2. *[5] Let f and g be two entire functions. Then for every $\alpha > 0$ and $0 < r < R$,*

$$\mu(r, f \circ g) \leq \frac{\alpha}{\alpha - 1}\mu\left(\frac{\alpha R}{R - r}\mu(R, g), f\right).$$

Lemma 3. *[2] Let f be a meromorphic function and g be transcendental entire. If $\rho_{f \circ g} < \infty$ then $\rho_f = 0$.*

Lemma 4. *[1] Let f be meromorphic and g be entire such that $\rho_f < \infty$ and $\rho_g = 0$.Then $\rho_{f \circ g} < \infty$.*

3. Theorems

In this section we present the main results of the paper.

Theorem 1. *Let f and g be entire functions such that $\rho_{f \circ g} = 0$. Also let $0 < \lambda^{**}_{f \circ g} \leq \rho^{**}_{f \circ g} < \infty$ and $0 < \lambda^{**}_f \leq \rho^{**}_f < \infty$. Then for any positive number A,*

$$\frac{\lambda^{**}_{f \circ g}}{A\rho^{**}_f} \leq \liminf_{r \to \infty} \frac{\log \mu(r, f \circ g)}{\log \mu(r^A, f)} \leq \frac{\lambda^{**}_{f \circ g}}{A\lambda^{**}_f} \leq \limsup_{r \to \infty} \frac{\log \mu(r, f \circ g)}{\log \mu(r^A, f)} \leq \frac{\rho^{**}_{f \circ g}}{A\lambda^{**}_f}.$$

Proof. Since $\rho_{f \circ g} = 0 < \infty$ by Lemma 3, $\rho_f = 0$. Now from the definition of ρ^{**}_f and λ^{**}_f in terms of maximum term we have for arbitrary positive ε and for all sufficiently large values of r,

$$\log \mu(r, f \circ g) \geqslant \left(\lambda^{**}_{f \circ g} - \varepsilon\right) \log r \tag{1}$$

and

$$\log \mu\left(r^A, f\right) \leq A\left(\rho^{**}_f + \varepsilon\right) \log r. \tag{2}$$

Now from (1) and (2) it follows for all sufficiently large values of r,

$$\frac{\log \mu\left(r, f \circ g\right)}{\log \mu\left(r^{A}, f\right)} \geqslant \frac{\lambda_{f \circ g}^{**} - \varepsilon}{A\left(\rho_{f}^{**} + \varepsilon\right)}.$$

As $\varepsilon\,(> 0)$ is arbitrary, we obtain that

$$\liminf_{r \to \infty} \frac{\log \mu\left(r, f \circ g\right)}{\log \mu\left(r^{A}, f\right)} \geqslant \frac{\lambda_{f \circ g}^{**}}{A\rho_{f}^{**}}. \tag{3}$$

Again for a sequence of values of r tending to infinity,

$$\log \mu\left(r, f \circ g\right) \leq \left(\lambda_{f \circ g}^{**} + \varepsilon\right) \log r \tag{4}$$

and for all sufficiently large values of r,

$$\log \mu\left(r^{A}, f\right) \geqslant A\left(\lambda_{f}^{**} - \varepsilon\right) \log r. \tag{5}$$

Combining (4) and (5) we get for a sequence of values of r tending to infinity,

$$\frac{\log \mu\left(r, f \circ g\right)}{\log \mu\left(r^{A}, f\right)} \leq \frac{\left(\lambda_{f \circ g}^{**} + \varepsilon\right)}{A\left(\lambda_{f}^{**} - \varepsilon\right)}.$$

Since $\varepsilon\,(> 0)$ is arbitrary it follows that

$$\liminf_{r \to \infty} \frac{\log \mu\left(r, f \circ g\right)}{\log \mu\left(r^{A}, f\right)} \leq \frac{\lambda_{f \circ g}^{**}}{A\lambda_{f}^{**}}. \tag{6}$$

Also for a sequence of values of r tending to infinity,

$$\log \mu\left(r^{A}, f\right) \leq A\left(\lambda_{f}^{**} + \varepsilon\right) \log r. \tag{7}$$

Now from (1) and (7) we obtain for a sequence of values of r tending to infinity,

$$\frac{\log \mu\left(r, f \circ g\right)}{\log \mu\left(r^{A}, f\right)} \geq \frac{\left(\lambda_{f \circ g}^{**} - \varepsilon\right)}{A\left(\lambda_{f}^{**} + \varepsilon\right)}.$$

As $\varepsilon\,(> 0)$ is arbitrary, we get from above that

$$\limsup_{r \to \infty} \frac{\log \mu\left(r, f \circ g\right)}{\log \mu\left(r^{A}, f\right)} \geq \frac{\lambda_{f \circ g}^{**}}{A\lambda_{f}^{**}}. \tag{8}$$

Also for all sufficiently large values of r,

$$\log \mu (r, f \circ g) \leq \left(\rho^{**}_{f \circ g} + \varepsilon\right) \log r. \tag{9}$$

So from (5) and (9) it follows for all sufficiently large values of r,

$$\frac{\log \mu (r, f \circ g)}{\log \mu (r^A, f)} \leq \frac{\left(\rho^{**}_{f \circ g} + \varepsilon\right)}{A\left(\lambda^{**}_f - \varepsilon\right)}.$$

Since $\varepsilon (> 0)$ is arbitrary we obtain that

$$\limsup_{r \to \infty} \frac{\log \mu (r, f \circ g)}{\log \mu (r^A, f)} \leq \frac{\rho^{**}_{f \circ g}}{A\lambda^{**}_f}. \tag{10}$$

Thus the theorem follows from $(3), (6), (8)$ and (10).

□

Similarly in view of Lemma 4, we may state the following theorem without proof for the right factor g of the composite function $f \circ g$:

Theorem 2. *Let f and g be two entire functions with $\rho_{f \circ g} = 0$. Also let $0 < \lambda^{**}_{f \circ g} \leq \rho^{**}_{f \circ g} < \infty$, $\rho_f < \infty$ and $0 < \lambda^{**}_g \leq \rho^{**}_g < \infty$. Then for any positive number A ,*

$$\frac{\lambda^{**}_{f \circ g}}{A\rho^{**}_g} \leq \liminf_{r \to \infty} \frac{\log \mu (r, f \circ g)}{\log \mu (r^A, g)} \leq \frac{\lambda^{**}_{f \circ g}}{A\lambda^{**}_g} \leq \limsup_{r \to \infty} \frac{\log \mu (r, f \circ g)}{\log \mu (r^A, g)} \leq \frac{\rho^{**}_{f \circ g}}{A\lambda^{**}_g}.$$

Theorem 3. *Let f and g be entire functions such that $\rho_{f \circ g} = 0$. Also let $0 < \lambda^{**}_{f \circ g} \leq \rho^{**}_{f \circ g} < \infty$ and $0 < \rho^{**}_f < \infty$. Then for any positive number A,*

$$\liminf_{r \to \infty} \frac{\log \mu (r, f \circ g)}{\log \mu (r^A, f)} \leq \frac{\rho^{**}_{f \circ g}}{A\rho^{**}_f} \leq \limsup_{r \to \infty} \frac{\log \mu (r, f \circ g)}{\log \mu (r^A, f)}.$$

Proof. In view of Lemma 3, $\rho_{f \circ g} = 0$ implies that $\rho_f = 0$.
From the definition of ρ^{**}_f in terms of maximum term, we get for a sequence of values of r tending to infinity,

$$\log \mu \left(r^A, f\right) \geqslant A\left(\rho^{**}_f - \varepsilon\right) \log r. \tag{11}$$

Now from (9) and (11) it follows for a sequence of values of r tending to infinity,

$$\frac{\log \mu (r, f \circ g)}{\log \mu (r^A, f)} \leq \frac{\rho^{**}_{f \circ g} + \varepsilon}{A\left(\rho^{**}_f - \varepsilon\right)}.$$

As $\varepsilon\,(>0)$ is arbitrary we obtain that

$$\liminf_{r\to\infty}\frac{\log\mu\left(r,f\circ g\right)}{\log\mu\left(r^{A},f\right)}\leq\frac{\rho_{f\circ g}^{**}}{A\rho_{f}^{**}}. \tag{12}$$

Again for a sequence of values of r tending to infinity,

$$\log\mu\left(r,f\circ g\right)\geqslant\left(\rho_{f\circ g}^{**}-\varepsilon\right)\log r. \tag{13}$$

So combining (2) and (13) we get for a sequence of values of r tending to infinity,

$$\frac{\log\mu\left(r,f\circ g\right)}{\log\mu\left(r^{A},f\right)}\geqslant\frac{\rho_{f\circ g}^{**}-\varepsilon}{A\left(\rho_{f}^{**}+\varepsilon\right)}.$$

Since $\varepsilon\,(>0)$ is arbitrary it follows that

$$\limsup_{r\to\infty}\frac{\log\mu\left(r,f\circ g\right)}{\log\mu\left(r^{A},f\right)}\geqslant\frac{\rho_{f\circ g}^{**}}{A\rho_{f}^{**}}. \tag{14}$$

Thus the theorem follows from (12) and (14). □

Theorem 4. *Let f and g be two entire functions such that $\rho_{f\circ g}=0$. Also let $0<\lambda_{f\circ g}^{**}\leq\rho_{f\circ g}^{**}<\infty$, $\rho_{f}<\infty$ and $0<\rho_{g}^{**}<\infty$. Then for any positive number A,*

$$\liminf_{r\to\infty}\frac{\log\mu\left(r,f\circ g\right)}{\log\mu\left(r^{A},g\right)}\leq\frac{\rho_{f\circ g}^{**}}{A\rho_{g}^{**}}\leq\limsup_{r\to\infty}\frac{\log\mu\left(r,f\circ g\right)}{\log\mu\left(r^{A},g\right)}.$$

The proof is omitted.

The following theorem is a natural consequence of Theorem 1 and Theorem 3.

Theorem 5. *Let f and g be entire functions such that $\rho_{f\circ g}=0$. Also let $0<\lambda_{f\circ g}^{**}\leq\rho_{f\circ g}^{**}<\infty$ and $0<\lambda_{f}^{**}\leq\rho_{f}^{**}<\infty$. Then for any positive number A,*

$$\begin{aligned}\liminf_{r\to\infty}\frac{\log\mu\left(r,f\circ g\right)}{\log\mu\left(r^{A},f\right)}&\leq\min\left\{\frac{\lambda_{f\circ g}^{**}}{A\lambda_{f}^{**}},\frac{\rho_{f\circ g}^{**}}{A\rho_{f}^{**}}\right\}\\&\leq\max\left\{\frac{\lambda_{f\circ g}^{**}}{A\lambda_{f}^{**}},\frac{\rho_{f\circ g}^{**}}{A\rho_{f}^{**}}\right\}\leq\limsup_{r\to\infty}\frac{\log\mu\left(r,f\circ g\right)}{\log\mu\left(r^{A},f\right)}.\end{aligned}$$

The proof is omitted.

Combining Theorem 2 and Theorem 4 we may state the following theorem.

Theorem 6. *Let f and g be entire functions with $\rho_{f\circ g} = 0$. Also let $0 < \lambda^{**}_{f\circ g} \le \rho^{**}_{f\circ g} < \infty$, $\rho_f < \infty$ and $0 < \lambda^{**}_g \le \rho^{**}_g < \infty$. Then for any positive number A,*

$$\liminf_{r\to\infty} \frac{\log \mu(r, f\circ g)}{\log \mu(r^A, g)} \le \min\left\{\frac{\lambda^{**}_{f\circ g}}{A\lambda^{**}_g}, \frac{\rho^{**}_{f\circ g}}{A\rho^{**}_g}\right\}$$
$$\le \max\left\{\frac{\lambda^{**}_{f\circ g}}{A\lambda^{**}_g}, \frac{\rho^{**}_{f\circ g}}{A\rho^{**}_g}\right\} \le \limsup_{r\to\infty} \frac{\log \mu(r, f\circ g)}{\log \mu(r^A, g)}.$$

Theorem 7. *Let f and g be two entire functions such that g is of order zero and $0 < \lambda_f \le \rho_f < \infty$ and $0 < \lambda^{**}_g \le \rho^{**}_g < \infty$. Then for any positive number A,*

$$\limsup_{r\to\infty} \frac{\log^{[2]} \mu(r, f\circ g)}{\log^{[2]} \mu(r^A, f)} \ge \max\left\{\frac{\lambda^{**}_g}{A}, \frac{\lambda_f \rho^{**}_g}{A\rho_f}\right\}.$$

Proof. Let us choose $\varepsilon\,(> 0)$ in such a way that

$$0 < \varepsilon < \min\left\{\rho_f, \lambda^{**}_g, \rho^{**}_g\right\}.$$

Now we obtain from Lemma 1 for a sequence of values of r tending to infinity,

$$\log^{[2]} \mu(r, f\circ g) \ge \log^{[2]}\left\{\frac{1}{4}\mu\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right), f\right)\right\}$$
$$\text{i.e., } \log^{[2]} \mu(r, f\circ g) \ge (\rho_f - \varepsilon)\log\left\{\frac{1}{8}\mu\left(\frac{r}{4}, g\right)\right\} + O(1)$$
$$\text{i.e., } \log^{[2]} \mu(r, f\circ g) \ge (\rho_f - \varepsilon)\log\frac{1}{8} + (\rho_f - \varepsilon)\log\mu\left(\frac{r}{4}, g\right) + O(1)$$
$$\text{i.e., } \log^{[2]} \mu(r, f\circ g) \ge (\rho_f - \varepsilon)\left(\lambda^{**}_g - \varepsilon\right)\log\frac{r}{4} + O(1)$$
$$\text{i.e., } \log^{[2]} \mu(r, f\circ g) \ge (\rho_f - \varepsilon)\left(\lambda^{**}_g - \varepsilon\right)\log r + O(1). \quad (15)$$

Also for all sufficiently large values of r,

$$\log^{[2]} \mu\left(r^A, f\right) \le (\rho_f + \varepsilon) A\log r. \quad (16)$$

So from (15) and (16) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} \mu(r, f\circ g)}{\log^{[2]} \mu(r^A, f)} \ge \frac{(\rho_f - \varepsilon)\left(\lambda^{**}_g - \varepsilon\right)\log r + O(1)}{(\rho_f + \varepsilon) A\log r}$$

$$\text{i.e. } \limsup_{r\to\infty} \frac{\log^{[2]} \mu(r, f\circ g)}{\log^{[2]} \mu(r^A, f)} \geq \frac{\lambda_g^{**}}{A}. \tag{17}$$

Also for a sequence of values of r tending to infinity it follows from Lemma1 that

$$\begin{aligned}
\log^{[2]} \mu(r, f\circ g) &\geq \log^{[2]} \left\{\frac{1}{4}\mu\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right), f\right)\right\} \\
\text{i.e., } \log^{[2]} \mu(r, f\circ g) &\geq (\lambda_f - \varepsilon)\log\left\{\frac{1}{8}\mu\left(\frac{r}{4}, g\right)\right\} + O(1) \\
\text{i.e., } \log^{[2]} \mu(r, f\circ g) &\geq (\lambda_f - \varepsilon)\log\frac{1}{8} + (\lambda_f - \varepsilon)\log\mu\left(\frac{r}{4}, g\right) + O(1) \\
\text{i.e., } \log^{[2]} \mu(r, f\circ g) &\geq (\lambda_f - \varepsilon)\left(\rho_g^{**} - \varepsilon\right)\log\frac{r}{4} + O(1) \\
\text{i.e., } \log^{[2]} \mu(r, f\circ g) &\geq (\lambda_f - \varepsilon)\left(\rho_g^{**} - \varepsilon\right)\log r + O(1). \qquad (18)
\end{aligned}$$

Now from (16) and (18) it follows for a sequence of values of r tending to infinity,

$$\frac{\log^{[2]} \mu(r, f\circ g)}{\log^{[2]} \mu(r^A, f)} \geq \frac{(\lambda_f - \varepsilon)\left(\rho_g^{**} - \varepsilon\right)\log r + O(1)}{(\rho_f + \varepsilon) A\log r}$$

$$\text{i.e. } \limsup_{r\to\infty} \frac{\log^{[2]} \mu(r, f\circ g)}{\log^{[2]} \mu(r^A, f)} \geq \frac{\lambda_f \rho_g^{**}}{A\rho_f}. \tag{19}$$

Thus the theorem follows from (17) and (19). □

In the line of Theorem 7 one can easily prove the following theorem.

Theorem 8. *Let f and g be two entire functions such that g is of order zero and $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g^{**} \leq \rho_g^{**} < \infty$. Then for any positive number A,*

$$\limsup_{r\to\infty} \frac{\log^{[2]} \mu(r, f\circ g)}{\log \mu(r^A, g)} \geq \max\left\{\frac{\lambda_f}{A}, \frac{\rho_f \lambda_g^{**}}{A\rho_g^{**}}\right\}.$$

The proof is omitted.

Theorem 9. *Let f and g be two entire functions such that g is of order zero and $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g^{**} \leq \rho_g^{**} < \infty$. Then for any positive number A,*

$$\limsup_{r\to\infty} \frac{\log^{[2]} \mu(r, f\circ g)}{\log^{[2]} \mu(r^A, f)} \leq \frac{\rho_f \rho_g^{**}}{A\lambda_f}.$$

Proof. By Lemma 2 it follows for all sufficiently large values of r that

$$\begin{aligned}
\log^{[2]} \mu(r, f \circ g) &\leq \log^{[2]} \mu\left(\frac{\alpha R}{R-r}\mu(R, g), f\right) + O(1) \\
\text{i.e.,}\ \log^{[2]} \mu(r, f \circ g) &\leq (\rho_f + \varepsilon) \log\left(\frac{\alpha R}{R-r}\mu(R, g)\right) + O(1) \\
\text{i.e.,}\ \log^{[2]} \mu(r, f \circ g) &\leq (\rho_f + \varepsilon) \log \mu(R, g) + O(1) \\
\text{i.e.,}\ \log^{[2]} \mu(r, f \circ g) &\leq (\rho_f + \varepsilon)\left(\rho_g^{**} + \varepsilon\right) \log R + O(1). \qquad (20)
\end{aligned}$$

Also for all sufficiently large values of r,

$$\log^{[2]} \mu\left(r^A, f\right) \geq (\lambda_f - \varepsilon) A \log r. \qquad (21)$$

Now from (20) and (21) we obtain for all sufficiently large values of r,

$$\begin{aligned}
\frac{\log^{[2]} \mu(r, f \circ g)}{\log^{[2]} \mu(r^A, f)} &\leq \frac{(\rho_f + \varepsilon)\left(\rho_g^{**} + \varepsilon\right) \log R + O(1)}{(\lambda_f - \varepsilon) A \log r} \\
\text{i.e.}\ \limsup_{r\to\infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log^{[2]} \mu(r^A, f)} &\leq \frac{\rho_f \rho_g^{**}}{A\lambda_f}.
\end{aligned}$$

This complets the proof. □

Theorem 10. *Let f and g be two entire functions such that g is of order zero and $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g^{**} \leq \rho_g^{**} < \infty$. Then for any positive number A,*

$$\limsup_{r\to\infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log \mu(r^A, g)} \leq \frac{\rho_f \rho_g^{**}}{A\lambda_g^{**}}.$$

We omit the proof of the theorem because it can be carried out in the line of Theorem 9.

The following theorem is a natural consequence of Theorem 7 and Theorem 9.

Theorem 11. *Let f and g be two entire functions such that g is of order zero and $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g^{**} \leq \rho_g^{**} < \infty$. Then for any positive number A,*

$$\max\left\{\frac{\lambda_g^{**}}{A}, \frac{\lambda_f \rho_g^{**}}{A\rho_f}\right\} \leq \limsup_{r\to\infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log^{[2]} \mu(r^A, f)} \leq \frac{\rho_f \rho_g^{**}}{A\lambda_f}.$$

Combining Theorem 8 and Theorem 10 we may state the following theorem.

Theorem 12. *Let f and g be two entire functions such that g is of order zero and $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g^{**} \leq \rho_g^{**} < \infty$. Then for any positive number A,*

$$\max\left\{\frac{\lambda_f}{A}, \frac{\rho_f \lambda_g^{**}}{A\rho_g^{**}}\right\} \leq \limsup_{r\to\infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log \mu(r^A, g)} \leq \frac{\rho_f \rho_g^{**}}{A\lambda_g^{**}}.$$

Theorem 13. *Let f and g be two entire functions such that g is of order zero and $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g^{**} \leq \rho_g^{**} < \infty$ Then for any positive number A,*

$$\liminf_{r\to\infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log^{[2]} \mu(r^A, f)} \geq \frac{\lambda_f \lambda_g^{**}}{A\rho_f}.$$

Proof. For all sufficiently large values of r we get from Lemma 1 that

$$\begin{aligned}
\log^{[2]} \mu(r, f \circ g) &\geq \log^{[2]} \left\{\frac{1}{4}\mu\left(\frac{1}{8}\mu\left(\frac{r}{4}, g\right), f\right)\right\} \\
\text{i.e., } \log^{[2]} \mu(r, f \circ g) &\geq (\lambda_f - \varepsilon) \log\left\{\frac{1}{8}\mu\left(\frac{r}{4}, g\right)\right\} + O(1) \\
\text{i.e., } \log^{[2]} \mu(r, f \circ g) &\geq (\lambda_f - \varepsilon) \log\frac{1}{8} + (\lambda_f - \varepsilon) \log \mu\left(\frac{r}{4}, g\right) + O(1) \\
\text{i.e., } \log^{[2]} \mu(r, f \circ g) &\geq (\lambda_f - \varepsilon)\left(\lambda_g^{**} - \varepsilon\right) \log\frac{r}{4} + O(1) \\
\text{i.e., } \log^{[2]} \mu(r, f \circ g) &\geq (\lambda_f - \varepsilon)\left(\lambda_g^{**} - \varepsilon\right) \log r + O(1) \qquad (22)
\end{aligned}$$

where we choose

$$0 < \varepsilon < \min\{\lambda_g, \rho_g\}.$$

Combining (16) and (22) it follows for all sufficiently large values of r,

$$\begin{aligned}
\frac{\log^{[2]} \mu(r, f \circ g)}{\log^{[2]} \mu(r^A, f)} &\geq \frac{(\lambda_f - \varepsilon)\left(\lambda_g^{**} - \varepsilon\right) \log r + O(1)}{(\rho_f + \varepsilon) A \log r} \\
\text{i.e., } \liminf_{r\to\infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log^{[2]} \mu(r^A, f)} &\geq \frac{\lambda_f \lambda_g^{**}}{A\rho_f}.
\end{aligned}$$

Thus the theorem follows. □

In the line of Theorem 13 one can easily prove the following theorem.

Theorem 14. *Let f and g be two entire functions such that g is of order zero and $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g^{**} \leq \rho_g^{**} < \infty$. Then for any positive number A,*

$$\liminf_{r\to\infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log \mu(r^A, g)} \geq \frac{\lambda_f . \lambda_g^{**}}{A . \rho_g^{**}}.$$

Theorem 15. *Let f and g be two entire functions such that g is of order zero and $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g^{**} \leq \rho_g^{**} < \infty$. Then for any positive number A,*

$$\liminf_{r\to\infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log^{[2]} \mu(r^A, f)} \leq \min\left\{\frac{\rho_g^{**}}{A}, \frac{\rho_f \lambda_g^{**}}{A\lambda_f}\right\}.$$

Proof. By Lemma 2 we have for a sequence of values of r tending to infinity,

$$\begin{aligned} \log^{[2]} \mu(r, f \circ g) &\leq \log^{[2]} \mu\left(\frac{\alpha R}{R-r}\mu(R,g), f\right) + O(1) \\ \text{i.e., } \log^{[2]} \mu(r, f \circ g) &\leq (\lambda_f + \varepsilon)\log\left(\frac{\alpha R}{R-r}\mu(R,g)\right) + O(1) \\ \text{i.e., } \log^{[2]} \mu(r, f \circ g) &\leq (\lambda_f + \varepsilon)\log \mu(R,g) + O(1) \\ \text{i.e., } \log^{[2]} \mu(r, f \circ g) &\leq (\lambda_f + \varepsilon)\left(\rho_g^{**} + \varepsilon\right)\log R + O(1). \qquad (23) \end{aligned}$$

Now from (21) and (23) we obtain for sequence of values of r tending to infinity,

$$\begin{aligned} \frac{\log^{[2]} \mu(r, f \circ g)}{\log^{[2]} \mu(r^A, f)} &\leq \frac{(\lambda_f + \varepsilon)\left(\rho_g^{**} + \varepsilon\right)\log R + O(1)}{(\lambda_f - \varepsilon) A \log r} \\ \text{i.e. } \liminf_{r\to\infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log^{[2]} \mu(r^A, f)} &\leq \frac{\rho_g^{**}}{A}. \qquad (24) \end{aligned}$$

Also for a sequence of values of r tending to infinity it follows from Lemma 2 that

$$\begin{aligned} \log^{[2]} \mu(r, f \circ g) &\leq \log^{[2]} \mu\left(\frac{\alpha R}{R-r}\mu(R,g), f\right) + O(1) \\ \text{i.e., } \log^{[2]} \mu(r, f \circ g) &\leq (\rho_f + \varepsilon)\log\left(\frac{\alpha R}{R-r}\mu(R,g)\right) + O(1) \\ \text{i.e., } \log^{[2]} \mu(r, f \circ g) &\leq (\rho_f + \varepsilon)\log \mu(R,g) + O(1) \\ \text{i.e., } \log^{[2]} \mu(r, f \circ g) &\leq (\rho_f + \varepsilon)\left(\lambda_g^{**} + \varepsilon\right)\log R + O(1). \qquad (25) \end{aligned}$$

Therefore from (21) and (25) it follows for sequence of values of r tending to infinity that,

$$\begin{aligned} \frac{\log^{[2]} \mu(r, f \circ g)}{\log^{[2]} \mu(r^A, f)} &\leq \frac{(\rho_f + \varepsilon)\left(\lambda_g^{**} + \varepsilon\right)\log R + O(1)}{(\lambda_f - \varepsilon) A \log r} \\ \text{i.e. } \liminf_{r\to\infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log^{[2]} \mu(r^A, f)} &\leq \frac{\rho_f \lambda_g^{**}}{A\lambda_f}. \qquad (26) \end{aligned}$$

Thus the theorem follows from (24) and (26). □

In the line of Theorem 15 one can easily prove the following theorem.

Theorem 16. *Let f and g be two entire functions such that g is of order zero and $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g^{**} \leq \rho_g^{**} < \infty$. Then for any positive number A,*

$$\liminf_{r\to\infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log \mu(r^A, g)} \leq \min \left\{ \frac{\lambda_f \rho_g^{**}}{A\lambda_g^{**}}, \frac{\rho_f}{A} \right\}.$$

The proof is omitted.

Combining Theorem 13 and Theorem 15 we may state the following theorem without proof.

Theorem 17. *Let f and g be two entire functions such that g is of order zero and $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g^{**} \leq \rho_g^{**} < \infty$. Then for any positive number A,*

$$\frac{\lambda_f \rho_g^{**}}{A\rho_f} \leq \liminf_{r\to\infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log^{[2]} \mu(r^A, f)} \leq \min \left\{ \frac{\rho_g^{**}}{A}, \frac{\rho_f \lambda_g^{**}}{A\lambda_f} \right\}.$$

Similarly the following theorem is a natural consequence of Theorem 14 and Theorem 16.

Theorem 18. *Let f and g be two entire functions such that g is of order zero and $0 < \lambda_f \leq \rho_f < \infty$ and $0 < \lambda_g^{**} \leq \rho_g^{**} < \infty$. Then for any positive number A,*

$$\frac{\lambda_f . \lambda_g^{**}}{A . \rho_g^{**}} \leq \liminf_{r\to\infty} \frac{\log^{[2]} \mu(r, f \circ g)}{\log \mu(r^A, g)} \leq \min \left\{ \frac{\lambda_f \rho_g^{**}}{A\lambda_g^{**}}, \frac{\rho_f}{A} \right\}.$$

References

[1] S.K. Datta, T. Biswas, On the definition of a meromorphic function of order zero, *Int. Math. Forum*, **4**, No. 37 (2009), 1851-1861.

[2] F. Gross, *Factorization of Meromorphic Functions*, U.S. Government Printing Office, Washington, D.C. (1972).

[3] W.K. Hayman, *Meromorphic Functions*, The Clarendon Press, Oxford (1964).

[4] L. Liao, C.C. Yang, On the growth of composite entire functions, *Yokohama Math. J.*, **46** (1999), 97-107.

[5] A.P. Singh, On maximum term of composition of entire functions, *Proc. Nat. Acad. Sci. India*, **59(A)**, Part I (1989), 103-115.

[6] A.P. Singh, M.S. Baloria, On maximum modulus and maximum term of composition of entire functions, *Indian J. Pure Appl. Math.*, **22**, No. 12 (1991), 1019-1026.

[7] G. Valiron, *Lectures on the General Theory of Integral Functions*, Chelsea Publishing Company (1949).