SOME GRAPH PARAMETERS OF FAN GRAPH

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Abstract: The fan graph $F_{m,n} \cong \overline{K}_m \vee P_n$, is the graph that $V(F_{m,n}) = V(\overline{K}_m) \cup V(P_n)$ and $E(F_{m,n}) = E(P_n) \cup \{uv | u \in V(\overline{K}_m), v \in V(P_n)\}$. Let $\mathcal{G}$ be a set of all simple graphs. The function $f : \mathcal{G} \rightarrow \mathbb{Z}^+$ is called graph parameter, if $G \cong H$, then $f(G) = f(H)$. In this paper, we determine generalizations of some graph parameters (clique number, independent number, vertex covering number and domination number) of fan graph.

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1. Introduction

In this paper must be simple graphs which can be trivial graph. The fan graph $F_{m,n} \cong \overline{K}_m \vee P_n$, is the graph that $V(F_{m,n}) = V(\overline{K}_m) \cup V(P_n)$ and $E(F_{m,n}) = E(P_n) \cup \{uv | u \in V(\overline{K}_m), v \in V(P_n)\}$. Clearly, $|V(F_{m,n})| = m + n$, $|E(F_{m,n})| = (n - 1) + mn$. 

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In [2], there are some properties about joined graph. We recall here.

**Theorem 1.1.** (see [2]) Any joined graphs are always connected.

**Theorem 1.2.** (see [2]) Let $G_1$ and $G_2$ be graph. Then $G_1 \lor G_2 = G_1 \cup G_2$.

Let $\mathcal{G}$ be a set of all simple graphs. The function $f : \mathcal{G} \to \mathbb{Z}^+$ is called a graph parameter, if $G \sim H$, then $f(G) = f(H)$. Next, we give the definitions about some graph parameters.

A complete subgraph of graph $G$ is called a *clique* of $G$, the maximum order of clique of $G$ is called the *clique number* of $G$, denoted by $\omega(G)$.

A subset $U$ of the vertex set $V(G)$ of $G$ is said to be an independent set of $G$ if the induced subgraph $G[U]$ is a empty graph. An independent set of $G$ with maximum number of vertices is called a maximum independent set of $G$. The number of vertices of maximum independent set of $G$ is called the independent number of $G$, denoted by $\alpha(G)$.

A vertex of graph $G$ is said to cover the edges incident with it, and a vertex cover of a graph $G$ is a set of vertices covering all the edges of $G$. The minimum cardinality of a vertex cover of a graph $G$ is called the vertex covering number of $G$, denoted by $\beta(G)$.

A dominating set (or domset) of graph $G$ is a subset $D$ of $V(G)$ such that each vertex of $V - D$ is adjacent to at least one vertex of $D$. The minimum cardinality of a dominating set of a graph $G$ is called the domination number of $G$, denoted by $\gamma(G)$.

Next, we are going to prove that clique number of fan graph.

**Theorem 1.3.** Let $F_{m,n} \cong \overline{K}_m \lor P_n$. Then $\omega(F_{m,n}) = 3$.

*Proof.* Since $\omega(\overline{K}_m) = 1$ and $\omega(P_n) = 2$. Then there are $K_1$ and $K_2$ as complete subgraphs of $\overline{K}_m$ and $P_n$ respectively. It is easy to see that $K_1 \lor K_2 = K_3$ is complete subgraph of $F_{m,n}$.

So $\omega(G) \geq 3$.

Suppose that $\omega(G) > 3$. Then there exists $v \in [V(\overline{K}_m) - V(K_1)] \cup [V(P_n) - V(K_2)]$ adjacent with $v_i$, $v_j$ and $v_{j+1}$. So $\omega(\overline{K}_m) > 1$ or $\omega(P_n) > 2$, this contradicts to the assumption.

Hence $\omega(F_{m,n}) = 3$. \qed
2. Independent Number of Fan Graph.

We begin this section with some remarks which show the character of an independent set.

**Remark 2.1.** $I(G) = \{v_1, v_2, ..., v_k\}$ is independent set of connected graph $G$ if:

1. $v_i$ is not adjacent with $v_j$ for all $i \neq j$ and $i, j = 1, 2, ..., k$ and
2. $V(G) - I(G) = \bigcup_{i=1}^{k} N(v_i)$.

**Lemma 2.2.** (see [1]) Let $G$ be a graph. Then $\alpha(G) = \omega(G)$.

**Theorem 2.3.** Let $F_{m,n} \cong \overline{K}_m \vee P_n$. Then $\alpha(F_{m,n}) = \max\{m, \lceil \frac{n}{2} \rceil\}$.

**Proof.** Since $\alpha(K_m) = m$, $\alpha(P_n) = \lceil \frac{n}{2} \rceil$. Assume that maximum independent set of $\overline{K}_m, P_n$ is

$$V(\overline{K}_m) = I_1 = \{v_1, v_2, v_3, ..., v_m\},$$

$$I_2 = \{u_1, u_3, u_5, ..., u_{2\lceil \frac{n}{2}\rceil - 1}\}.$$

Suppose a vertex $u_k \in I_2$. Because $u_k$ is not adjacent with another vertices in $I_2$, and $\bigcup_{u_k \in I_2} N_{F_{m,n}}(u_k) = V(\overline{K}_m) \cup (V(P_n) - I_2)$. Thus $I_2$ is independent set of $F_{m,n}$. 

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**Figure 1:** $K_1 \vee K_2 = K_3$
Similarly, we get \( I_1 \) is independent set of \( F_{m,n} \).

Hence \( \alpha(F_{m,n}) \geq \max\{m, \lceil \frac{n}{2} \rceil \} \).

Suppose that \( \alpha(F_{m,n}) > \max\{m, \lceil \frac{n}{2} \rceil \} \). Then there exists \( u \) (or \( v \)) \( \in V(F_{m,n}) - (I_1 \cup I_2) \) is not adjacent with another vertices in \( I_2 \) (or \( I_1 \)), it is not true.

Hence \( \alpha(F_{m,n}) = \max\{m, \lceil \frac{n}{2} \rceil \} \).

![Figure 2: The case that \( I_2 \) is independent set of \( F_{m,n} \).](image)

On the other hand, we can show by Theorem 1.2 and Lemma 2.2 as follows:

\[
\alpha(F_{m,n}) = \alpha(K_m \lor P_n) \\
= \omega(K_m \lor P_n) \\
= \omega(K_m \cup \overline{P_n}) \\
= max\{\omega(K_m), \omega(\overline{P_n})\} \\
= max\{\alpha(K_m), \alpha(P_n)\} \\
= max\{m, \lceil \frac{n}{2} \rceil \}.
\]

3. Vertex Covering Number of Fan Graph.

We begin this section by giving Lemma 3.1 that shows a relation of independent number and vertex covering number.

**Lemma 3.1.** (see [1]) Let \( G \) be a simple graph with order \( n \). Then \( \alpha(G) + \beta(G) = n \).
Theorem 3.2. Let $F_{m,n} \cong \overline{K_m} \vee P_n$. Then $\beta(F_{m,n}) = \min\{n, m + \lfloor \frac{n}{2} \rfloor\}$.

Proof. We get $\beta(\overline{K_m}) = 0$ and $\beta(P_n) = \lfloor \frac{n}{2} \rfloor$. Assume that minimum vertex covering set of $P_n$ is $C = \{u_2, u_4, u_6, ..., u_{2\lfloor \frac{n}{2} \rfloor}\}$.

Suppose a vertex $u_k \in C$. So $N_{F_{m,n}}(u_k) = V(\overline{K_m}) \cup (V(P_n) - C)$, all edges such that adjacent with $u_k$ are $\{u_kv_i$ for all $i = 1, 2, 3, ..., m\} \cup \{u_ku_j$ for some $j = 1, 3, 5, ..., 2\lfloor \frac{n}{2} \rfloor + 1\}$.

Then all edges, which are adjacent with $u_k$ for all $k = 2, 4, 6, ..., \lfloor \frac{n}{2} \rfloor$ are $A = \{u_kv_i$ for all $i = 1, 2, 3, ..., m\} \cup \{u_ku_j$ for all $j = 1, 3, 5, ..., 2\lfloor \frac{n}{2} \rfloor + 1\}$.

Let $B = \{v_iu_j$ for all $i = 1, 2, 3, ..., m$ and $j = 1, 3, 5, ..., 2\lfloor \frac{n}{2} \rfloor + 1\}$.

So we have $A \cup B = E(F_{m,n})$ and also $V(\overline{K_m}) \cup C$ is vertex covering set of $F_{m,n}$.

In the same, vertex in $V(\overline{K_m})$ adjacent with all vertex in $V(P_n)$, so $V(P_n)$ is the vertex covering of $F_{m,n}$.

Hence $\beta(F_{m,n}) \leq \min\{n, m + \lfloor \frac{n}{2} \rfloor\}$.

Suppose that $\beta(F_{m,n}) < \min\{n, m + \lfloor \frac{n}{2} \rfloor\}$. Then there exist vertices in $V(\overline{K_m}) \cup C$ which is not adjacent with $u_k$ for all $k = 1, 3, 5, ..., 2\lfloor \frac{n}{2} \rfloor + 1$. But is not true. Hence $\beta(F_{m,n}) = \min\{n, m + \lfloor \frac{n}{2} \rfloor\}$.

On the other hand, we can show by Theorem 2.3 and Lemma 3.1, we can also show that

$$\beta(F_{m,n}) + \alpha(F_{m,n}) = m + n$$

$$\beta(F_{m,n}) = (m + n) - \alpha(F_{m,n})$$
\[= (m + n) - \max\{m, \left\lceil \frac{n}{2} \right\rceil\}\]
\[= (m + n) + \min\{-m, -\left\lceil \frac{n}{2} \right\rceil\}\]
\[= \min\{(m + n) - m, (m + n) - \left\lceil \frac{n}{2} \right\rceil\}\]
\[= \min\{n, m + \left\lceil \frac{n}{2} \right\rceil\}.
\]

4. Domination Number of Fan Graph

Next, we show a domination number of Fan Graph.

**Theorem 4.1.** Let \(F_{m,n} \cong \overline{K}_m \lor P_n\). Then:

\[
\gamma(F_{m,n}) = \begin{cases} 
1, & m = 1 \text{ or } n = 1 \\
2, & m \neq 1 \text{ and } n \neq 1.
\end{cases}
\]

**Proof.** Suppose \(m = 1\), the vertex \(v \in V(\overline{K}_m)\), we have \(N_{F_{m,n}}(v) = \{u \in V(P_n)\}\). In the same, if \(n = 1\), the vertex \(u \in V(P_n)\), we get \(N_{F_{m,n}}(u) = \{v \in V(\overline{K}_m)\}\). Hence \(\gamma(F_{m,n}) = 1\), where \(m = 1 \text{ or } n = 1\).

Suppose \(m, n \neq 1\). Choose \(v \in V(\overline{K}_m)\), we have \(N_{F_{m,n}}(v) = \{u \in V(P_n)\}\), and choose \(u \in V(P_n)\), we get \(N_{F_{m,n}}(u) = \{v \in V(\overline{K}_m)\}\). Hence \(\gamma(F_{m,n}) = 2\), where \(m, n \neq 1\).

Figure 4: The case that \(m = 1 \text{ or } n = 1\).
Figure 5: The case that \( m \neq 1 \) and \( n \neq 1 \).

References

