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CERTAIN SUBCLASS OF BOUNDED STARLIKE FUNCTIONS

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Abstract: Let $\mathcal{R}_{\alpha,\beta}(\gamma,\delta)$ be the subclass of normalized analytic functions f and satisfy Re $e^{i\delta}\{\alpha f'(z)+\beta z f''(z)\}>\gamma$ in the open unit disk $D=\{z\in\mathbb{C}:|z|<1\}$ for some $\alpha>0,\beta>0$ and $\gamma\in\mathbb{R}(0\le\gamma<\alpha)$ where $\alpha\cos\delta-\gamma>0$. In this paper, we find the extreme points of $\mathcal{R}_{\alpha,\beta}(\gamma,\delta)$ and then obtain sharp bound for a_n and bound for $f(z)$.

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1. Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ defined by

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \tag{1}$$

which are analytic in the open unit disk with the normalizations $f(0)=f'(0)-1=0$. A function $f(z)\in\mathcal{A}$ is said to be member of the subclass $\mathcal{R}_{\alpha,\beta}(\gamma,\delta)$ of $\mathcal{A}$ if it satisfies

$$\text{Re } e^{i\delta}\{\alpha f'(z)+\beta z f''(z)\}>\gamma,\quad z\in D. \tag{2}$$

for some $\alpha>0,\beta>0$ and $\gamma\in\mathbb{R}(0\le\gamma<\alpha)$ where $\alpha\cos\delta-\gamma>0$.

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In a later paper, some researchers already considered many of the subclasses of $\mathcal{R}_{\alpha,\beta}(\gamma,\delta)$ such as [1] which introduced the class of $\mathcal{R}_{\alpha,\beta}(\gamma,0)$, [2] that initiated the class of $\mathcal{R}_{1,\beta}(\gamma,0)$ and followed by [3] and [4] for $\mathcal{R}_{1,1}(\gamma,0)$. The extremal function will guarantee the sharp results for each basic properties. With that, we will find one of extensively basic properties which is coefficient bound or bound for a_n and also the bound of $f(z)$ for the class $\mathcal{R}_{\alpha,\beta}(\gamma,\delta)$.

2. Results

Theorem 1. *If the function $f\in\mathcal{A}$ is in the class of $\mathcal{R}_{\alpha,\beta}(\gamma,\delta)$ then there exists $p\in P$ define by*

$$\frac{e^{i\delta}(\alpha f'(z)+\beta zf''(z))-\gamma-i\alpha\sin\delta}{\alpha\cos\delta-\gamma}=p(z)\quad(z\in D).$$

Proof. We relate function in $\mathcal{R}_{\alpha,\beta}(\gamma,\delta)$ to the class of functions with positive real part, P in form of $p(z)=1+\sum_{n=1}^{\infty}c_nz^n$. By differentiating (1) and then substitute in (2), we can see that the result follows. □

Theorem 2. *Let $f\in\mathcal{R}_{\alpha,\beta}(\gamma,\delta)$. Then for some probability measure μ on the unit circle $X:|x|=1$,*

$$f(z)=(1-\frac{2Ae^{-i\delta}}{\alpha})z+2e^{-i\delta}A\bar{x}\int_{|x|=1}\sum_{n=0}^{\infty}\frac{(xz)^{n+1}}{(n+1)(n\beta+\alpha)}d\mu(x)\qquad(3)$$

where $A=\alpha\cos\delta-\gamma$. Conversely, if $f(z)$ is given by (3), then $f\in\mathcal{R}_{\alpha,\beta}(\gamma,\delta)$. For fixed α,β,γ and δ, $\mathcal{R}_{\alpha,\beta}(\gamma,\delta)$ and the probability measures, μ defined on X are one-to-one by expression (3).

Proof. By the aid of Carathéodory expressions of functions in P, we have

$$\frac{e^{i\delta}(\alpha f'(z)+\beta zf''(z))-\gamma-i\alpha\sin\delta}{\alpha\cos\delta-\gamma}=\int_{|x|=1}\frac{1+xz}{1-xz}d\mu(x).$$

Since $\frac{1+xz}{1-xz}=1+2\sum_{n=1}^{\infty}x_nz^n$, it follows that

$$e^{i\delta}(\alpha f'(z)+\beta zf''(z))=\alpha e^{i\delta}+2A\int_{|x|=1}\sum_{n=1}^{\infty}x_nz^nd\mu(x).\qquad(4)$$

So we obtain

$$\int_0^z e^{i\delta}(\frac{\alpha f'(\zeta)}{\beta} + \zeta f''(\zeta))\zeta^{\frac{\alpha}{\beta}-1} d\zeta$$
$$= \int_0^z \{\alpha e^{i\delta} + 2A \int_{|x|=1} \sum_{n=1}^{\infty} x_n z^n d\mu(x)\} \frac{\zeta^{\frac{\alpha}{\beta}-1}}{\beta} d\zeta$$

that is

$$f'(z) = 1 + 2e^{-i\delta} A \int_{|x|=1} \sum_{n=1}^{\infty} \frac{x_n z^n}{n\beta + \alpha} d\mu(x).$$

or

$$f'(z) = 1 - \frac{2e^{-i\delta} A}{\alpha} + 2e^{-i\delta} A \int_{|x|=1} \sum_{n=0}^{\infty} \frac{x_n z^n}{n\beta + \alpha} d\mu(x). \tag{5}$$

Integrating (5) again, we have

$$f(z) = (1 - \frac{2Ae^{-i\delta}}{\alpha})z + 2Ae^{-i\delta}\bar{x} \int_{|x|=1} \sum_{n=0}^{\infty} \frac{x_{n+1} z^{n+1}}{(n+1)(n\beta + \alpha)} d\mu(x).$$

Therefore, this completes the proof of this theorem. □

Remark 3. With specific value of parameters α, β, γ and δ, (3) can be reduced to the results obtained by [1] and [2]. Note that (3) is also true for $\mathcal{R}_{1,1}(\gamma, 0)$, if $f(z)$ is analytic in D and satisfy $Re\{f'(z) + zf''(z)\} > \gamma$ where $\gamma < 1$ and $z \in D$, then

$$f(z) = (2\gamma - 1)z + 2(1 - \gamma)\bar{x} \int_{|x|=1} \sum_{n=0}^{\infty} \frac{x_{n+1} z^{n+1}}{(n+1)^2} d\mu(x)$$
$$= \int_{|x|=1} (\int_0^z \frac{(2\gamma - 1)\zeta + 2(\gamma - 1)\bar{x} \log 1 - x\zeta}{\zeta} d\zeta) d\mu(x)$$

which was obtained by [4].

Corollary 4. *The extreme points of* $\mathcal{R}_{\alpha,\beta}(\gamma, \delta)$ *are*

$$f_x(z) = (1 - \frac{2Ae^{-i\delta}}{\alpha})z + 2Ae^{-i\delta}\bar{x} \sum_{n=0}^{\infty} \frac{x_{n+1} z^{n+1}}{(n+1)(n\beta + \alpha)}, \tag{6}$$

also, its derivatives are

$$f'_x(z) = 1 + 2e^{-i\delta} A \sum_{n=0}^{\infty} \frac{x_n z^n}{(n\beta + \alpha)}$$

where $|x| = 1$ *and* $A = \alpha \cos\delta - \gamma$.

Proof. Using the notation $f_x(z)$, (3) can be expressed as

$$f_\mu(z) = \int_{|x|=1} f_x(z)d\mu(x).$$

By Theorem 2, the map $\mu \to f_\mu$ is one-to-one. According to [5], the assertion follows. Then, we differentiate the results of $f_x(z)$ in order to get the $f'_x(z)$. □

Theorem 5. *If $f \in \mathcal{R}_{\alpha,\beta}(\gamma,\delta)$ and $A = \alpha\cos\delta - \gamma$, then*

$$|a_n| \le \frac{2A}{((n-1)\beta+\alpha)n}$$

where $n = 2, 3, 4, ...$ Equality holds for the function $f(z)$ given by (6) and the results are sharp.

Proof. Note that the coefficient estimates are maximized at the extreme points. From (6), it can be represented as

$$f(z) = z + 2e^{-i\delta}A\sum_{n=2}^{\infty}\frac{x_{n-1}z^n}{n((n-1)\beta+\alpha)}, |x| = 1. \tag{7}$$

By comparing (1) and (7), the result follows. □

Theorem 6. *If $f \in \mathcal{R}_{\alpha,\beta}(\gamma,\delta)$, then for $|z| = r < 1$ and $|x| = 1$,*

$$|f(z)| \le r + 2A\sum_{n=2}^{\infty}\frac{r^n}{n((n-1)\beta+\alpha)}. \tag{8}$$

where $A = \alpha\cos\delta - \gamma$.

Proof. Note that $|x| = 1$, $|e^{-i\delta}| = 1$ and $|z| = r < 1$, from (7)

$$\begin{aligned}|f(z)| &= |z + 2e^{-i\delta}A\sum_{n=2}^{\infty}\frac{x^{n-1}z^n}{n((n-1)\beta+\alpha)}|\\ &\le |z| + 2|e^{-i\delta}|A\sum_{n=2}^{\infty}\frac{|x|^{n-1}|z|^n}{n((n-1)\beta+\alpha)}\\ &= r + 2A\sum_{n=2}^{\infty}\frac{r^n}{n((n-1)\beta+\alpha)}.\end{aligned}$$

□

The proof is complete.

Corollary 7. *If* $\beta > 0$ *and* $\frac{\alpha}{\beta} = m(m = 2, 3, 4, ...)$, *then* $|f(z)| < 1+\frac{2A \log m}{\beta(m-1)}$ *where* $A = \alpha \cos \delta - \gamma$.

Proof. If $\beta > 0$ and $\frac{\alpha}{\beta} = m(m = 2, 3, 4, ...)$, from (8), we have

$$|f(z)| = r + \frac{2A}{\beta} \sum_{n=2}^{\infty} \frac{r^n}{n(n+m-1)},$$

or

$$\begin{aligned} |f(z)| &\leq r + \frac{2Ar^2}{\beta} \sum_{n=2}^{\infty} \frac{1}{n(n+m-1)} \\ &= r + \frac{2Ar^2}{\beta(m-1)} \sum_{n=2}^{\infty} (\frac{1}{n} - \frac{1}{n+m-1}) \\ &= r + \frac{2Ar^2}{\beta(m-1)} \sum_{n=2}^{m} \frac{1}{n} \\ &< r + \frac{2A \log m}{\beta(m-1)} r^2. \end{aligned}$$

Since $r \to 1$, therefore we have

$$|f(z)| < 1 + \frac{2A \log m}{\beta(m-1)}.$$

□

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