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TOPOLOGY AND VALUATION RINGS GENERATED ON A SKEW FIELD BY A G-VALUATION

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Abstract: The basic idea in this paper is the existence of torsion in the value group of a generalized valuation defined on a skew field and named G-valuation. The existence of torsion causes a chaos in several parts of the whole theory and our attempt is to handle this disorder. It is interesting that the construction of the G-value group is based on the completion of a partially ordered group on the Kurepa's completion, which differs a little from the MacNeille's completion to which is based a classical valuation, for instance a semi-valuation. Next we examine whether basic conclusions for the topology on a valuated field are in valid here. Finally we study the "valuation rings" defined by a G-valuation

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1. Introduction

The main subject of this paper is the study of a skew field endowed with a topology defined by a generalized valuation. This generalized valuation has been exposed to some papers of the authors, for instance in [10] and [11], under the name of a *G-valuation*. The first characteristic of a G-valuation is that its value group has torsion. Every classical valuation has value group which is

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torsion free. This characteristic property of the value group of a G-valuation, causes a chaos and it is our task to handle this disorder and our target is to study what remains in valid from the classical results concerning topology. This is the object of the third paragraph. A second characteristic of a G-valuation is the relation which it has with the *completion of a partially ordered (p.o.) set, the Kurepa's completion.* The classical valuations are related with the well-known *MacNeille's completion.* The relative results with completions, are given below.

1.1. Given a p.o. structure $(E, \leq)$ ($\leq$ the order relation and E the carrier relation) and a subset M of E we put $x \leq M$ (respectively $x < M$) for all $x \in E$ which are greater (resp. smaller) than all the elements of M. We put $M^+ = \{y \in E | M < y\}$ and respectively $M^- = \{x \in E | x < M\}$.

A couple (A, B) of E's subsets is called *MacNeille's cut* iff: (1) $A \cap B = \emptyset$, (2) *there is no element of E smaller than all the elements of B which does not belong to A*, (3) *there is no element of E greater than all the elements of A which does not belong to B*. The subset A is said to be *the lower class of the cut*, the subset B *the upper class.* For $M \subseteq E$, $M \neq E$ we form the cut as follows: we take M^+, after that, the subset $(M^+)^-$ and the cut is the couple $((M^+)^-, M^+)$. Of course, for a given $M \subset E$, we can take firstly the subset M^-, after that the subset $(M^-)^+$ etc. We will, always, use the first case, that is the cut $((M^+)^-, M^+)$. Moreover, if there is an element e of E which is maximal of the class $(M^+)^-$ and minimal class of the class M^+, then we identify the two cuts with $e \in E$, itself. Thus, the MacNeille's completion is

$$\hat{E} = E \cup \Lambda(E),$$

where $\Lambda(E)$ is the set of the cuts without end points neither to the *lower* nor to the *upper* class. These latter cuts are called gaps.

If $\leq$ is the ordering in E, we extend it, (in brief), with the same symbol, in the set $\hat{E}$ of the MacNeille's completion as follows:

For $x \in E, y = (A, B) \in \Lambda(E),$ it is $x < y$ iff $x \in A,\ x > y$ iff $x \in B.$
For $(A, B) \notin E, (C, D) \notin E:\ (A, B) \leq (C, D)$ iff $A \subseteq C.$

The characteristic property of a valuation v on a field is the *triangular property*, which makes the valuations *non-archimedean* and gives the most of the properties in valuated fields. It says:

$$\text{“}\textit{If } v(x) \geq \gamma, v(y) \geq \gamma, \textit{ then } v(x+y) \geq \gamma\text{”}, \tag{1}$$

$\gamma \in G_+ = \{g | g \geq 0\}, x, y \in K$. It has been proved (cf. for instance [6]), the equivalence of (1) with the expression:

$$v(x+y) \geq \min\{v(x), v(y)\}. \tag{2}$$

If the value group is a p.o. group, then the minimum becomes infimum and thus the type (1) changes into the relation:

$$v(x+y) \geq \inf_{\hat{E}}\{v(x), v(y)\} \quad (\textit{the ultrametric property})$$

Besides, it has been proved that the MacNeille's completion is a $\wedge$-semi-lattice. So, the relation (2) expresses the contemporary type of a valuation.

In an irrelevant time, J. Ohm wrote in [14]:

"*If* $\alpha_0, \alpha_2, \ldots, \alpha_n$ *are elements of a p.o. set* E*, we define the expression*

$$\alpha_0 \geq \inf_{\hat{E}}\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$$

if and only if $\alpha_0 \geq \alpha$ *for all* $\alpha \in E$ *such that*

$$\alpha \leq \alpha_1, \alpha \leq \alpha_2, \ldots, \alpha \leq \alpha_n."$$

The $\inf_{\hat{E}}$ means the infimum into the MacNeille's completion of E and so, we get the equivalence of (1) and (2).

1.2. M. Krasner in [12] introduced the "*semireal numbers*" (*nombres semiriées*) imitating G. Kurepa, who in his *Thèse de Doctorat*, (Paris, 1935), has achieved in the same way a completion of a totally ordered group. Krasner used systematically the *semireal numbers* in many subjects (cf. [12], [13]). In the Krasner's consideration every real number, say e, is said to be of the *kind* 0, writing $e = e^0$, but e may be as well of the *kind* $+$, symbolized by e^+, as if the number e has a limit from the right, and similarly it may have a *kind* $-$, symbolized by e^-, as if e has a limit from the left. So, every real number e in the set of *semireal numbers* is substituted by three numbers, e^0, e^+, e^-. L. Docas and N. Varouchakis (cf. for instance in [7] and [16]) also gave a generalization of the *Kurepa's completion*. G. Burns (in [4]) disagreed with some aspects of these generalizations, mainly with some identifications in the *Kurepa's completion* and so it is stopped the discussion on the subject.

Because of this disagreement on the *Kurepa's completion*, we give in brief a description of this completion.

We start again from a p.o. structure $(E, \leq)$. The *Kurepa's completion* $(E_{ku}, \overline{\leq})$ is formed as follows:

The carrier E_{Ku} is constituted from three kinds of elements: a) from the elements of E itself, b) from the upper classes of the MacNeille's cuts which are *gaps* and c) from the elements of the set

$$(\leftarrow, e]^{+} \tag{3}$$

where $(\leftarrow, e]$ is the set of all the elements of E which are smaller than or equal to e and $e \in E$. If $\bar{e} \in E_{Ku}$ then it is an upper class of a *gap* or it is the upper class of a MacNeille's cut of the form (3) or it is an element of E. In all the cases, we may think as $\bar{e} = B_{\bar{e}}$.

The ordering $\overline{\leq}$ is defined by the definition 1.2 below (in brief again as the MacNeille's one). In the following definition, if $\overline{x}, \overline{y} \in E_{Ku}$, we may thing them as $\overline{x} = B_{\overline{x}}$, $\overline{y} = B_{\overline{y}}$. (Later we will change the symbol $\overline{\leq}$ into the initial symbol $\leq$.

Definition 1.1. We take $\overline{x} \overline{\leq} \overline{y}$ if and only if:

(a) $\overline{x} \in E$, $\overline{y} \in E$ and $\overline{x} < \overline{y}$ in E.

(b) $\overline{x} \in E$, $\overline{y} = B_{\overline{y}}$ and $B_{\overline{y}} \subset B_{\overline{x}}$ with $B_{\overline{x}} \neq B_{\overline{y}}$.

(c) $\overline{x} = B_{\overline{x}}$, $\overline{y} \in E$ and $\overline{y} \in B_{\overline{x}}$.

(d) $\overline{x} = B_{\overline{x}}$, $\overline{y} = B_{\overline{y}}$ and $B_{\overline{y}} \subset B_{\overline{x}}$ with $B_{\overline{x}} \neq B_{\overline{y}}$.

(e) $\overline{x} = B_{\overline{x}}$ and $\overline{y} \in B_{\overline{x}}$.

Remark 1.1.

(1) According to the above relation (e), if $B_{\overline{x}}$ has a minimum element $\overline{y}$, it is $\overline{x} < \overline{y}$.

(2) From now on, we write $\leq$ instead of $\overline{\leq}$ (as well as $<$ instead of $\overline{<}$).

(3) The order in the *MacNeille's* and in the *Kurepa's* completions are extensions of the given order of E.

(4) It has been proved that the *Kurepa's completion* of an ordered space is a $\wedge$-semi-lattice.

In the *Kurepa's completion* the triangular property of a valuation means the following:

$$\text{“ } \textit{If } w(x) > \gamma, w(y) > \gamma, \textit{ then } w(x+y) > \gamma\text{”} \tag{4}$$

$\gamma \in G_+$, $x, y \in K$, w the valuation. On the other hand, the above J. Ohm's writings (cf. the Section 1.1), may take the form:

"*If* $\alpha_0, \alpha_2, \ldots, \alpha_n$ *are elements of a p.o. set* E, *we define the expression*

$$\alpha_0 > \inf_{\overline{E}}\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$$

if and only if $\alpha_0 > \alpha$ *for all* $\alpha \in E$ *such that*

$$\alpha < \alpha_1, \alpha < \alpha_2, \ldots, \alpha < \alpha_n.\text{”}$$

The symbol $\inf_{\overline{E}}$ means the infimum into the *Kurepa's completion.* This notation confirms the equivalence of the relation (4)with the relation:

$$w(x+y) > \inf_{\overline{w(K)}}\{w(x), w(y)\} \quad (\textit{the ultrametric property}) \tag{5}$$

In the paragraph 2 we will give the typical proof of the equivalence of the relations (4) and (5).

1.3. We start with a *right Ore domain* R: It means that R is a non-commutative ring and there is a subset S of R, which is multiplicatively closed and contains 1, that is $1 \in S$ and if $a, b \in S$, then $a.b \in S$. The set $R^* = R \setminus \{0\}$ is non-void, there holds

$$s.R^* \cap \rho.S \neq \{0\},$$

for a couple, at least, of elements $\rho \in R^*$, $s \in S$, and if $s.r = 0$, $s \in S$, $r \in R^*$, then $r.t = 0$, for some $t \in S$ (*Ore condition*). Then there exists a field K, *the field of fractions of* R. We introduce in R an evident equivalence and we embed R into K by the natural homomorphism. We put $K^* = K \setminus \{0\}$ and we, also, suppose that there are not zero divisors.

1.4. The well-known construction of a *semi-valuation* or a *Krull valuation*, in the commutative case, begins with an integral domain R, its quotient field K, the set R^* of non zero elements of R and with the multiplicative group $U(R)$ of the units of R. Then, a preorder is defined on K^* by taking:

$$(K^*)^+ = R^*.$$

The *canonical map* (*the natural homomorphism*)

$$w : K^* \to K^*/U(R)$$

of K^* onto the associated ordered (from the above preorder $(*)$) group $K^*/U(R)$, written additively, is a *semi-valuation* or a *Krull valuation.*

1.5. Let R be a right Ore domain and, as always, $U(R)$ be the multiplicative group of its units and K the field of fractions of R. A valuation w, is a homomorphism of K^* onto an ordered group. Since $1 \in R$, it implies that $w(1) = w(1) + w(1)$, i.e. $w(1) = 0$.

If for instance $x^n = 1$, $x \in K$, $n > 1$, then $w(x^n) = nw(x) = 0$, hence, if $w(x)$ has order different of n, is equal to 0. This fact is a reason for the kernel of w to be zero. But, on the other hand, if the points have *infinite order*, the kernel of w is not obligatory zero. So, we say that the value group is in general a mixed p.o. group, that is the group may have *torsion.*

We suppose now that the value group of a G-valuation has torsion, that is the multiplicative group $U(R)$ has not w-image 0. Then, any element $x \in R \backslash U(R)$ has a positive image; in fact, if x, y are in $U(R)$ with $w(x) > w(y)$, then it must $xy^{-1} \in R \setminus U(R)$, impossible, since $U(R)$ is a multiplicative group. It means that $w(x), w(y)$ are parallel, that is they are incomparable, one another. Thus, it is possible a subgroup of $U(R)$, say C_0, to have w-image the 0 and all the other elements of $U(R)$ to have w-images points parallel to 0 and one another. If, in addition, we suppose that the value group is *splitting*, then it gets the form

$$G = G_0 \oplus \Gamma,$$

where G_0 is a subgroup of G whose all the elements are parallel to zero and one another and Γ is a subgroup of representatives of the quotient group G/G_0. We will preserve this expression of the value group through all the paper.

It is possible for one to thing that G_0 reduces to one only point, the zero, in which case the G-valuation coincides with the *semi-valuation.*

1.6. In conclusion, three points are the crucial ones in the present paper: First, the *Kurepa's completion* which forms the value group of a G-valuation. We refer to a G-valuation to the first two paragraphs. Second, the fact that by a G-valuation they are valid some classical topological results which hold in a classical valuation. It is the object of study in the paragraph 3. Finally, we give in the paragraph 4, an alternative presentation of the terms *topological ring* and of *valuation ring* via a valuation.

We recall three more modifications.

(a) A subring A of a skew field K is said to be *total*, if for each $\alpha \in K^*$, either $\alpha \in A$ or $\alpha^{-1} \in A$. A subring A of a field K is said to be invariant if $c^{-1}Ac \subseteq A$, for any c in K. An *invariant* and *total* subring is called valuation *ring* of the field.

(b) A valuation on a skew field is called *abelian*, if its *value group* is abelian.

(c) The Wedderburn's theorem on finite fields (cf. [5], [6]) states that every finite field is commutative. Another relative statement is due to Jacobson (cf. [6]): a field K whose K^* is periodic, is commutative.

2. G-Valuation on a Skew Field

In the next paragraphs, without any reminding, we consider a *right Ore domain* R, its field of fractions K and $U(R)$ the multiplicative group of the units of R. We, also, suppose that there are not zero divisors. In general, the suppositions are as we have described them in Sections 1.3-1.5.

The definition of a classical valuation has as follows (cf. [6]):

Definition 2.1. Let R be a right Ore domain and $\overline{G} = G \cup \{\infty\}$, where G is an abelian totally ordered group and ∞ is an element attached to G, greater than all the $\alpha \in G$ and such that $\alpha + \infty = \infty + \alpha = \infty + \infty = \infty$. A valuation v is a map from the field K of fractions of R on the set $\overline{G}$, provided that for every x, y in R, the following hold:

(v_1) $v(x.y) = v(x) + v(y)$.

(v_2) If $v(x) \geq \gamma$ and $v(y) \geq \gamma$, then $v(x+y) \geq \gamma$, for every $\gamma \in G_+^* = \{g \in G | g > 0\}$.

(v_3) $v(x) = \infty$ iff $x = 0$.

(v_4) $v(-1) = 0$ if the order $v(-1)$ is not 2.

The property (v_4) gives $v(x) = v(-x)$. The (v_2) property is called *triangular property*.

Remark 2.1. According to what we have said in Section 1.1, (cf. also the lemma 2.1 below) the demand (v_2) may be written as:

$$(v_2)' \quad (\forall x \in K)(\forall y \in K)[w(x+y) \geq \inf_{\widehat{w(K)}}\{w(x), w(y)\},$$

where $\inf_{\widehat{w(K)}}\{w(x), w(y)\}$ is the infimum of $\{w(x), w(y)\}$ in the MacNeille's completion of the set $w(K)$.

There also hold:

Proposition 2.1. *For every domain R with K its field of fractions, every abelian valuation v on R has a unique extension on K.*

Proposition 2.2. *For every abelian valuation v of a skew field K, the set*

$$V = \{x \in K | v(x) \geq 0\}$$

is a valuation ring.

Lemma 2.1. *In any p.o. set E the following properties are equivalent for a, b, c being fixed elements of E:*

(i) $(\forall d \in E)[a \geq d, b \geq d \to c \geq d]$.

(ii) $c \geq \inf_{\hat{E}}\{a, b\}$.

The $\inf_{\hat{E}}\{a, b\}$ means the infimum $\{a, b\}$ in the MacNeille's completion of the set E.

Referred to the sets $R, K, U(R)$ and G, we give the definition:

Definition 2.2. Let R be a right Ore domain, K be the field of its fractions and $\overline{G}$ be as in the definition 2.1. A G-valuation is a map of K onto $\overline{G}$ which fulfills the following (x, y in R):

(w_1) $w(x.y) = w(x) + w(y)$.

(w_2) If $w(x) > \gamma$ and $w(y) > \gamma$, then $w(x+y) > \gamma$, for $\gamma \in G^*_+$ (the triangular).

(w_3) $w(x) = \infty$ iff $x = 0$.

(w_4) $w(-1) = 0$, if the order of $w(-1)$ is not 2.

The property (w_4) also gives that $w(x) = w(-x)$.

Remark 2.2. The proposition 2.1 is in valid, so the G-valuation w may be referred to a non commutative field K, the field of fractions of R.

2.8. The suppositions and notations referring to $R, K, U(R)$ and $\overline{G}$ are as above. Firstly there is a subgroup C_0 of $U(R)$, which contains the elements 1 and -1 and which has $\{0\}$ as *w-image*. So, the whole process refers to the quotient group K^*/C_0. We also suppose that the group G is splitting, thus

$$G = G_0 \oplus \Gamma,$$

where G_0 has elements parallel to 0 and one another and Γ is a group of representatives of the group G/G_0. Finally, we remark that the elements of $U(R)$ correspond to the subgroup G_0 via the G-valuation w. The elements $x, y \in U(R)$ do not fulfill the relations neither $w(x) > w(y)$ nor $w(x) < w(y)$. The elements of $R^* \backslash U(R)$ are characterized as *positive elements.* That is, if, say $x \in R \backslash U(R)$, then its w-image is greater than the elements of the subgroup G_0. Moreover, the elements of the set $xU(R)$, $x \in R$, have w-images greater that the elements of G_0. In fact: if $y \in U(R)$, then $x.y \in R \setminus U(R)$.

Remark 2.3. According to what we have said in Section 2.8, there hold $G_+ = \{g \in G | g \in G_0 \text{ or } g > G_0\}$ and $G_+^* = G_+ \setminus G_0$. Dually, the symbols G_-, G_-^*.

Lemma 2.2. *In any p.o. set E the following properties are equivalent (a, b, c are fixed elements of E):*

(i) $(\forall d \in E)[a > d, b > d \rightarrow c > d]$.

(ii) $c > \inf_{\overline{E}}\{a, b\}$.

The $\inf_{\overline{E}}\{a, b\}$ means the infimum of $\{a, b\}$ in the *Kurepa's completion* of E.

Proof. We suppose that (i) holds: the elements a and b are greater than d and then c being greater than d, belongs to the upper class of a and b, thus $c > \inf_{\overline{E}}\{a, b\}$. We suppose that (ii) holds: the element c belongs to the upper class of a and b. So, if a and b are greater than d, d belongs to the lower class of a and b, hence $c > d$. □

Remark 2.4. According to lemma 2.2 the relation (w_2) of the definition 2.2 changes into:

$$(\mathrm{w}_2)^* \quad (\forall x \in K)(\forall y \in K)[w(x + y) > \inf_{\overline{w(K)}}\{w(x), w(y)\}],$$

where $\inf_{\overline{w(K)}}\{w(x), w(y)\}$ is the infimum of $\{w(x), w(y)\}$ in the *Kurepa's completion* of the set $w(K)$. Thus, it becomes true the Ohm's remark we have cited in Section 1.2. Moreover, it has been proved that the *Kurepa's completion* of a p.o. set is a $\wedge$-semi-lattice, so it is meaningful the symbol $\inf_{\overline{w(K)}}\{w(a), w(b)\}$.

An impressive result is the following:

Theorem 2.1. *For a p.o. set $(E, \leq)$ the following statements are equivalent:*

(a) *A complete* $\wedge$*-semi-lattice* $\hat{E}$ *is isomorphic to the MacNeille's completion of* E*.*

(b) *There exists an infimum-preserving embedding* j *of* E *into* $\hat{E}$ *such that for any complete* $\wedge$*-semi-lattice* L *and any infimum preserving map* $h : E \to L$*, there exists a unique* $\wedge$*-homomorphism* $\hat{h}$ *such that the following diagram commutes:*

$$\begin{array}{ccc} E & \overset{j}{\to} & \hat{E} \\ \| & & \downarrow \hat{h} \\ E & \overset{h}{\to} & L \end{array}$$

Proof. We suppose the condition (a): there is an embedding j of E to $\hat{E}$ and h is an embedding of E into L as well. We put:

$$\hat{h}(j(\alpha)) = h(\alpha), \quad \text{for every } \alpha \in E.$$

On the other hand the infimum-preserving function j, as well as the function h, define the unique homomorphism $\hat{h}$ assigning to every infimum of $j(\alpha_i)$, $i \in I$, the infimum of $h(\alpha_i)$, $i \in I$. Thus, an embedding $\hat{h}$ of $\hat{E}$ into L has been defined.

The inverse statement is evident since $\hat{E}$ is the infimum of the $\wedge$-semi-lattice completions of E; this set is embedded to every$\wedge$-semi-lattice hyper set of E; so $\hat{E}$ is isomorphic to the *MacNeille's completion* of E. □

Remark 2.5. The theorem 2.1 says that-under some conditions-the MacNeille's completion is the finest of all the completions of a p.o. set which are $\wedge$-semi-lattice. The *Kurepa's completion* does not coincide with the MacNeille's one. It is larger than this. For instance, the MacNeille's completion of **Q** (the set of rational numbers) is the set **R** (the real numbers). The *Kurepa's completion* of **Q** is not **R**. Similarly, the MacNeille's completion of **R** is **R** too; The *Kurepa's completion* of **R** is a subset of the *semireal numbers* as Krasner has defined them.

Remark 2.6. With the below statements we define the operation in the *Kurepa's completion* of a p.o. group. We consider the $(G_{\overline{Ku}}, \overline{+}, \overline{\leq})$, the Kurepa's completion of a p.o. group $(G, +, \leq)$. If $\overline{x}$ is an element of the completion, we write X for the element which determines $\overline{x}$; respectively, we write Y and Z for the elements $\overline{y}$ and $\overline{z}$; finally, we write, as usual $X+Y = \{a+b | a \in X,\ b \in Y\}$, ($X$ and Y are subset of G or $G_{\overline{Ku}}$).

Definition 2.3. If $\overline{x}, \overline{y} \in G_{\overline{Ku}}$ then $\overline{x} \overline{+} \overline{y}$ is the infimum of $X+Y$ (i.e. $\inf_{\overline{Ku}}(X+Y)$. The notion is well defined, as one can prove. If $\overline{x} \in G_{\overline{Ku}}$, $\overline{y} \notin G_{\overline{Ku}}$ or $\overline{x}, \overline{y} \notin G_{\overline{Ku}}$, the sets $\overline{x}+Y$ or $X+Y$, respectively, have not endpoints.

Proposition 2.3. *If $\overline{x}, \overline{y}, \overline{z}$ are elements of $G_{\overline{Ku}}$ and $\overline{x} \leq \overline{y}$, then $\overline{x} \overline{+} \overline{z} \overline{\leq} \overline{y} \overline{+} \overline{z}$.*

Proof. We work with the usual operations between elements and the usual relations between sets. Thus, let $\overline{x} \leq \overline{y}$. It means that $X \subseteq Y$. Since $Z = \cup_{i \in I} \tau_i$ then

$$X + Z = \cup_{k \in K} a_k \overline{+} \cup_{i \in I} \tau_i \subseteq \cup_{r \in R} b_r \overline{+} \cup_{i \in I} \tau_i,$$

with $X = \cup_{k \in K} a_k$, $Y = \cup_{r \in R} b_r$; finally $\overline{x} \overline{+} \overline{z} \overline{\leq} \overline{y} \overline{+} \overline{z}$. □

From this point on we don't make use of the dash over the elements, the addition + and the relation $\leq$.

Among others we have the following properties:

Proposition 2.4. *Let K be a field G-valuated by a G-valuation w. There hold:*

(i) $w(x^{-1}) = -w(x)$, for every $x \in K^*$.

(ii) If $w(x) > \gamma$, $w(y) > \gamma$, then $w(x-y) > \gamma$, $\gamma \in G^*_+$.

(iii) If the field K is finite and the group G is torsion free, then the only possible G-valuation is the trivial one, (i.e. it is the valuation which fulfils $w(0) = \infty$ and $w(x) = 0$ for all $x \in K^*$).

The proofs are the same as in the commutative case.

Proposition 2.5. *Let K be a field G-valuated by a G-valuation w. Then:*

(i) If x, y, a are elements of G with $x \parallel y$ (i.e. they are not comparable), then $x + a \parallel y + a$.

(ii) If $w(x) < w(y)$, then $w(x+y) = w(x)$ or $w(x+y) \parallel w(x)$.

Proof. (i) If $x + a < y + a$, then $x < y + a - a$, an absurd.

(ii) There holds: $w(x+y) > \inf_{\overline{Ku}}\{w(x), w(y)\}$, that is $w(x+y)$ belongs to the upper class of $\{w(x), w(y)\}$. Besides $w(x) = w(x+y-y) \leq \inf_{\overline{Ku}}\{w(-y), w(x+y)\}$ or $w(x) \leq \inf_{\overline{Ku}}\{w(y), w(x+y)\}$ so as well, $w(x)$ is not contained to the lower class of $\{w(x+y), w(y)\}$. Hence either $w(x+y) \leq w(x)$ and $w(x) \leq w(x+y)$, in which case $w(x+y) = w(x)$ or $w(x+y)$ is not comparable with $w(x)$.

□

Proposition 2.6. *Let K be a skew field G-valuated by a G-valuation w. Then:*

(i) If G_0 is the torsion subgroup of G, then all the elements of this subgroup are parallel one another.

(ii) If a point of G is strictly larger than all the negative elements of G and strictly smaller than all the positive elements of it, then it is parallel to zero.

Proof. (i) Let be $u \in G_0$, n the order of u and $u > 0$. Then $nu = 0 > 0$, an absurd, hence $u \parallel 0$. Besides, if $v \in G_0$ and $u > v$ then $u - v \in G_0$ and $u - v > 0$, an absurd. The same hold when $u < 0$.

(ii) It is obvious.

□

Lastly, we give a property of $\inf_{\overline{Ku}}$.

Proposition 2.7. *Let $(G, +, \leq)$ be an abelian p.o. torsion free group and $(G_{\overline{Ku}}, +, \leq)$ be its Kurepa's completion. Then:*

$$\inf_{\overline{Ku}}\{a+d, b+d\} = \inf_{\overline{Ku}}\{a, b\} + d,$$

$\inf_{\overline{Ku}}$ is the infimum in $G_{\overline{Ku}}$ and $a, b, d \in G$. If G_0 is the torsion subgroup of G, then the theorem is valid for the quotient group G/G_0, provided that there exists a group of representatives for this quotient group.

Proof. Let be $s = \inf_{\overline{Ku}}\{a, b\}$. If $a \parallel b$, then $s + d$ is smaller that $a + d$ and $b + d$ and $s + d \leq \inf_{\overline{Ku}}\{a+d, b+d\}$. If, say $a < b$, then $s = a$ or $s \parallel a$, that is, $s + d = a + d$ or $s + d \parallel a + d$. In both of the cases $s + d \leq \inf_{\overline{Ku}}\{a+d, b+d\}$.

Inversely: Let K be the first member of our relation. If $a + d \parallel b + d$, then $K - d \leq a$, $K - d \leq b$ and finally $K - d \leq \inf_{\overline{Ku}}\{a, b\}$. If $a < b$ then $K = a + d$ or $K \parallel a + d$. So, $K - d < b$ or $K - d \parallel a$. In both of the cases there holds: $K - d \leq \inf_{\overline{G}}\{a, b\}$. □

3. Topology by a G-Valuation

We consider again a right Ore domain R, as we have supposed it in the introduction of the paragraph 2. We also consider the field K of R's fractions, a G-valuation w defined on R and after that on K with value group G. We suppose that G is splitting, so there holds:

$$G = G_0 \oplus \Gamma,$$

where G_0 is the torsion part of G whose the elements are parallel to 0 and one another and Γ is a group of representatives of the quotient group of G by G_0.

The following theorem is a well known result of valuated fields (cf. [2], [3], [6]) and it is in valid for G-valuated fields, too.

Theorem 3.1. *Let K be a skew field valuated by a valuation w and $(G, +, \leq)$ be a totally ordered abelian group which is the value group of w. Then the family*

$$\mathbf{V}_\beta = (V_\beta(0)), \text{ where } V_\beta(0) = \{x \in K | w(x) > \beta\}, \quad \text{for } \beta \in G_+^*,$$

defines a neighborhood system of 0 *for a topology, for which the set*

$$\mathbf{V} = \{x \in K | w(x) \geq 0\}$$

is a topological ring. Moreover, $\mathbf{V}$ *is a valuation ring and K is a topological field for the same topology.*

Theorem 3.2. *With the notation and suppositions of the Theorem 3.1, there hold:*

(a) *The set $C_0^0 = \{x \in K | w(x) = 0\}$ is a multiplicative topological group.*

(b) *The set $P = \{x \in K | w(x) > 0\}$ is a maximal ideal of* $\mathbf{V}$.

We can restate the above theorem with a slight modification of the suppositions.

Theorem 3.3. *Let K be a skew field G-valuated by a G-valuation w, whose the value group G is partially ordered, abelian and torsion free. Beginning from the relation:*

$$G = G_+ \cup G_-,$$

we prove that the family

$$\mathbf{V}_\beta = (V_\beta(0))_{\beta>0}, \text{ where } V_\beta(0) = \{x \in K | w(x) > \beta\}, \quad \text{for } \beta \in G_+^*,$$

defines a neighborhood system of $\{0\}$ *for a topology, for which the set*

$$\mathbf{V} = \{x \in K | w(x) \geq 0\}$$

is a topological ring. Moreover $\mathbf{V}$ *is a valuation ring and* K *is a topological field.*

Proof. • The family $\mathbf{V}_\beta$ define a topology

(i) The sets $V_\beta(0)$, $\beta > 0$ constitute a filter base.

In fact; if β_1, β_2 in G^*_+ such that $w(x_1) > \beta_1$, $w(x_2) > \beta_2$, then $\beta_1 + \beta_2$ is greater than β_1 and β_2 or equal to them, thus the meet of the two subsets $V_{\beta_1}(0)$ and $V_{\beta_2}(0)$ is of the form $V_\beta(0)$ and hence, it is non void.

(ii) $(\forall \beta \in G^*_+)(\exists \delta \in G^*_+)[V_\delta(0) + V_\delta(0) \subseteq V_\beta(0)]$.

In fact; if x_1, x_2 are in $V_\beta(0)$, then $w(x_1) > \beta$, $w(x_2) > \beta$, whence $w(x_1 + x_2) > \beta$ and $V_\beta(0)$ is a group. Thus $V_\beta(0) + V_\beta(0) = V_\beta(0)$.

(iii) The group $V_\beta(0)$ is symmetrical.

(iv) $(\forall a \in K)(\forall V_\beta(0) \in \mathbf{V}_\beta)[a + V_\beta(0) - a \in \mathbf{V}_\beta]$.

• $\mathbf{V}$ is a topological ring.

(i) $\mathbf{V}$ is a topological group.

If $w(x_1) > 0$ and $w(x_2) > 0$, then $w(x_1 + x_2) > 0$. If $w(x_1) = 0$ and $w(x_2) \geq 0$, then it is impossible $w(x_1 + x_2) = \rho < 0$. So, it is a group. Moreover, the above (ii) proposition assures the topological character of the structure.

(ii) $\mathbf{V}$ is a topological ring.

Evidently, it is a ring: if $w(x_1) \geq 0$ and $w(x_2) \geq 0$, then $w(x_1 x_2) \geq 0$. Furthermore, it is a topological ring: let $V_\beta(x_1) = \{x_1\} + V_\beta(0)$, $V_\beta(x_2) = \{x_2\} + V_\beta(0)$, be neighborhoods of the $\mathbf{V}$'s elements x_1, x_2, respectively. It is enough to be proved:

$$(\forall \beta \in G^*_+)(\exists \delta \in G^*_+)[V_\delta(x_1).V_\delta(x_2) \subseteq V_\beta(x_1 x_2)]$$

or

$$(\forall \beta \in G^*_+)(\exists \delta \in G^*_+)[\{x_1 x_2\} + \{x_1\}V_\delta(0) + V_\delta(0)\{x_2\} + V_\delta(0)V_\delta(0) \subseteq \{x_1 x_2\} + V_\delta(0)].$$

Let be $z = \{x_1\}z_1 + z_2\{x_2\} + z_3 z_4$, where z_1, z_2, z_3 and z_4 are elements of $V_\delta(0)$. So,

$$\begin{aligned} w(z) &> \inf_{\overline{Ku}}\{w(x_1) + w(z_1), w(z_2) + w(x_2), w(z_3) + w(z_4)\} \\ &\geq \inf_{\overline{Ku}}\{w(x_1) + \delta, \delta + w(x_2), \delta + \delta\}. \end{aligned}$$

But (by the Proposition 2.7)

$$\inf_{\overline{Ku}}\{w(x_1) + \delta, \delta + w(x_2), \delta + \delta\} = \delta + \inf_{\overline{Ku}}\{w(x_1), w(x_2), \delta\}.$$

The elements x_1, x_2 are fixed elements of K and β of G. So, there is a δ such that

$$\delta + \inf_{\overline{Ku}}\{w(x_1), w(x_2), \delta\} \geq \beta.$$

- **V** is a valuation ring.

(i) **V** is a ring (as it has been already proved).

(ii) **V** is a total ring.

If $x \notin \mathbf{V}$, then $w(x) < 0$ and $w(x^{-1}) > 0$.

(iii) **V** is an invariant ring.

If $w(x) > 0$ and $c \in K$, then $w(cxc^{-1}) = w(c) + w(x) + w(c-1) = w(x)$ and the proof i over.

- K is a topological field.

It is $V_\beta(x) = \{x\} + V_\beta(0)$, hence, for every $y \in V_\beta(x)$, there holds $w(x-y) < \beta$ (cf. proposition 2.4) (ii) and moreover the fact that the element x is fixed). We will show that:

$$(\forall x \in K^*)(\forall \beta \in G_+^*)(\exists \delta \in G_+^*)[(V_\delta(x))^{-1} \subseteq V_\beta(x^{-1})]. \tag{6}$$

The relation (6) can be written as $V_\delta(x) \subseteq (V_\beta(x^{-1}))^{-1}$.

Let $y \in V_\delta(x)$, that is

$$w(x - y) > \delta. \tag{7}$$

It is enough to be proved that $y \in (V_\beta(x^{-1}))^{-1}$ or $y^{-1} \in V_\beta(x^{-1})$ or $w(1/x - 1/y) > \beta$ or $w(x - y) - w(x) - w(y) > \beta$ or (because of (7))

$$\delta > w(x) + w(y) + \beta. \tag{8}$$

On the other hand $w(x) > w(y)$ or $w(x) \parallel w(y)$. In both of cases it is enough to hold the relation (8) or, more convenient,

$$\delta > 2w(x) + \beta,$$

which means that there exist the element δ. But, such a δ exists. □

It is not difficult to be proved:

Theorem 3.4. *With the suppositions of the Theorem 3.3 there hold:*

(i) The set $C_0^0 = \{x \in K | w(x) = 0\}$ is a topological group.

(ii) The set $P = \{x \in K | w(x) > 0\}$ is a maximal ideal if $\mathbf{V}$.

(The sets P, C_0^0 constitute a partition of the valuation ring $\mathbf{V}$*).*

We generalize now the Theorems 3.3 and 3.4 supposing the more general case of the group G.

Theorem 3.5. *Let K be a skew field G-valuated by a G-valuation w, whose the value group G is an abelian p.o. group which is splitting, that is*

$$G = G_0 \oplus \Gamma,$$

where G_0 and Γ are as in the introduction of the paragraph. Then, the family $\mathbf{V}_\beta = (V_\beta(0))_{\beta \in G_+^*}$, where $V_\beta(0) = \{x \in K | w(x) > \beta\}$, $\beta \in G_+^*$, constitutes a neighborhood system of 0 for a topology. Moreover, the same family is a neighborhood system of every point e of G_0, the set

$$\mathbf{V} = \cup\{V_\beta(0) | \beta \in G_+^*\}$$

is a topological ring, in fact a valuation ring, and K is a topological field.

Proof. • The family $\mathbf{V}_\beta$ defines a topology.

(i) The family $\mathbf{V}_\beta$ is a filter.
If $\mathbf{V}\beta = (V_\beta(0))_{\beta \in G_+^*}$, then there is a $\beta > \beta_1, \beta_2$ etc.

(ii) $(\forall \beta \in G_+^*)(\exists \delta \in G_+^*)[V_\delta(0) + V_\delta(0) \subseteq V_\beta(0)]$.
The set $V_\beta(0)$ is a group: if $x, y \in R$ with $w(x) \in G \backslash G_0$ and $w(y) \in G \backslash G_0$, then $w(x+y) \in G \setminus G_0$ and the set $V_\beta(0)$ is a group. So, we may put $V_\beta(0) = V_\delta(0)$.

(iii) The group $V_\beta(0)$ is symmetrical.

(iv) $(\forall a \in K)(\forall V_\beta(0) \in \mathbf{V}_\beta)[a + V_\beta(0) - a \in \mathbf{V}]$.

In a similar way, as in the demonstration of the Theorem 3.3, we prove that the family $\mathbf{V}_\beta$ is a topological ring (in fact a valuation ring) and K is a topological field.

It is also evident, that the same family constitutes a neighborhood system for every element G of and that the topology is not T_0. □

We also have the following:

Theorem 3.6. *With the suppositions of Theorem 3.5 there hold:*

(i) The set $C_0^0 = \{x \in K | w(x) \in G_0\}$ *is a topological group.*

(ii) The set $P = \{x \in K | w(x) \in G_+^*\}$ *is a maximal ideal of* $\mathbf{V}$.

4. An Alternative Approach of Valuation Rings

As we have already said, this paragraph is devoted to an alternative approach of a valuation ring of a G-valuated non-commutative field. (For the commutative case cf. [1], [11], [15]).

In the whole process we will face the following four problems:

1^{st} *The extension of the torsion free p.o. group* G *or the extension of the group* G/G_0*, (if* G *is mixed), to a totally ordered one.*

After *Szpilrajn* (cf. [15]) every p.o. set may be extended to a totally ordered one and this property may be transferred as well to the theory of groups. More precisely, a splitting p.o. ordered group (and a torsion free p.o. group as well) may be extended to a totally ordered group (cf. [8], [15]).

The basic result is the following (Szpilrajn makes use of the Zermelo's *maximal principle.* cf. [11] and [15]):

Theorem 4.1. *For every abelian torsion free p.o. ordered group* $(G, +, \leq)$ *there is a totally ordered group* $(G, +, \overline{\leq})$*, where* $\overline{\leq}$ *is an extension of* $\leq$*. If the group is mixed and splitting, that is if* $G = G_0 \oplus \Gamma$*, where* G_0 *and* Γ *are as in the introduction of the paragraph 2, then, we firstly extend the group* G/G_0 *to a totally ordered group.*

We write $\leq$ instead of $\overline{\leq}$.

It is well-known that the theorem of *Lorenzen-Simbireva-Everett* (formulated by each one of them in different papers and in different periods, cf. [8]) states that

"*An abelian group is extended to a totally ordered group if and only if it is torsion free*"

According to the above discussion the mixed and splitting group may be considered as a p.o. group and thus the Theorem 4.1 constitutes a generalization of the afore theorem.

2^{nd}. *The “solenoidal” form of a splitting ring.*

We suppose again that the value group G is a partially ordered, mixed and splitting group:

$$G = G_0 \oplus \Gamma,$$

where G_0 and Γ are as in the beginning of the paragraph 2. We also mention the symbols $U(R)$, w and $\overline{G}$. Now, we describe the subgroup G_0 as

$$G_0 = \{e_i^0 \in G | i \in I\},$$

I a set of indices. Let us put G_γ for any class of the quotient group G/G_0 which is the correspondence class to an element $\gamma \in \Gamma$. For the sake of similarity we symbolize by A_0 the subgroup $U(R)$, which is represented via w to the subgroup G_0 and by A_γ the subset of K^*, which is represented via w to the class G_γ.

We have:

$$C_0^0 = \{x \in K | w(x) = 0\}$$

By analogy:

$$C_i^0 = \{x \in K | w(x) \in e_i^0\}$$

and since the quotient group $K^*/U(R)$ is isomorphic to Γ,

$$C_i^\gamma = \{x \in K | w(x) = e_i^\gamma\},$$

where we have supposed that each of the classes A_γ has as w-image the set $\{e_i^\gamma \in G | i \in I\}$. This result comes from one of the theorems of isomorphism for groups. In the present case it has the form:

$$K^* \Big/ U(R) \cong K^*/C_0^0 \Big/ U(R)/C_0^0 \tag{9}$$

with the left and the right hands of the relation (9) being isomorphic. Thus, the classes G_γ are in an one to one correspondence and the same is true for the different C_i^γ where γ is fixed.

We symbolize by $x_{i,\rho}^\gamma$ any element of K, where $\gamma \in \Gamma$ represents the class of G/G_0 to which belongs the w-image of the element, the index i corresponds to the element e_i^γ, when $w(x_{i,\rho}^\gamma) = e_i^\gamma$ and ρ is the current index on the set C_i^γ to which belongs the afore element.

We remark that

$$x_{i,\rho}^\gamma = x_{0,\rho}^\gamma . k_i^\gamma, \quad k_i^\gamma \in K^*,$$

in fact, $k_i^\gamma = x_{i,\rho}^\gamma . (x_{0,\rho}^\gamma)^{-1}$. In particular, there holds: $x_{i,\rho}^0 = x_{0,\rho}^0 . k_i^0$, $k_i^0 = x_{i,\rho}^0 . (x_{0,\rho}^0)^{-1}$. Then

$$w(x_{i,\rho}^\gamma) = w(x_{0,\rho}^\gamma) + w(k_i^\gamma) = e_i^\gamma + \gamma,$$

where we have put $w(x^{\gamma}_{0,\rho}) = e^{\gamma}_i$ and $w(k^{\gamma}_i) = \gamma$.

According to the above discussion we may write:

$$U(R) = A_0 = C^0_0 \otimes (U(R)/C^0_0)$$

and

$$K^* = U(R) \otimes (K^*/U(R)),$$

that is the set $U(R)$ is as a "torrus" and the set K^* may be considered as a union of "torri", that is the ring has the form of a "solenoidal".

3^{rd}. *The G-valuation and the order during the extension of $\leq$.*

Let G be a group partially ordered by $\leq$, torsion free and G-valuated by w. We extend $\leq$ into a linear order, say $\leq_l$. We assign to every step of the extension of the order, an ordinal number for the new order, say σ. This ordinal number may be a regular ordinal (that is, there exists the $\sigma+1$ ordinal) or it is a limit one. In any case, there is an extension over the group structure G_σ and over $\leq_\sigma$,the two structures being compatible (cf. [11], [15]). We symbolize by w_σ the G-valuation the assigned to $\leq_\sigma$ and we define the value of this valuation by

$$w_\sigma(x) = w(x), \quad x \in K^*,\ K^* \text{ as above.} \tag{10}$$

The preservation of (10) and the compatibility of structures are the only limitations we have in all the steps of the procedure, while the open sets are designated by the initial G-valuation w.

As we have seen in the Theorems 3.3 and 3.5, this defines on K a topology τ_σ (we name σ the topology which the G-valuation w induces), with a subbase of open subsets, the subsets of the form

$$\{x \in K | w_\sigma(x) > \beta, \beta \in G^*_+\}.$$

Every $\hat{O} \in \tau_\sigma$ evidently contains an open $O \in \tau$, thus $\tau_\sigma \subseteq \tau$. It means that τ gives more open sets than the ones which τ_σ gives. In fact, the topology τ is the supremum of all the topologies τ_σ.

A natural question has been arisen: how many are these τ_σ? The answer is that they are so many as the *dimension* of the partially ordered group G is (cf. [9]). We give the relative definition:

Definition 4.1. Let S be a set and let L be any collection of linear orders defined on S. We define a partial order $P(\leq)$ as follows: for any two elements x_1 and x_2 of S we put $x_1 \ll x_2$ if and only if $x_1 < x_2$ in in every linear order of the collection L. A partial order, so obtained, will be said to *be realized by the linear orders of L*.

By the dimension of a partial order P defined on a set S we meant the smallest cardinal number $\mathbf{m}$ such that P is realized by $\mathbf{m}$ linear orders on S.

4^{th}. *The valuation rings of a field.*

We have already said, in the Theorem 3.1, that given a valuated field whose the value group G is a totally ordered group, the set

$$\mathbf{V} = \{x \in K | w(x) \geq 0\}$$

is a valuation ring and the set

$$\mathbf{P} = \{x \in K | w(x) > 0\}$$

is a maximal ideal of $\mathbf{V}$.

We consider, now, a G-valuated field K whose the value group is a splitting p.o. group G, that is, the value group is such that

$$G = G_0 \oplus \Gamma,$$

where the meaning of G_0 and Γ have been given previously. We preserve the notation of the previous sections. In addition, referring to the final level of the extension, (that is to the linear order), we make use of the symbols $w_l, \tau_l, \leq_l$ for the corresponded G-valuation, for the topology and for the order, respectively.

We consider this last step of the extension $(G, \leq_l, \tau_l)$. As we have seen in the proof of Theorem 3.3, the set

$$\mathbf{V}_l \{x \in K | w_l(x) \geq_l 0\}$$

is a *valuation ring* and the set

$$\mathbf{P}_l = \{x \in K | w_l(x) >_l 0\}$$

is a *maximal ideal* of $\mathbf{V}_l$. The interesting thing is that in any level σ of the extension, the set

$$\mathbf{V}_\sigma \{x \in K | w_\sigma(x) \in G_+\}$$

is a *valuation ring* (the w_σ is the G-valuation in the level σ) and the set

$$\mathbf{P}_\sigma = \{x \in K | w_\sigma(x) \in G_+^*\}$$

is a *maximal ideal* of $\mathbf{V}_\sigma$.

If we come now back to the initial structure, that is to the field K, to the G-valuation w and to the p.o. group $(G, \leq, \tau)$ the sets have the same properties, as they have in the step of the total ordering. The structures $\mathbf{V}_\sigma$ and $\mathbf{P}_\sigma$ are changed into the subsets of $\mathbf{V}$, but they remain a valuation ring and a maximal ideal of it, respectively, i.e. this ring $\mathbf{V}$ is a *total* and *invariant ring* as well.

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