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ON SOME CYCLIC HOMOGENEOUS POLYNOMIAL INEQUALITIES OF DEGREE FOUR IN REAL VARIABLES UNDER CONSTRAINTS

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Abstract: We find the best lower and upper bounds of the ratio

$$\frac{x^3y+y^3z+z^3x}{(x^2+y^2+z^2)^2}$$

for all real numbers x, y, z satisfying $k(x^2+y^2+z^2)=xy+yz+zx$, or $k(x^2+y^2+z^2)\geq xy+yz+zx$, or $k(x^2+y^2+z^2)\leq xy+yz+zx$, where k is a given real number. The obtained results generalize the known cyclic inequalities

$$(x^2+y^2+z^2)^2\geq 3(x^3y+y^3z+z^3x)$$

and

$$(x^2+y^2+z^2)^2+\frac{8}{\sqrt{7}}(x^3y+y^3z+z^3x)\geq 0,$$

which hold for all real x, y, z.

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1. Introduction

Consider the fourth degree cyclic homogeneous polynomial

$$f_4(x,y,z)=\sum x^4+A\sum x^2y^2+Bxyz\sum x+C\sum x^3y+D\sum xy^3,$$

where A,B,C,D are real constants, and $\sum$ denotes a cyclic sum over x, y and z.

In the particular case $f_4(1,1,1)=0$, the following theorem expresses the necessary and sufficient conditions that the inequality $f_4(x,y,z)\geq 0$ holds for any real numbers x,y,z (see [3], [4]).

Theorem 1.1. *For*

$$1+A+B+C+D=0,$$

the cyclic inequality $f_4(x,y,z)\geq 0$ holds for all real numbers x,y,z if and only if

$$3(1+A)\geq C^2+CD+D^2.$$

In particular, for $A=2$, $B=0$, $C=-3$ and $D=0$, we get the elegant inequality (see [1], [2])

$$(x^2+y^2+z^2)^2\geq 3(x^3y+y^3z+z^3x), \tag{1.1}$$

with equality for $x=y=z$, and for

$$\frac{x}{\sin^2\frac{4\pi}{7}}=\frac{y}{\sin^2\frac{2\pi}{7}}=\frac{z}{\sin^2\frac{\pi}{7}}$$

(or any cyclic permutation).

On the other hand, the following similar inequality holds for all real numbers x,y,z (see [5]):

$$(x^2+y^2+z^2)^2+\frac{8}{\sqrt{7}}(x^3y+y^3z+z^3x)\geq 0, \tag{1.2}$$

with equality when x,y,z are proportional to the roots of the equation

$$w^3-w^2-\left(1+\frac{\sqrt{7}}{2}\right)w+\frac{1}{4}\left(1+\frac{3}{\sqrt{7}}\right)=0$$

and satisfy $(x-y)(y-z)(z-x)\geq 0$.

In this paper, we give a common generalization of the inequalities (1.1) and (1.2).

2. Main Result

Our results are based on the following lemma.

Lemma 2.1. *If α, β, a, b are real numbers, $\alpha \geq 0$, $a \geq 0$ and $a^2 \geq b^2$, then*

$$\alpha\sqrt{a^2 - b^2} \leq a\sqrt{\alpha^2 + \beta^2} + \beta b,$$

with equality if and only if

$$\beta a + b\sqrt{\alpha^2 + \beta^2} = 0.$$

The proof of this lemma is fairly simple. Since

$$a\sqrt{\alpha^2 + \beta^2} + \beta b \geq |\beta| a + \beta b \geq |\beta||b| + \beta b \geq 0,$$

the inequality in Lemma 2.1 is true if and only if

$$\alpha^2(a^2 - b^2) \leq (a\sqrt{\alpha^2 + \beta^2} + \beta b)^2,$$

which is equivalent to the obvious inequality

$$(\beta a + b\sqrt{\alpha^2 + \beta^2})^2 \geq 0.$$

Let us denote

$$\alpha_{\mathsf{k}} = \frac{1 + 13k - 5k^2 - 2(1-k)\sqrt{7(1-k)(1+2k)}}{27},$$

$$\beta_{\mathsf{k}} = \frac{1 + 13k - 5k^2 + 2(1-k)\sqrt{7(1-k)(1+2k)}}{27},$$

$$r_1 = 5k - 2 + (1-k)\sqrt{\frac{1-k}{7(1+2k)}},$$

$$r_2 = 5k - 2 - (1-k)\sqrt{\frac{1-k}{7(1+2k)}},$$

where $k \in (-1/2, 1]$.

Theorem 2.2. *If x, y, z are real numbers such that*

$$k(x^2 + y^2 + z^2) = xy + yz + zx, \quad k \in \left[\frac{-1}{2}, 1\right],$$

then

$$\alpha_{\mathsf{k}} \le \frac{x^3y + y^3z + z^3x}{(x^2+y^2+z^2)^2} \le \beta_{\mathsf{k}}.$$

If $k \in \left(\frac{-1}{2}, 1\right]$, *then the left inequality is an equality when* x, y, z *are proportional to the roots of the equation*

$$w^3 - w^2 + \frac{k}{1+2k}w - \frac{r_1}{27(1+2k)} = 0 \tag{2.1}$$

and satisfy

$$(x-y)(y-z)(z-x) \ge 0,$$

while the right inequality is an equality when x, y, z *are proportional to the roots of the equation*

$$w^3 - w^2 + \frac{k}{1+2k}w - \frac{r_2}{27(1+2k)} = 0 \tag{2.2}$$

and satisfy

$$(x-y)(y-z)(z-x) \le 0.$$

If $k = \frac{-1}{2}$, *then* $\alpha_{\mathsf{k}} = \beta_{\mathsf{k}} = \frac{-1}{4}$, *and equality holds when* $x + y + z = 0$.

Theorem 2.3. *If* x, y, z *are real numbers such that*

$$k(x^2+y^2+z^2) \ge xy + yz + zx, \quad k \in \left[\frac{-1}{2}, \frac{2}{3}\right],$$

then

$$\beta_{\mathsf{k}}(x^2+y^2+z^2)^2 \ge x^3y + y^3z + z^3x.$$

If $k \in \left(\frac{-1}{2}, \frac{2}{3}\right]$, *then equality holds when* x, y, z *are proportional to the roots of the equation*

$$w^3 - w^2 + \frac{k}{1+2k}w - \frac{r_2}{27(1+2k)} = 0$$

and satisfy

$$(x-y)(y-z)(z-x) \le 0.$$

If $k = \frac{-1}{2}$, *then equality holds when* $x + y + z = 0$.

Theorem 2.4. *If x, y, z are real numbers such that*

$$k(x^2+y^2+z^2) \le xy+yz+zx, \quad k \in \left[\frac{1-\sqrt{7}}{4}, 1\right],$$

then

$$\alpha_k(x^2+y^2+z^2)^2 \le x^3y+y^3z+z^3x,$$

with equality when x, y, z are proportional to the roots of the equation

$$w^3 - w^2 + \frac{k}{1+2k}w + \frac{r_1}{27(1+2k)} = 0$$

and satisfy

$$(x-y)(y-z)(z-x) \ge 0.$$

Remark 2.5. For all $k \in (-1/2, 1]$, the equations (2.1) and (2.2) in Theorem 2.2, as well as the similar equations in Theorems 2.3 and 2.4, have all roots real. To prove this claim, we write the equations (2.1) and (2.2) as $f_1(w) = 0$ and $f_2(w) = 0$, respectively. From

$$f_1'(w) = f_2'(w) = 3w^2 - 2w + \frac{k}{1+2k},$$

it follows that the derivatives of $f_1(w)$ and $f_2(w)$ have the same real roots, namely

$$w_1 = \frac{1}{3}\left(1 - \sqrt{\frac{1-k}{1+2k}}\right), \quad w_2 = \frac{1}{3}\left(1 + \sqrt{\frac{1-k}{1+2k}}\right).$$

To prove that the equation (2.1) has all roots real, we only need to show that $f_1(w_1) \ge 0$ and $f_1(w_2) \le 0$. Since

$$3w_1^2 - 2w_1 + \frac{k}{1+2k} = 0, \quad 3w_2^2 - 2w_2 + \frac{k}{1+2k} = 0,$$

we have

$$\begin{aligned} 9f_1(w_1) &= 3w_1\left(2w_1 - \frac{k}{1+2k}\right) - 9w_1^2 + \frac{9k}{1+2k}w_1 - \frac{r_1}{3(1+2k)} \\ &= -3w_1^2 + \frac{6k}{1+2k}w_1 - \frac{r_1}{3(1+2k)} \\ &= \left(-2w_1 + \frac{k}{1+2k}\right) + \frac{6k}{1+2k}w_1 - \frac{r_1}{3(1+2k)} \end{aligned}$$

$$= \frac{k}{1+2k} - \frac{2(1-k)}{1+2k} w_1 - \frac{r_1}{3(1+2k)}$$
$$= \frac{1-k}{3(1+2k)} \left(2 - \frac{1}{\sqrt{7}}\right) \geq 0$$

and

$$9f_1(w_2) = \frac{k}{1+2k} - \frac{2(1-k)}{1+2k} w_2 - \frac{r_1}{3(1+2k)}$$
$$= \frac{1-k}{3(1+2k)} \left(-2 - \frac{1}{\sqrt{7}}\right) \leq 0.$$

Similarly, we can prove that the equation (2.2) has also all roots real. We find

$$9f_2(w_1) = \frac{1-k}{3(1+2k)} \left(2 + \frac{1}{\sqrt{7}}\right) \geq 0$$

and

$$9f_2(w_2) = \frac{1-k}{3(1+2k)} \left(-2 + \frac{1}{\sqrt{7}}\right) \leq 0.$$

Remark 2.6. Using the substitution

$$t = \sqrt{\frac{1-k}{1+2k}}, \quad t \geq 0,$$

we get

$$\alpha_{\mathsf{k}} = \frac{1+3t^2-2\sqrt{7}t^3-3t^4}{3(1+2t^2)^2},$$
$$\beta_{\mathsf{k}} = \frac{1+3t^2+2\sqrt{7}t^3-3t^4}{3(1+2t^2)^2}.$$

Since

$$\alpha_{\mathsf{k}} + \frac{\sqrt{7}}{8} = \frac{(2t-3-\sqrt{7})^2[18t^2+2(\sqrt{7}-1)t+2+\sqrt{7}]}{48(2+\sqrt{7})(1+2t^2)^2} \geq 0$$

and

$$\beta k = \frac{1}{3} - \frac{t^2(1-\sqrt{7}t)^2}{3(1+2t^2)^2} \leq \frac{1}{3},$$

from Theorem 2.2 we get the known inequalities (see [5])

$$\frac{-\sqrt{7}}{8} \leq \frac{x^3y+y^3z+z^3x}{(x^2+y^2+z^2)^2} \leq \frac{1}{3}, \tag{2.3}$$

which are true for all real x, y, z.

Since $\beta_{\mathsf{k}} = 1/3$ for $t = 0$ (when $k = 1$ and $r_2 = 3$), the right inequality (2.3) is an equality when $x = y = z$. In addition, since $\beta_{\mathsf{k}} = 1/3$ for $t = 1/\sqrt{7}$ (when $k = 2/3$ and $r_2 = 9/7$), the right inequality (2.3) is also an equality when x, y, z are proportional to the roots of the equation

$$49w^3 - 49w^2 + 14w - 1 = 0$$

and satisfy

$$(x-y)(y-z)(z-x) \leq 0.$$

Using the substitution $t = w/7$, we get that equality holds when x, y, z are proportional to the roots of the equation

$$t^3 - 7t^2 + 14t - 7 = 0$$

and satisfy

$$(x-y)(y-z)(z-x) \leq 0.$$

Since the equation in t has the roots

$$t_1 = 4\sin^2\frac{4\pi}{7}, \quad t_2 = 4\sin^2\frac{2\pi}{7}, \quad t_3 = 4\sin^2\frac{\pi}{7},$$

equality holds when

$$\frac{x}{\sin^2\frac{4\pi}{7}} = \frac{y}{\sin^2\frac{2\pi}{7}} = \frac{z}{\sin^2\frac{\pi}{7}}$$

(or any cyclic permutation).

Since $\alpha_{\mathsf{k}} = -\sqrt{7}/8$ for $t = (3+\sqrt{7})/2$, when $k = (1-\sqrt{7})/4$ and $r_1 = -27\sqrt{7}/28$, the left inequality (2.3) is an equality when x, y, z are proportional to the roots of the equation

$$w^3 - w^2 - \left(1 + \frac{\sqrt{7}}{2}\right)w + \frac{1}{4}\left(1 + \frac{3}{\sqrt{7}}\right) = 0$$

and satisfy $(x-y)(y-z)(z-x) \geq 0$.

Remark 2.7. For $k = 2/3$, from Theorem 2.2 we get that

$$x^2 + y^2 + z^2 = \frac{3}{2}(xy + yz + zx)$$

involves

$$\frac{53}{243} \leq \frac{x^3y + y^3z + z^3x}{(x^2+y^2+z^2)^2} \leq \frac{1}{3}.$$

The left inequality is an equality when x, y, z are proportional to the roots of the equation

$$w^3 - w^2 + \frac{2}{7}w - \frac{29}{1323} = 0$$

and satisfy $(x-y)(y-z)(z-x) \ge 0$.

The right inequality is an equality when x, y, z are proportional to the roots of the equation

$$w^3 - w^2 + \frac{2}{7}w - \frac{1}{49} = 0$$

and satisfy $(x-y)(y-z)(z-x) \le 0$.

Remark 2.8. Applying Theorem 2.2 for $k = 1/5$, it follows that

$$x^2 + y^2 + z^2 = 5(xy + yz + zx)$$

involves

$$\frac{-1}{25} \le \frac{x^3y + y^3z + z^3x}{(x^2+y^2+z^2)^2} \le \frac{197}{675}.$$

The left inequality is an equality when x, y, z are proportional to the roots of the equation

$$w^3 - w^2 + \frac{1}{7}w + \frac{1}{49} = 0$$

and satisfy $(x-y)(y-z)(z-x) \ge 0$.

The right inequality is an equality when x, y, z are proportional to the roots of the equation

$$w^3 - w^2 + \frac{1}{7}w + \frac{43}{1323} = 0$$

and satisfy $(x-y)(y-z)(z-x) \le 0$.

Remark 2.9. For $k = -1/6$, from Theorem 2.2 we get that

$$x^2 + y^2 + z^2 + 6(xy + yz + zx) = 0$$

involves

$$\frac{-1}{4} \le \frac{x^3y + y^3z + z^3x}{(x^2+y^2+z^2)^2} \le \frac{149}{972}.$$

The left inequality is an equality when x, y, z are proportional to the roots of the equation

$$w^3 - w^2 - \frac{1}{4}w + \frac{1}{8} = 0$$

and satisfy $(x-y)(y-z)(z-x) \ge 0$.

The right inequality is an equality when x, y, z are proportional to the roots of the equation

$$w^3 - w^2 - \frac{1}{4}w + \frac{41}{216} = 0$$

and satisfy $(x-y)(y-z)(z-x) \leq 0$.

Remark 2.10. For $k = -2/5$, from Theorem 2.2 we get that

$$x^2 + y^2 + z^2 + \frac{5}{2}(xy + yz + zx) = 0$$

involves

$$\frac{-223}{675} \leq \frac{x^3y + y^3z + z^3x}{(x^2+y^2+z^2)^2} \leq \frac{-1}{25}.$$

The left inequality is an equality when x, y, z are proportional to the roots of the equation

$$w^3 - w^2 - 2w + \frac{13}{27} = 0$$

and satisfy $(x-y)(y-z)(z-x) \geq 0$.

The right inequality is an equality when x, y, z are proportional to the roots of the equation

$$w^3 - w^2 - 2w + 1 = 0$$

and satisfy $(x-y)(y-z)(z-x) \leq 0$.

Remark 2.11. By Theorem 2.2, we get the following best lower and upper bounds of the cyclic expression (see [6])

$$E = x^3y + y^3z + z^3x, \quad x, y, z \in \mathbf{R}$$

for fixed $p = x + y + z$ and $q = xy + yz + zx$, namely

$$f_1(p,q) \leq x^3y + y^3z + z^3x \leq f_2(p,q),$$

where

$$f_1(p,q) = \frac{p^4 + 9p^2q - 27q^2 - 2(p^2-3q)\sqrt{7p^2(p^2-3q)}}{27},$$

$$f_2(p,q) = \frac{p^4 + 9p^2q - 27q^2 + 2(p^2-3q)\sqrt{7p^2(p^2-3q)}}{27}.$$

For $p = 0$, we have $E = -q^2$.

For $p \neq 0$, the expression E attains its lower bound when

$$27xyz = -2p^3 + 9pq + (p^2 - 3q)\sqrt{\frac{p^2-3q}{7}}$$

and $(x-y)(y-z)(z-x) \geq 0$.

For $p \neq 0$, the expression E attains its upper bound when

$$27xyz = -2p^3 + 9pq - (p^2 - 3q)\sqrt{\frac{p^2 - 3q}{7}}$$

and $(x-y)(y-z)(z-x) \leq 0$.

3. Proof of Theorem 2.2

Consider first the case $k = -1/2$, when $\alpha_k = \beta_k = -1/4$, and the hypothesis $k(x^2+y^2+z^2) = xy+yz+zx$ is equivalent to $x+y+z=0$. We need to show that $x+y+z=0$ involves

$$(x^2+y^2+z^2)^2 + 4(x^3y+y^3z+z^3x) = 0.$$

Since

$$x^3y+y^3z+z^3x - xy^3 - yz^3 - zx^3 = -(x+y+z)(x-y)(y-z)(z-x) = 0,$$

we have

$$\begin{aligned} 2(x^3y+y^3z+z^3x) &= x^3y+y^3z+z^3x + (xy^3+yz^3+zx^3) \\ &= (xy+yz+zx)(x^2+y^2+z^2), \end{aligned}$$

and hence

$$(x^2+y^2+z^2)^2 + 4(x^3y+y^3z+z^3x) = (x^2+y^2+z^2)(x+y+z)^2 = 0.$$

Consider now that $k \in (-1/2, 1]$, and denote

$$p = x+y+z, \quad q = xy+yz+zx, \quad r = xyz.$$

For $p=0$, from the hypothesis $k(x^2+y^2+z^2) = xy+yz+zx$ we get $xy+yz+zx = 0$. Since

$$xy+yz+zx = x(y+z)+yz = -(y+z)^2 + yz = -y^2 - yz - z^2,$$

it follows that $y^2+yz+z^2 = 0$, which involves $x=y=z=0$. Therefore, it suffices to consider further that $k \in (-1/2, 1]$ and $p \neq 0$. Since the statement remains unchanged by replacing x, y, z with $-x, -y, -z$, respectively, it suffices to consider the case $p > 0$. In addition, due to homogeneity, we may assume that $p = 1$, which implies

$$q = \frac{k}{1+2k}.$$

(a) Write the left desired inequality as

$$2\alpha_{\mathsf{k}}(x^2+y^2+z^2)^2 \le \sum xy(x^2+y^2) + (\sum x^3y - \sum xy^3).$$

Since

$$x^2+y^2+z^2 = p^2-2q = 1-2q,$$

$$\sum xy(x^2+y^2) = q(p^2-2q) - pr = q - 2q^2 - r,$$

$$\sum x^3y - \sum xy^3 = -p(x-y)(y-z)(z-x) \ge -p\sqrt{(x-y)^2(y-z)^2(z-x)^2}$$

$$= -p\sqrt{\frac{4(p^2-3q)^3-(2p^3-9pq+27r)^2}{27}} = -\sqrt{\frac{4(1-3q)^3-(2-9q+27r)^2}{27}},$$

it suffices to prove that

$$2\alpha_{\mathsf{k}}(1-2q)^2 \le q-2q^2-r-\sqrt{\frac{4(1-3q)^3-(2-9q+27r)^2}{27}}.$$

Applying Lemma 2.1 for

$$\alpha = \frac{1}{\sqrt{27}}, \quad \beta = \frac{-1}{27}, \quad a = 2(1-3q)\sqrt{1-3q}, \quad b = 2-9q+27r,$$

we get

$$\sqrt{\frac{4(1-3q)^3-(2-9q+27r)^2}{27}} + r + \frac{2-9q}{27} \le \frac{4(1-3q)\sqrt{7(1-3q)}}{27},$$

with equality for

$$(1-3q)\sqrt{\frac{1-3q}{7}} - 2 + 9q - 27r = 0.$$

Thus, it suffices to show that

$$2\alpha_{\mathsf{k}}(1-2q)^2 \le q - 2q^2 + \frac{2-9q}{27} - \frac{4(1-3q)\sqrt{7(1-3q)}}{27},$$

$$27\alpha_{\mathsf{k}}(1-2q)^2 \le 1 + 9q - 27q^2 - 2(1-3q)\sqrt{7(1-3q)}.$$

This inequality is equivalent to

$$27\alpha_{\mathsf{k}} \le 1 + 13k - 5k^2 - 2(1-k)\sqrt{7(1-k)(1+2k)},$$

which is an identity.

For $p = 1$, equality holds when $(x-y)(y-z)(z-x) \ge 0$, $q = k/(1+2k)$ and

$$27r = (1-3q)\sqrt{\frac{1-3q}{7}} - 2 + 9q = \frac{r_1}{1+2k},$$

where

$$r_1 = 5k - 2 + (1-k)\sqrt{\frac{1-k}{7(1+2k)}}.$$

Therefore, equality holds when x, y, z are proportional to the roots of the equation

$$w^3 - w^2 + \frac{k}{1+2k}w - \frac{r_1}{27(1+2k)} = 0$$

and satisfy $(x-y)(y-z)(z-x) \ge 0$.

(b) Write the right desired inequality as

$$2\beta_{\mathsf{k}}(x^2+y^2+z^2)^2 \ge \sum xy(x^2+y^2) + (\sum x^3y - \sum xy^3).$$

Since

$$x^2+y^2+z^2 = p^2 - 2q = 1 - 2q,$$

$$\sum xy(x^2+y^2) = q(p^2-2q) - pr = q - 2q^2 - r,$$

$$\sum x^3y - \sum xy^3 = -p(x-y)(y-z)(z-x) \le p\sqrt{(x-y)^2(y-z)^2(z-x)^2}$$

$$= p\sqrt{\frac{4(p^2-3q)^3 - (2p^3 - 9pq + 27r)^2}{27}} = \sqrt{\frac{4(1-3q)^3 - (2-9q+27r)^2}{27}},$$

it suffices to prove that

$$2\beta_{\mathsf{k}}(1-2q)^2 \ge q - 2q^2 - r + \sqrt{\frac{4(1-3q)^3 - (2-9q+27r)^2}{27}}.$$

Applying Lemma 2.1 for

$$\alpha = \frac{1}{\sqrt{27}}, \quad \beta = \frac{1}{27}, \quad a = 2(1-3q)\sqrt{1-3q}, \quad b = 2 - 9q + 27r,$$

we get

$$\sqrt{\frac{4(1-3q)^3-(2-9q+27r)^2}{27}}-r-\frac{2-9q}{27}\leq\frac{4(1-3q)\sqrt{7(1-3q)}}{27},$$

with equality for

$$(1-3q)\sqrt{\frac{1-3q}{7}}+2-9q+27r=0.$$

Thus, it suffices to show that

$$2\beta_{\mathsf{k}}(1-2q)^2\geq q-2q^2+\frac{2-9q}{27}+\frac{4(1-3q)\sqrt{7(1-3q)}}{27},$$

$$27\beta_{\mathsf{k}}(1-2q)^2\geq 1+9q-27q^2+2(1-3q)\sqrt{7(1-3q)}.$$

This inequality is equivalent to

$$27\beta_{\mathsf{k}}\geq 1+13k-5k^2+2(1-k)\sqrt{7(1-k)(1+2k)},$$

which is an identity.

For $p=1$, equality holds when $(x-y)(y-z)(z-x)\leq 0$, $q=k/(1+2k)$ and

$$27r=9q-2-(1-3q)\sqrt{\frac{1-3q}{7}}=\frac{r_2}{1+2k},$$

where

$$r_2=5k-2-(1-k)\sqrt{\frac{1-k}{7(1+2k)}}.$$

Therefore, equality holds when x,y,z are proportional to the roots of the equation

$$w^3-w^2+\frac{k}{1+2k}w-\frac{r_2}{27(1+2k)}=0$$

and satisfy $(x-y)(y-z)(z-x)\leq 0$.

4. Proof of Theorem 2.3

Using the same way as in the proof of the right inequality in Theorem 2.2, we need to show that $p=1$ and

$$q\leq\frac{k}{1+2k}$$

involve

$$27\beta_k(1-2q)^2 \geq 1+9q-27q^2+2(1-3q)\sqrt{7(1-3q)}.$$

Using the substitution

$$t=\sqrt{7(1-3q)},$$

the inequality becomes

$$3(7+2t^2)^2\beta_k \geq 49+21t^2+14t^3-3t^4;$$

that is,

$$3\beta_k \geq f(t),$$

where

$$f(t)=\frac{49+21t^2+14t^3-3t^4}{(7+2t^2)^2}.$$

From $q \leq \dfrac{k}{1+2k}$ and $k \leq 2/3$, we get

$$t \geq t_2=\sqrt{\frac{7(1-k)}{1+2k}} \geq 1.$$

Since

$$f'(t)=\frac{14t(t-1)(7-14t-2t^2)}{(7+2t^2)^2} \leq 0,$$

$f(t)$ is strictly decreasing on $[t_2,\infty)$. Therefore, we have

$$3\beta_k-f(t) \geq 3\beta_k-f(t_2)=0.$$

For $p=1$, equality holds when $(x-y)(y-z)(z-x) \leq 0$, $t=t_2=\sqrt{\dfrac{7(1-k)}{1+2k}}$, $q=\dfrac{7-t^2}{21}=\dfrac{k}{1+2k}$ and

$$27r=9q-2-(1-3q)\sqrt{\frac{1-3q}{7}}=\frac{r_2}{1+2k},$$

where

$$r_2=5k-2-(1-k)\sqrt{\frac{1-k}{7(1+2k)}}.$$

Therefore, equality holds when x,y,z are proportional to the roots of the equation

$$w^3-w^2+\frac{k}{1+2k}w-\frac{r_2}{27(1+2k)}=0$$

and satisfy $(x-y)(y-z)(z-x) \leq 0$.

5. Proof of Theorem 2.4

Using the same way as in the proof of the left inequality in Theorem 2.2, we need to show that $p = 1$ and

$$q \geq \frac{k}{1+2k}$$

involves

$$27\alpha_k(1-2q)^2 \leq 1 + 9q - 27q^2 - 2(1-3q)\sqrt{7(1-3q)}.$$

Using the substitution

$$t = \sqrt{7(1-3q)},$$

the inequality becomes

$$3(7+2t^2)^2\alpha_k \leq 49 + 21t^2 - 14t^3 - 3t^4;$$

that is,

$$3\alpha_k \leq f(t),$$

where

$$f(t) = \frac{49 + 21t^2 - 14t^3 - 3t^4}{(7+2t^2)^2}.$$

From $q \geq \dfrac{k}{1+2k}$ and $k \geq \dfrac{1-\sqrt{7}}{4}$, we get

$$t \leq t_1 = \sqrt{\frac{7(1-k)}{1+2k}} \leq \frac{7+3\sqrt{7}}{2}.$$

Since

$$f'(t) = \frac{14t(1+t)(2t^2-14t-7)}{(7+2t^2)^2} \leq 0,$$

$f(t)$ is strictly decreasing on $[0, t_1]$. Therefore, we have

$$3\alpha_k - f(t) \leq 3\alpha_k - f(t_1) = 0.$$

For $p = 1$, equality holds when $(x-y)(y-z)(z-x) \geq 0$, $t = t_1 = \sqrt{\dfrac{7(1-k)}{1+2k}}$, $q = \dfrac{7-t^2}{21} = \dfrac{k}{1+2k}$ and

$$27r = 9q - 2 + (1-3q)\sqrt{\frac{1-3q}{7}} = \frac{r_1}{1+2k},$$

where

$$r_1 = 5k - 2 + (1-k)\sqrt{\frac{1-k}{7(1+2k)}}.$$

Therefore, equality holds when x, y, z are proportional to the roots of the equation

$$w^3 - w^2 + \frac{k}{1+2k}w - \frac{r_1}{27(1+2k)} = 0$$

and satisfy $(x-y)(y-z)(z-x) \geq 0$.

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