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INVERSE NODAL PROBLEM FOR STURM-LIOUVILLE OPERATORS WITH COULOMB POTENTIAL

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Abstract: Inverse nodal problems consist in constructing operators from the given nodes (zeros) of their eigenfunctions. In this study, we have estimated nodal points and nodal lengths for the Sturm-Liouville operators with Coulomb potential. Furthermore, by using nodal points (zeros of eigenfunctions), we have shown that the potential function of the Sturm-Liouville operators with Coulomb potential can be established uniquely.

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1. Introduction

In spectral theory, the inverse problem is the usual name for any problem in which it is necessary to ascertain the spectral data that will determine a different operator uniquely and a method of construction for this operator from the data. A problem of this kind was first formulated and investigated by Ambartsumyan in 1929 [1]. Since 1946, various forms of the inverse problem

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have been considered by numerous authors Borg [2], and Levitan [3], etc. Later, the inverse problems having specified singularities were considered by a number of authors $[4-6]$.

Recently, a new class of inverse problems has attracted the attention of researchers. This is the so-called inverse nodal problem. The inverse nodal problem was initiated by McLaughlin [8], which is different from the classical inverse spectral theory of Gelfand and Levitan [7]. She found that the nodal set of the Dirichlet problem for the Sturm-Liouville equation can determine q up to a constant. Later, Hald and McLaughlin [9] and Browne and Sleeman [10] showed that only the knowledge of nodal points can determine the potential function of the regular Sturm-Liouville problem. Yang [11] showed that this uniqueness result is valid for any q. In the past few years, the inverse nodal problem of the regular and singular Sturm-Liouville problem has been investigated by several authors $[8-15]$.

In this work, we are concerned with the inverse problem for Sturm-Liouville operators with Coulomb potential, using a new kind of spectral data.

Before giving the main results, we mention some physical properties of the Sturm-Liouville operators with Coulomb potential. Learning about the motion of electrons moving under Coulomb potential is of significance in quantum theory. Solving these types of problems provides us to find energy levels not only hydrogen atom but also single valance electron atoms such as sodium.For hydrogen atom, Coulomb potential is given by $U = \frac{-e^2}{r}$, where r is the radius of the nucleus, e is electronic charge. According to this, we use time dependent Schrödinger equation;

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + U(x,y,z)\Psi, \quad \int_{R^3} |\Psi|^2\, dxdydz = 1,$$

where Ψ is the wave function, $\hbar$ is Planck's constant and m is the mass of electron. In this equation, if it is applied Fourier transform

$$\widetilde{\Psi} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\lambda t}\Psi dt,$$

it will convert to energy equation dependent to situation as following;

$$\frac{\hbar^2}{2m}\bigtriangledown^2\widetilde{\Psi} + \widetilde{U}\widetilde{\Psi} = E\widetilde{\Psi}.$$

Therefore, energy equation in the field with Coulomb potential become as following;

$$-\frac{\hbar^2}{2m}\bigtriangledown^2\widetilde{\Psi} + \left(E + \frac{e^2}{r}\right)\widetilde{\Psi} = 0.$$

If this hydrogen atom is substituted to other potential area, then energy equation become as following;

$$-\frac{\hbar^2}{2m}\bigtriangledown^2\widetilde{\Psi}+\left(E+\frac{e^2}{r}+q(x,y,z)\right)\widetilde{\Psi}=0.$$

If we make the necessary transformation, then we can get a Sturm-Liouville equation with Coulomb potential

$$-y''+\left[\frac{A}{x}+q(x)\right]y=\lambda y$$

where λ is a parameter which corresponds to the energy [16].

We consider the singular Sturm-Liouville problems

$$-y''+\left[\frac{A}{x}+q(x)\right]y=\lambda y,\ (0<x\leq\pi),\quad \lambda=\rho^2, \tag{1}$$

$$y(0)=0,\ y'(\pi)-Hy(\pi)=0, \tag{2}$$

and

$$-y''+\left[\frac{A}{x}+\widetilde{q}(x)\right]y=\lambda y,\ (0<x\leq\pi), \tag{3}$$

$$y(0)=0,\ y'(\pi)-\widetilde{H}y(\pi)=0, \tag{4}$$

in which the functions $q(x)$ and $\widetilde{q}(x)$ are assumed to be real valued and square integrable and A, H and $\widetilde{H}$ are finite real numbers. Next, we denote by $\varphi(x,\lambda)$ the solution of (1), and we denote by $\widetilde{\varphi}(x,\lambda)$ the solution of (3) satisfying the initial conditions (2) and (4), respectively.

It is well known that there exists a function $K(x,t)$ such that

$$\widetilde{\varphi}(x,\lambda)=\varphi(x,\lambda)+\int_0^x K(x,t)\varphi(t,\lambda)dt. \tag{5}$$

The function $K(x,t)$ satisfies the equation

$$\frac{\partial^2 K}{\partial x^2}-\left[\frac{A}{x}+\widetilde{q}(x)\right]K=\frac{\partial^2 K}{\partial t^2}-\left[\frac{A}{t}+q(t)\right]K, \tag{6}$$

and the conditions

$$K(x,x)=\frac{1}{2}\int_0^x\left[\widetilde{q}(t)-q(t)\right]dt,\ K(x,0)=0. \tag{7}$$

This problem can be solved by using the Riemann method [17].

Let $\lambda_0(q,\lambda) < \lambda_1(q,\lambda) < \cdots \to \infty$ be the eigenvalues of the problem (1), (2) and $0 < x_1^n < x_2^n < \cdots < x_j^n < \pi, \;\; j = 1, 2, \ldots, n-1$ be nodal points of the n th eigenfunction. It is shown that the set of all nodal points $\left\{x_j^n\right\}$ is dense in $(0,\pi)$; in fact a judicious choice of one nodal point x_j^n for each $y_n, \; n > 1$ also forms a dense set in $(0,\pi)$.

2. Main Results

In this section, our purpose is to develop asymptotic expressions for the points x_j^n and l_j^n (nodal length) at which y_n the eigenfunction corresponding to the eigenvalue λ_n of the problem (1), (2) vanishes.

Theorem 1. *We consider the equation*

$$-y'' + \left[\frac{A}{x} + q\left(x\right)\right] y = \lambda y, \tag{8}$$

with the boundary conditions

$$y(0) = 0, \tag{9}$$

$$y'(\pi) - Hy(\pi) = 0. \tag{10}$$

Then, the nodal points and nodal length of the problem (8)-(10) are

$$x_j^n = \frac{j\pi}{n} + O(\frac{1}{n^2}), \;\; (j = 1, 2, \ldots, n-1, \;\; n = 1, 2, \ldots), \tag{11}$$

$$l_j^n = \frac{\pi}{n} + O(\frac{1}{n^2}). \tag{12}$$

Proof. As known [4], the eigenvalues and eigenfunctions of the problem (8)-(10) satisfy the asymptotics

$$\rho_n^2 = \lambda_n = n + \frac{1}{2} + \frac{A}{2\pi}\frac{\ln(n+\frac{1}{2})}{(n+\frac{1}{2})} + \frac{c_0}{(n+\frac{1}{2})} + O(\frac{1}{n^2}), \;\; n \to \infty, \tag{13}$$

$$\begin{aligned}\varphi_n(x) \;\; &= \;\; \sin(n+\frac{1}{2})x + \frac{A}{2\pi}\frac{\ln(n+\frac{1}{2})x}{(n+\frac{1}{2})}\cos(n+\frac{1}{2})x + \frac{A\pi}{4}\frac{\sin(n+\frac{1}{2})x}{(n+\frac{1}{2})} \qquad (14)\\ &-\frac{\cos(n+\frac{1}{2})x}{(n+\frac{1}{2})}\beta(x) - \frac{A}{2}\frac{\cos(n+\frac{1}{2})x}{(n+\frac{1}{2})}\ln(n+\frac{1}{2})x + O(\frac{1}{n^2}),\end{aligned}$$

where,

$$c_0 = \frac{1}{\pi}\left(AM_1 - H + \frac{A\ln\pi}{2} + \frac{1}{2}\int_0^{\pi} q(t)dt\right),$$

$$\beta(x) = AM_1 + \frac{1}{2}\int_0^{x} q(t)dt,$$

$$M_1 = M + \frac{\sin 2}{2},\ M = \int_0^1 \frac{\sin^2\xi}{\xi}d\xi\ .$$

Hence, we use the classical estimate

$$\left|\varphi_n(x) - \sin(n+\frac{1}{2})x\right| < \frac{T}{n}$$

where T is a constant. Thus $\varphi_n(x)$ will vanish in the intervals whose end points are solution to

$$\sin(n+\frac{1}{2})x = \pm\frac{T}{n}.$$

This equation can also be written as

$$(n+\frac{1}{2})x = \arcsin\left(\pm\frac{T}{n}\right).$$

Using the Taylors expansion for $\arcsin\left(\pm\frac{T}{n}\right)$, we get

$$(n+\frac{1}{2})x = j\pi \pm \frac{T}{n} + O\left(\frac{1}{n^3}\right),$$

$$x_j^n = \frac{j\pi}{n} + O\left(\frac{1}{n^2}\right),\ \ (j = 1, 2, \ldots, n-1,\ n = 1, 2, \ldots).$$

The nodal length is

$$l_j^n = x_{j+1}^n - x_j^n,$$

$$l_j^n = \left(\frac{(j+1)}{n}\right)\pi - \left(\frac{j}{n}\right)\pi + O\left(\frac{1}{n^2}\right) = \frac{\pi}{n} + O\left(\frac{1}{n^2}\right).$$

This completes the proof. □

Now, we will give a uniqueness theorem. It says that the potential function $q(x)$ for a singular Sturm-Liouville problem is uniquely determined by a dense subset of the nodes. We mentioned that this theorem was given for regular Sturm-Liouville problems by McLaughlin [8], Hald and McLaughlin [9] and Browne and Sleeman [10].

Theorem 2. *Suppose that $q(x)$ is integrable. Then, H and $q - \int_0^\pi q$ are uniquely determined by any dense set of nodal points.*

Proof. Assume that we have two problems of the type (8)-(10) with H, $\widetilde{H}$ and q, $\widetilde{q}$. Let the nodal points x_j^n, $\widetilde{x}_j^n$ satisfying $x_j^n = \widetilde{x}_j^n$ form a dense set in $(0,\pi)$. We take solutions of (8)-(10) y_n for (H,q) and $\widetilde{y}_n$ for $(\widetilde{H},\widetilde{q})$. It follows from (8) that

$$\left(y_n'\widetilde{y}_n - \widetilde{y}_n' y_n\right)' = \left(q - \widetilde{q} + \widetilde{\lambda}_n - \lambda_n + \frac{A}{x} - \frac{A}{x}\right) y_n\widetilde{y}_n. \tag{15}$$

Recall that $(\widetilde{\lambda}_n - \lambda_n)$ are uniformly bounded in n and the $y_n\widetilde{y}_n$ are uniformly bounded in n and $x \in (0,\pi)$. We select a subsequence of nodes from the dense set. To show that $H = \widetilde{H}$, we integrate both sides of (15) from x_j^n to π and choose a subsequence that tends to π, and we see that

$$\left(H - \widetilde{H}\right) y_n(\pi)\widetilde{y}_n(\pi) = \int_{x_j^n}^{\pi} \left(q - \widetilde{q} + \widetilde{\lambda}_n - \lambda_n\right) y_n\widetilde{y}_n dx.$$

From this results, we can say $H = \widetilde{H}$.Let $x_j^n = \widetilde{x}_j^n$ and integrate both sides of (15) from 0 to x_j^n, we obtain

$$\int_0^{x_j^n} \left(y_n'\widetilde{y}_n - \widetilde{y}_n' y_n\right)' dx = \int_0^{x_j^n} \left(q - \widetilde{q} + \widetilde{\lambda}_n - \lambda_n\right) y_n\widetilde{y}_n dx,$$

$$\left[y_n'(x_j^n)\widetilde{y}_n(x_j^n) - y_n(x_j^n)\widetilde{y}_n'(x_j^n)\right] = \int_0^{x_j^n} \left(q - \widetilde{q} + \widetilde{\lambda}_n - \lambda_n\right) y_n\widetilde{y}_n dx,$$

$$0 = \int_0^{x_j^n} \left(q - \widetilde{q} + \widetilde{\lambda}_n - \lambda_n\right) y_n\widetilde{y}_n dx.$$

From the asymptotic forms of $\widetilde{\lambda}_n$ and λ_n, we have

$$0 = \int_0^{x_j^n} \left(q - \widetilde{q} + \widetilde{c}_0 - c_0 + H - \widetilde{H}\right) y_n\widetilde{y}_n dx,$$

where

$$c_0 = \frac{1}{\pi}\left[-H + \frac{1}{2}\int_0^\pi q(t)dt\right],\ \widetilde{c}_0 = \frac{1}{\pi}\left[-\widetilde{H} + \frac{1}{2}\int_0^\pi \widetilde{q}(t)dt\right]$$

We take a sequence x_j^n accumulating at an arbitrary $x \in (0,\pi)$. Hence,

$$0 = \int_0^x \left(q - \widetilde{q} - \int_0^\pi (q - \widetilde{q})ds\right) y_n\widetilde{y}_n dt$$

this holds for all x. We can therefore conclude that $q - \int_0^\pi q(s)ds$ is uniquely determined by a dense set of nodes. This completes the proof. □

Corollary 3. *For the problem (8)-(10), the potential q is uniquely determined by a dense set of nodes and the constant c_0.*

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