AN IDENTITY OF THE TWISTED $q$-EULER POLYNOMIALS ASSOCIATED WITH THE $p$-ADIC $q$-INTEGRALS ON $\mathbb{Z}_p$

Cheon Seoung Ryoo
Department of Mathematics
Hannam University
Daejeon, 306-791, KOREA

Abstract: In [6], we studied the twisted $q$-Euler numbers and polynomials. By using these numbers and polynomials, we investigate the alternating sums of powers of consecutive integers. By applying the symmetry of the fermionic $p$-adic $q$-integral on $\mathbb{Z}_p$, we give recurrence identities the twisted $q$-Euler polynomials.

AMS Subject Classification: 11B68, 11S40, 11S80
Key Words: Euler numbers and polynomials, $q$-Euler numbers and polynomials, $q$-Euler numbers and polynomials, alternating sums, the twisted $q$-Euler polynomials

1. Introduction

Throughout this paper, we always make use of the following notations: $\mathbb{C}$ denotes the set of complex numbers, $\mathbb{Z}_p$ denotes the ring of $p$-adic rational integers, $\mathbb{Q}_p$ denotes the field of $p$-adic rational numbers, and $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$.

Let $\nu_p$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of $q$-extension, $q$ is considered in many ways such as an
indeterminate, a complex number \( q \in \mathbb{C} \), or \( p \)-adic number \( q \in \mathbb{C}_p \). If \( q \in \mathbb{C} \) one normally assume that \( |q| < 1 \). If \( q \in \mathbb{C}_p \), we normally assume that \( |q - 1|_p < p^{-1} \) so that \( q^x = \exp(x \log q) \) for \( |x|_p \leq 1 \). Throughout this paper we use the notation:

\[
[x]_q = \frac{1 - q^x}{1 - q}, \quad [x]_{-q} = \frac{1 - (-q)^x}{1 + q} \tag{1.1}
\]

Hence, \( \lim_{q \to 1} [x] = x \) for any \( x \) with \( |x|_p \leq 1 \) in the present \( p \)-adic case. For \( g \in UD(\mathbb{Z}_p) = \{ g \mid g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function} \} \), the \( p \)-adic \( q \)-integral was defined by

\[
I_{-q}(g) = \int_{\mathbb{Z}_p} g(x)d\mu_{-q}(x) = \lim_{N \to \infty} \frac{[2]_q}{1 + q^{p^N}} \sum_{x=0}^{p^N-1} g(x)(-q)^x, \quad \text{see} \ [1-8]. \tag{1.2}
\]

If we take \( g_1(x) = g(x+1) \) in (1.1), then we easily see that

\[
qI_{-q}(g_1) + I_{-q}(g) = [2]_q g(0). \tag{1.3}
\]

Let \( T_p = \bigcup_{N \geq 1} C_{p^N} = \lim_{N \to \infty} C_{p^N} \), where \( C_{p^N} = \{ \zeta \mid \zeta^{p^N} = 1 \} \) is the cyclic group of order \( p^N \). For \( \zeta \in T_p \), we denote by \( \phi_\zeta : \mathbb{Z}_p \to \mathbb{C}_p \) the locally constant function \( x \mapsto \zeta^x \).

In [6], we defined the twisted \( q \)-Euler numbers and polynomials and investigate their properties. For \( q \in \mathbb{C}_p \) with \( |1 - q|_p \leq 1 \), and \( \zeta \in T_p \), the twisted \( q \)-Euler polynomials \( \tilde{E}_{n,q,\zeta}(x) \) are defined by

\[
\tilde{F}_{q,\zeta}(x, t) = \sum_{n=0}^{\infty} \tilde{E}_{n,q,\zeta}(x) \frac{t^n}{n!} = \frac{[2]_q}{\zeta q e^t + 1} e^{xt}. \tag{1.4}
\]

The following elementary properties of the \( q \)-Euler numbers \( \tilde{E}_{n,q,\zeta} \) and polynomials \( \tilde{E}_{n,q,\zeta}(x) \) are readily derived form (1.1), (1.2), (1.3) and (1.4) (see, for details, [6]). We, therefore, choose to omit details involved.
Theorem 1. (Witt formula). For $q \in \mathbb{C}_p$ with $|1 - q| < 1$ and $\zeta \in \mathbb{Z}_p$, we have

$$\tilde{E}_{n,q,\zeta} = \int_{\mathbb{Z}_p} \zeta^x x^n d\mu_{-q}(x),$$

$$\tilde{E}_{n,q,\zeta}(x) = \int_{\mathbb{Z}_p} \zeta^x (x + y)^n d\mu_{-q}(y).$$

Theorem 2. For any positive integer $n$, we have

$$\tilde{E}_{n,q,\zeta}(x) = \sum_{k=0}^{n} \binom{n}{k} \tilde{E}_{k,q,\zeta} x^{n-k}.$$

In this paper, by using the symmetry of $p$-adic $q$-integral on $\mathbb{Z}_p$, we obtain the recurrence identities the twisted $q$-Euler polynomials.

2. The Alternating Sums of Powers of Consecutive $q$-Integers

Let $q$ be a complex number with $|q| < 1$ and $\zeta$ be the $p^N$-th root of unity. By using (1.3), we give the alternating sums of powers of consecutive $q$-integers as follows:

$$\sum_{n=0}^{\infty} \tilde{E}_{n,q,\zeta} \frac{t^n}{n!} = \frac{[2]_q}{\zeta q e^t + 1} = [2]_q \sum_{n=0}^{\infty} (-1)^n \zeta^n q^n e^{nt}.$$

From the above, we obtain

$$- \sum_{n=0}^{\infty} (-1)^n \zeta^n q^n e^{(n+k)t} + \sum_{n=0}^{\infty} (-1)^{n-k} \zeta^{n-k} q^{n-k} e^{nt} = \sum_{n=0}^{k-1} (-1)^{n-k} \zeta^{n-k} q^{n-k} e^{nt}.$$

Thus, we have

$$- [2]_q \sum_{n=0}^{\infty} (-1)^n \zeta^n q^n e^{(n+k)t} + [2]_q (-1)^{-k} \zeta^{-k} q^{-k} \sum_{n=0}^{\infty} (-1)^n \zeta^n q^n e^{nt}$$

$$= [2]_q (-1)^{-k} \zeta^{-k} q^{-k} \sum_{n=0}^{k-1} (-1)^n \zeta^n q^n e^{nt}. \quad (2.1)$$

By using (1.3) and (1.4), and (2.1), we obtain

$$- \sum_{j=0}^{\infty} \tilde{E}_{j,q,\zeta}(k) \frac{t^j}{j!} + (-1)^{-k} \zeta^{-k} q^{-k} \sum_{j=0}^{\infty} \tilde{E}_{j,q,\zeta} \frac{t^j}{j!}$$
\[
[2]q \sum_{j=0}^{\infty} \left( (-1)^{-k} \zeta^{-k} q^{-k} \sum_{n=0}^{k-1} (-1)^n \zeta^n q^n j^j \right) \frac{t^j}{j!}.
\]

By comparing coefficients of \( \frac{t^j}{j!} \) in the above equation, we obtain

\[
\sum_{n=0}^{k-1} (-1)^n \zeta^n q^n j^j = \frac{(-1)^{k+1} \zeta^k q^k E_{j,q}\zeta(k) + E_{j,q}\zeta}{[2]q}.
\]

By using the above equation we arrive at the following theorem:

**Theorem 3.** Let \( k \) be a positive integer and \( q \in \mathbb{C} \) with \( |q| < 1 \). Then we obtain

\[
\tilde{T}_{j,q}\zeta(k-1) = \sum_{n=0}^{k-1} (-1)^n \zeta^n q^n j^j = \frac{(-1)^{k+1} \zeta^k q^k \tilde{E}_{j,q}\zeta(k) + \tilde{E}_{j,q}\zeta}{[2]q}.
\]

**Corollary 4.** For \( \zeta = 1 \), we have

\[
\lim_{q \to 1} \tilde{T}_{j,q}\zeta(k-1) = \sum_{n=0}^{k-1} (-1)^n q^n j^j = \frac{(-1)^{k+1} E_j(k) + E_j}{2},
\]

where \( E_j(x) \) and \( E_j \) denote the Euler polynomials and Euler numbers, respectively.

Next, we assume that \( q \in \mathbb{C}_p \) and \( \zeta \in T_p \). We obtain recurrence identities the second \( q \)-Euler polynomials and the \( q \)-analogue of alternating sums of powers of consecutive integers. By using (1.1), we have

\[
q^n I_{-q}(g(x)) + (-1)^{n-1} I_{-q}(g) = [2]q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l),
\]

where \( g_n(x) = g(x + n) \). If \( n \) is odd from the above, we obtain

\[
q^n I_{-q}(g(x)) + I_{-q}(g) = [2]q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l) \quad \text{(cf. [1-5])}.
\]

(2.2)

It will be more convenient to write (2.2) as the equivalent integral form

\[
q^n \int_{\mathbb{Z}_p} g(x+n) d\mu_{-q}(x) + \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = [2]q \sum_{k=0}^{n-1} (-1)^k q^k g(k).
\]

(2.3)
Substituting \( g(x) = \zeta^x e^{xt} \) into the above, we obtain
\[
\zeta^n q^n \int_{Z_p} \zeta^x e^{(x+n)t} d\mu_q(x) + \int_{Z_p} \zeta^x e^{xt} d\mu_q(x) = [2]_q \sum_{j=0}^{n-1} (-1)^j \zeta^j q^j e^{jt}.
\] (2.4)

After some elementary calculations, we have
\[
\int_{Z_p} \zeta^x e^{xt} d\mu_q(x) = \frac{[2]_q}{\zeta q e^t + 1},
\] (2.5)

\[
\int_{Z_p} \zeta^x e^{(x+n)t} d\mu_q(x) = e^{nt} \frac{[2]_q}{\zeta q e^t + 1}.
\]

By using (2.4) and (2.5), we have
\[
\zeta^n q^n \int_{Z_p} \zeta^x e^{(x+n)t} d\mu_q(x) + \int_{Z_p} \zeta^x e^{xt} d\mu_q(x) = \frac{[2]_q (1 + \zeta^n q^n e^{nt})}{\zeta q e^t + 1}.
\]

From the above, we get
\[
\frac{[2]_q (1 + \zeta^n q^n e^{nt})}{\zeta q e^t + 1} = \frac{[2]_q \int_{Z_p} \zeta^x e^{xt} d\mu_q(x)}{\int_{Z_p} \zeta^x q^{(n-1)x} e^{ntx} d\mu_q(x)}.
\] (2.6)

By substituting Taylor series of \( e^{xt} \) into (2.4), we obtain
\[
\sum_{m=0}^{\infty} \left( \zeta^n q^n \int_{Z_p} \zeta^x (x+n)^m d\mu_q(x) + \int_{Z_p} \zeta^x x^m d\mu_q(x) \right) \frac{t^m}{m!} = \sum_{m=0}^{\infty} \left( [2]_q \sum_{j=0}^{n-1} (-1)^j \zeta^j q^j j^m \right) \frac{t^m}{m!}.
\]

By comparing coefficients \( \frac{t^m}{m!} \) in the above equation, we obtain
\[
\zeta^n q^n \sum_{k=0}^{m} \binom{m}{k} n^{m-k} \int_{Z_p} \zeta^x x^k d\mu_q(x) + \int_{Z_p} \zeta^x x^m d\mu_q(x) = [2]_q \sum_{j=0}^{n-1} (-1)^j \zeta^j q^j j^m.
\]

By using Theorem 3, we have
\[
\zeta^n q^n \sum_{k=0}^{m} \binom{m}{k} n^{m-k} \int_{Z_p} \zeta^x x^k d\mu_q(x) + \int_{Z_p} \zeta^x x^m d\mu_q(x) = [2]_q \tilde{T}_{m,q,\zeta}(n-1).
\] (2.7)

By using (2.6) and (2.7), we arrive at the following theorem:
Theorem 5. Let \( n \) be odd positive integer. Then we have
\[
\frac{\int_{\mathbb{Z}_p} \zeta^n e^{xt} d\mu_q(x)}{\int_{\mathbb{Z}_p} \zeta^{nx} q^{(n-1)x} e^{ntx} d\mu_q(x)} = \sum_{m=0}^{\infty} \left( \tilde{T}_{m,q} \zeta(n-1) \right) \frac{t^m}{m!}.
\]

Let \( w_1 \) and \( w_2 \) be odd positive integers. By (2.5), Theorem 5, and after some elementary calculations, we obtain the following theorem.

Theorem 6. Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we have
\[
\frac{\int_{\mathbb{Z}_p} \zeta^{w_1x} e^{w_2xt} d\mu_q(x)}{\int_{\mathbb{Z}_p} \zeta^{w_1w_2x} q^{(w_1w_2-1)x} e^{w_1w_2tx} d\mu_q(x)} = \frac{[2]_{q^{w_2}}}{[2]_q} \sum_{m=0}^{\infty} \left( \tilde{T}_{m,q} \zeta w_2 (w-1)w_2^m \right) \frac{t^m}{m!}.
\]

(2.8)

By (1.1), we obtain
\[
\frac{\int_{\mathbb{Z}_p} \zeta^{w_1x} e^{w_2xt} d\mu_q(x)}{\int_{\mathbb{Z}_p} \zeta^{w_1x} e^{w_2xt} q^{w_1w_2-1}x e^{w_1w_2tx} d\mu_q(x)} = \frac{\zeta^{w_1x} e^{w_2xt} d\mu_q(x)}{\int_{\mathbb{Z}_p} \zeta^{w_1x} e^{w_2xt} q^{w_1w_2-1}x e^{w_1w_2tx} d\mu_q(x)} = \frac{\zeta^{w_1x} e^{w_2xt} d\mu_q(x)}{\int_{\mathbb{Z}_p} \zeta^{w_1x} e^{w_2xt} q^{w_1w_2-1}x e^{w_1w_2tx} d\mu_q(x)}.
\]

(2.9)

By using (2.8) and (2.9), after elementary calculations, we obtain
\[
a = \left( \int_{\mathbb{Z}_p} \zeta^{w_1x} e^{w_1x + w_1w_2x} t d\mu_q(w_1x) \right)
\times \left( \frac{\int_{\mathbb{Z}_p} \zeta^{w_1x} e^{w_1x + w_1w_2x} t d\mu_q(w_1x)}{\int_{\mathbb{Z}_p} \zeta^{w_1w_2x} q^{w_1w_2-1}x e^{w_1w_2tx} d\mu_q(x)} \right)
\]
\[
= \left( \sum_{m=0}^{\infty} \tilde{E}_{m,q} w_1 \zeta w_1 (w_2x) w_1^m \frac{t^m}{m!} \right) \left( \frac{[2]_{q^{w_2}}}{[2]_q} \sum_{m=0}^{\infty} \tilde{T}_{m,q} \zeta w_2 (w_1-1)w_2^m \frac{t^m}{m!} \right).
\]

(2.10)

By using Cauchy product in the above, we have
\[
a = \sum_{m=0}^{\infty} \left( \frac{[2]_{q^{w_2}}}{[2]_q} \sum_{j=0}^{m} \binom{m}{j} \tilde{E}_{j,q} w_1 \zeta w_1 (w_2x) w_1^j \tilde{T}_{m-j,q} \zeta w_2 (w_1-1)w_2^{m-j} \right) \frac{t^m}{m!}.
\]

(2.11)

By using the symmetry in (2.10), we obtain
AN IDENTITY OF THE TWISTED $Q$-EULER...

$$a = \left( \int_{\mathbb{Z}_p} \zeta^{w_1 x_2} e^{(w_2 x_2 + w_1 w_2 x)} d\mu_{-q} \right) (x_2)$$

$$\times \left( \int_{\mathbb{Z}_p} \zeta^{w_1 x_1} e^{x_1 w_1 t} d\mu_{-q} (x_1) \right)$$

$$= \left( \sum_{m=0}^{\infty} \tilde{E}_{m,q^w_1,\zeta^w_2} (w_1 x) w_2^m \frac{t^m}{m!} \right) \left( \frac{[2]_{q^w_1}}{[2]_q} \sum_{m=0}^{\infty} \tilde{T}_{m,q^w_1,\zeta^w_1} (w_2 - 1) w_1^m \frac{t^m}{m!} \right).$$

Thus we obtain

$$a = \sum_{m=0}^{\infty} \left( \frac{[2]_{q^w_1}}{[2]_q} \sum_{j=0}^{m} \binom{m}{j} \tilde{E}_{j,q^w_1,\zeta^w_2} (w_1 x) w_2^j \tilde{T}_{m-j,q^w_2,\zeta^w_2} (w_1 - 1) w_1^{m-j} \right) \frac{t^m}{m!}.$$  \hspace{1cm} (2.12)

By comparing coefficients $\frac{t^m}{m!}$ in the both sides of (2.11) and (2.12), we arrive at the following theorem.

**Theorem 7.** Let $w_1$ and $w_2$ be odd positive integers. Then we obtain

$$[2]_{q^w_2} \sum_{j=0}^{m} \binom{m}{j} \tilde{E}_{j,q^w_1,\zeta^w_1} (w_2 x) w_1^j \tilde{T}_{m-j,q^w_2,\zeta^w_2} (w_1 - 1) w_1^{m-j},$$

where $\tilde{E}_{k,q,\zeta}(x)$ and $\tilde{T}_{m,q,\zeta}(k)$ denote the twisted $q$-Euler polynomials and the $q$-analogue of alternating sums of powers of consecutive integers, respectively.

By using Theorem 2, we have the following corollary:

**Corollary 8.** Let $w_1$ and $w_2$ be odd positive integers. Then we obtain

$$[2]_{q^w_1} \sum_{j=0}^{m} \sum_{k=0}^{j} \binom{m}{j} \binom{j}{k} w_1^{m-k} w_2^j x^{j-k} \tilde{E}_{k,q,\zeta^w_2} \tilde{T}_{m-j,q^w_1,\zeta^w_1} (w_2 - 1)$$

$$= [2]_{q^w_2} \sum_{j=0}^{m} \sum_{k=0}^{j} \binom{m}{j} \binom{j}{k} w_2^{m-k} x^{j-k} \tilde{E}_{k,q,\zeta^w_1} \tilde{T}_{m-j,q^w_2,\zeta^w_2} (w_1 - 1).$$
By using (2.9), we have

\[
a = \left( e^{w_1 w_2 x t} \int_{Z_p} ^{w_1 w_2 x} e^{x_1 w_1 t} d \mu_{-q w_1} (x_1) \right) \left( \frac{\int_{Z_p} \zeta^{w_2 x_1} e^{x_2 w_2 t} d \mu_{-q w_2} (x_2)}{\int_{Z_p} \zeta^{w_1 w_2 x} e^{x_1 w_1 t} d \mu_{-q} (x)} \right)
\]

\[
= \frac{[2]_{q^{w_2}}}{[2]_{q}} \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j q^{w_2 j}} \int_{Z_p} \zeta^{w_1 x_1} e^{x_1 w_1 t} \left( x_1 + w_2 x + j \frac{w_2}{w_1} \right) d \mu_{-q w_1} (x_1)
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{[2]_{q^{w_2}}}{[2]_{q}} \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j q^{w_2 j}} \tilde{E}_{n,q^{w_2},\zeta^{w_2}} \left( w_2 x + j \frac{w_2}{w_1} \right) \right) \frac{t^n}{n!}.
\]

By using the symmetry property in (2.13), we also have

\[
a = \left( e^{w_1 w_2 x t} \int_{Z_p} \zeta^{w_2 x_2} e^{x_2 w_2 t} d \mu_{-q w_2} (x_2) \right) \left( \frac{\int_{Z_p} \zeta^{w_1 x_1} e^{x_1 w_1 t} d \mu_{-q w_1} (x_1)}{\int_{Z_p} \zeta^{w_1 w_2 x} e^{x_1 w_1 t} d \mu_{-q} (x)} \right)
\]

\[
= \frac{[2]_{q^{w_1}}}{[2]_{q}} \sum_{j=0}^{w_2-1} (-1)^j \zeta^{w_1 j q^{w_1 j}} \int_{Z_p} \zeta^{w_2 x_2} e^{x_1 w_1 t} \left( x_2 + w_2 x + j \frac{w_1}{w_2} \right) d \mu_{-q w_2} (x_2)
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{[2]_{q^{w_1}}}{[2]_{q}} \sum_{j=0}^{w_2-1} (-1)^j \zeta^{w_1 j q^{w_1 j}} \tilde{E}_{n,q^{w_2},\zeta^{w_2}} \left( w_1 x + j \frac{w_1}{w_2} \right) \right) \frac{t^n}{n!}.
\]

By comparing coefficients \( \frac{t^n}{n!} \) in the both sides of (2.13) and (2.14), we have the following theorem.

**Theorem 9.** Let \( w_1 \) and \( w_2 \) be odd positive integers. Then we have

\[
[2]_{q^{w_2}} \sum_{j=0}^{w_1-1} (-1)^j \zeta^{w_2 j q^{w_2 j}} \tilde{E}_{n,q^{w_2},\zeta^{w_2}} \left( w_2 x + j \frac{w_2}{w_1} \right) w_1^n \tag{2.15}
\]

**Corollary 10.** Let \( w_1 \) and \( w_2 \) be odd positive integers. If \( q \to 1 \) and \( \zeta = 1 \), we have

\[
\sum_{j=0}^{w_1-1} (-1)^j E_n \left( w_2 x + j \frac{w_2}{w_1} \right) w_1^n = \sum_{j=0}^{w_2-1} (-1)^j E_n \left( w_1 x + j \frac{w_1}{w_2} \right) w_2^n.
\]
Substituting $w_1 = 1$ into (2.15), we arrive at the following corollary.

**Corollary 11.** Let $w_2$ be odd positive integer. Then we obtain

$$
\tilde{E}_{n,q,\zeta}(x) = \left[\begin{array}{c} 2 \\ q \end{array}\right]_{q^w_2} \sum_{j=0}^{w_2-1} (-1)^j \zeta^j q^j \tilde{E}_{n,q,w_2,\zeta} \left(\frac{x+j}{w_2}\right) w^n_2.
$$

**References**


