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SOME GRAPH PARAMETERS OF FAN GRAPH

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Abstract: The fan graph $F_{m,n} \cong \overline{K}_m \vee P_n$, is the graph that $V(F_{m,n}) = V(\overline{K}_m) \cup V(P_n)$ and $E(F_{m,n}) = E(P_n) \cup \{uv | u \in V(\overline{K}_m), v \in V(P_n)\}$. Let $\Im$ be a set of all simple graphs. The function $f : \Im \to Z^+$ is called graph parameter, if $G \cong H$, then $f(G) = f(H)$. In this paper, we determine generalizations of some graph parameters (clique number, independent number, vertex covering number and domination number) of fan graph.

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Key Words: joined graph, clique number, independent number, vertex covering number, domination number

1. Introduction

In this paper must be simple graphs which can be trivial graph. The fan graph $F_{m,n} \cong \overline{K}_m \vee P_n$, is the graph that $V(F_{m,n}) = V(\overline{K}_m) \cup V(P_n)$ and $E(F_{m,n}) = E(P_n) \cup \{uv | u \in V(\overline{K}_m), v \in V(P_n)\}$. Clearly, $| V(F_{m,n}) | = m + n$, $| E(F_{m,n}) | = (n - 1) + mn$.

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In [2], there are some properties about joined graph. We recall here.

Theorem 1.1. (see [2]) *Any joined graphs are always connected.*

Theorem 1.2. (see [2]) *Let G_1 and G_2 be graph. Then* $\overline{G_1 \vee G_2} = \overline{G_1} \cup \overline{G_2}$.

Let $\Im$ be a set of all simple graphs. The function $f : \Im \to Z^+$ is called graph parameter, if $G \cong H$, then $f(G) = f(H)$. Next, we give the definitions about some graph parameters.

A complete subgraph of graph G is called a *clique* of G, the maximum order of clique of G is called the *clique number* of G, denoted by $\omega(G)$.

A subset U of the vertex set $V(G)$ of G is said to be an *independent set* of G if the induced subgraph $G[U]$ is a empty graph. An independent set of G with maximum number of vertices is called a *maximum independent set* of G. The number of vertices of maximum independent set of G is called the *independent number* of G, denoted by $\alpha(G)$.

A vertex of graph G is said to cover the *edges incident* with it, and a vertex cover of a graph G is a set of vertices covering all the edges of G. The minimum cardinality of a vertex cover of a graph G is called the *vertex covering number* of G, denoted by $\beta(G)$.

A dominating set (or domset) of graph G is a subset D of $V(G)$ such that each vertex of $V - D$ is adjacent to at least one vertex of D. The minimum cardinality of a dominating set of a graph G is called the *domination number* of G, denoted by $\gamma(G)$.

Next, we are going to prove that clique number of fan graph.

Theorem 1.3. *Let $F_{m,n} \cong \overline{K}_m \vee P_n$. Then $\omega(F_{m,n}) = 3$.*

Proof. Since $\omega(\overline{K}_m) = 1$ and $\omega(P_n) = 2$. Then there are K_1 and K_2 as complete subgraphs of $\overline{K}_m$ and P_n respectively. It is easy to see that $K_1 \vee K_2 = K_3$ is complete subgraph of $F_{m,n}$.

So $\omega(G) \geq 3$.

Suppose that $\omega(G) > 3$. Then there exists $v \in [V(\overline{K}_m) - V(K_1)] \cup [V(P_n) - V(K_2)]$ adjacent with v_i, v_j and v_{j+1}. So $\omega(\overline{K}_m) > 1$ or $\omega(P_n) > 2$, this contradicts to the assumption.

Hence $\omega(F_{m,n}) = 3$.

□

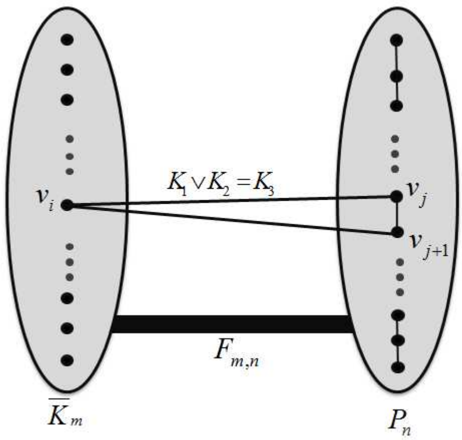

Figure 1: $K_1 \vee K_2 = K_3$

2. Independent Number of Fan Graph.

We begin this section with some remarks which show the character of an independent set.

Remark 2.1. $I(G) = \{v_1, v_2, ..., v_k\}$ is independent set of connected graph G if:

(1) v_i is not adjacent with v_j for all $i \neq j$ and $i, j = 1, 2, ..., k$ and

(2) $V(G) - I(G) = \bigcup_{i=1}^{k} N(v_i)$.

Lemma 2.2. (see [1]) *Let G be a graph. Then $\alpha(G) = \omega(\overline{G})$.*

Theorem 2.3. *Let $F_{m,n} \cong \overline{K}_m \vee P_n$. Then $\alpha(F_{m,n}) = max\{m, \lceil \frac{n}{2} \rceil\}$.*

Proof. Since $\alpha(\overline{K}_m) = m, \alpha(P_n) = \lceil \frac{n}{2} \rceil$. Assume that maximum independent set of $\overline{K}_m, P_n$ is

$$V(\overline{K}_m) = I_1 = \{v_1, v_2, v_3, ..., v_m\},$$

$$I_2 = \{u_1, u_3, u_5, ..., u_{2\lceil \frac{n}{2} \rceil - 1}\}.$$

Suppose a vertex $u_k \in I_2$. Because u_k is not adjacent with another vertices in I_2, and $\bigcup_{u_k \in I_2} N_{F_{m,n}}(u_k) = V(\overline{K}_m) \cup (V(P_n) - I_2)$. Thus I_2 is independent set of $F_{m,n}$.

Similarly, we get I_1 is independent set of $F_{m,n}$.

Hence $\alpha(F_{m,n}) \geq max\{m, \lceil \frac{n}{2} \rceil\}$.

Suppose that $\alpha(F_{m,n}) > max\{m, \lceil \frac{n}{2} \rceil\}$. Then there exists u(or v) $\in V(F_{m,n}) - (I_1 \cup I_2)$ is not adjacent with another vertices in $I_2(orI_1)$, it is not true.

Hence $\alpha(F_{m,n}) = max\{m, \lceil \frac{n}{2} \rceil\}$.

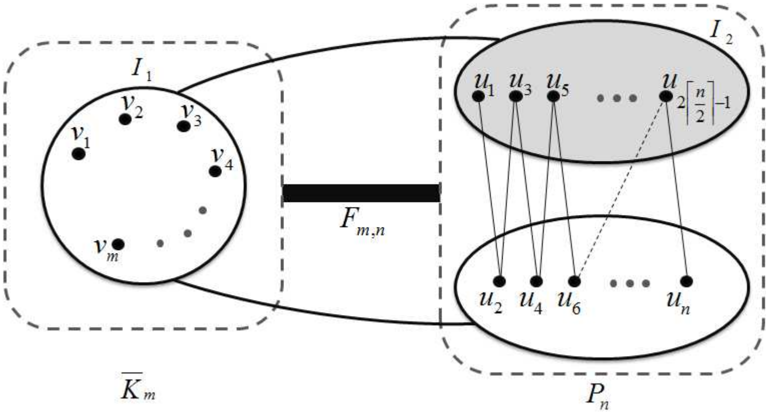

Figure 2: The case that I_2 is independent set of $F_{m,n}$.

On the other hand, we can show by Theorem 1.2 and Lemma 2.2 as follows:

$$\begin{aligned}
\alpha(F_{m,n}) &= \alpha(\overline{K}_m \vee P_n) \\
&= \omega(\overline{\overline{K}_m \vee P_n}) \\
&= \omega(K_m \cup \overline{P}_n) \\
&= max\{\omega(K_m), \omega(\overline{P}_n)\} \\
&= max\{\alpha(\overline{K_m}), \alpha(P_n)\} \\
&= max\{m, \lceil \frac{n}{2} \rceil\}.
\end{aligned}$$

□

3. Vertex Covering Number of Fan Graph.

We begin this section by giving Lemma 3.1 that shows a relation of independent number and vertex covering number.

Lemma 3.1. (see [1]) *Let G be a simple graph with order n. Then $\alpha(G) + \beta(G) = n$.*

Theorem 3.2. *Let $F_{m,n} \cong \overline{K}_m \vee P_n$. Then $\beta(F_{m,n}) = min\{n, m + \lfloor \frac{n}{2} \rfloor\}$.*

Proof. We get $\beta(\overline{K}_m) = 0$ and $\beta(P_n) = \lfloor \frac{n}{2} \rfloor$. Assume that minimum vertex covering set of P_n is $C = \{u_2, u_4, u_6, ..., u_{2\lfloor \frac{n}{2} \rfloor}\}$.

Suppose a vertex $u_k \in C$. So $N_{F_{m,n}}(u_k) = V(\overline{K}_m) \cup (V(P_n) - C)$, all edges such that adjacent with u_k are $\{u_k v_i$ for all $i = 1, 2, 3, ..., m\} \cup \{u_k u_j$ for some $j = 1, 3, 5, ..., 2\lfloor \frac{n}{2} \rfloor + 1\}$.

Then all edges, which are adjacent with u_k for all $k = 2, 4, 6, ..., \lfloor \frac{n}{2} \rfloor$ are $A = \{u_k v_i$ for all $i = 1, 2, 3, ..., m\} \cup \{u_k u_j$ for all $j = 1, 3, 5, ..., 2\lfloor \frac{n}{2} \rfloor + 1\}$.

Let $B = \{v_i u_j$ for all $i = 1, 2, 3, ..., m$ and $j = 1, 3, 5, ..., 2\lfloor \frac{n}{2} \rfloor + 1\}$.

So we have $A \cup B = E(F_{m,n})$ and also $V(\overline{K}_m) \cup C$ is vertex covering set of $F_{m,n}$.

In the same, vertex in $V(\overline{K}_m)$ adjacent with all vertex in $V(P_n)$, so $V(P_n)$ is the vertex covering of $F_{m,n}$.

Hence $\beta(F_{m,n}) \leq min\{n, m + \lfloor \frac{n}{2} \rfloor\}$.

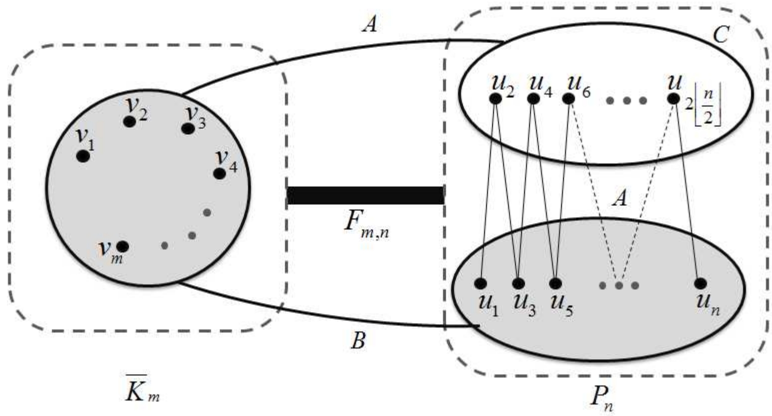

Figure 3: The case that $V(\overline{K}_m) \cup C$ is vertex covering set of $F_{m,n}$.

Suppose that $\beta(F_{m,n}) < min\{n, m + \lfloor \frac{n}{2} \rfloor\}$. Then there exist vertices in $V(\overline{K}_m) \cup C$ which is not adjacent with u_k for all $k = 1, 3, 5, ..., 2\lfloor \frac{n}{2} \rfloor + 1\}$. But is not true. Hence $\beta(F_{m,n}) = min\{n, m + \lfloor \frac{n}{2} \rfloor\}$.

On the other hand, we can show by Theorem 2.3 and Lemma 3.1, we can also show that

$$\beta(F_{m,n}) + \alpha(F_{m,n}) = m + n$$
$$\beta(F_{m,n}) = (m + n) - \alpha(F_{m,n})$$

$$= (m+n) - max\{m, \lceil \frac{n}{2} \rceil\}$$
$$= (m+n) + min\{-m, -\lceil \frac{n}{2} \rceil\}$$
$$= min\{(m+n) - m, (m+n) - \lceil \frac{n}{2} \rceil\}$$
$$= min\{n, m + \lfloor \frac{n}{2} \rfloor\}.$$

□

4. Domination Number of Fan Graph

Next, we show a domination number of Fan Graph.

Theorem 4.1. *Let $F_{m,n} \cong \overline{K}_m \vee P_n$. Then:*

$$\gamma(F_{m,n}) = \begin{cases} 1, & m = 1 \text{ or } n = 1 \\ 2, & m \neq 1 \text{ and } n \neq 1. \end{cases}$$

Proof. Suppose $m = 1$, the vertex $v \in V(\overline{K}_m)$, we have $N_{F_{m,n}}(v) = \{u \in V(P_n)\}$. In the same, if $n = 1$, the vertex $u \in V(P_n)$, we get $N_{F_{m,n}}(u) = \{v \in V(\overline{K}_m)\}$. Hence $\gamma(F_m, n) = 1$, where $m = 1$ or $n = 1$.

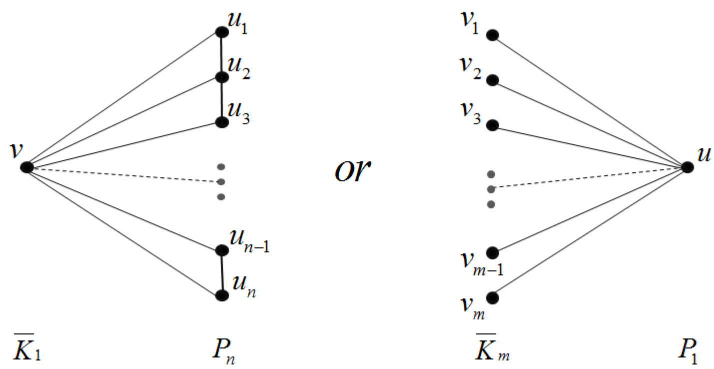

Figure 4: The case that $m = 1$ or $n = 1$.

Suppose $m, n \neq 1$. Choose $v \in V(\overline{K}_m)$, we have $N_{F_{m,n}}(v) = \{u \in V(P_n)\}$, and choose $u \in V(P_n)$, we get $N_{F_{m,n}}(u) = \{v \in V(\overline{K}_m)\}$. Hence $\gamma(F_{m,n}) = 2$, where $m, n \neq 1$.

□

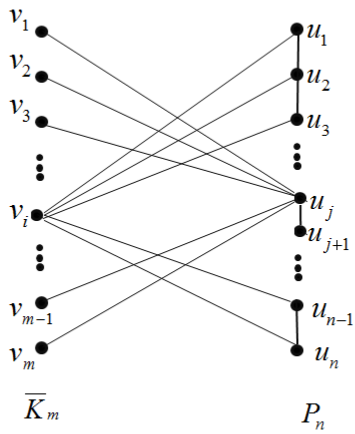

Figure 5: The case that $m \neq 1$ and $n \neq 1$.

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