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# THE FORMULATION OF DIRICHLET SERIES OF THE NUMBER THEORETIC FUNCTIONS

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**Abstract:** We study the behaviour of the Dirichlet series for the special cases of the theoretic functions. The abscissa of convergence from the number theoretic functions with the different conditions will be proved.

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**Key Words:** Dirichlet series, Riemann zeta function, abscissa of convergence

## 1. Introduction

**Notation.**

1. Let $\mathbb{P}$ denotes the set of all primes and $\mathcal{P}$ will denote subsets of $\mathbb{P}$.
2. The symbols $\mathbb{N}_0, \mathbb{N}, \mathbb{R}, \mathbb{C}$ denote the natural numbers including zero $\{0, 1, 2, 3, \cdots\}$, natural numbers $\{1, 2, 3, \cdots\}$, real numbers and complex numbers, respectively.
3. $|J|$ denotes the number of element of set $J$

**Definition 1.1.** A Dirichlet series is any series of the form

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$$\sum_{n=1}^{\infty} \frac{a_n}{n^s},$$

where $s$ and $a_n$ are complex numbers, $n = 1, 2, 3, \cdots$. If $a_n = 1$ for all $n$ then the Dirichlet series is

$$\sum_{n=1}^{\infty} \frac{1}{n^s},$$

which is the Riemann zeta function (see details in [1]).

**Definition 1.2.** A number theoretic function or Arithmetic function is a function whose domain is the set of positive integers (see details in [2]).

**Theorem 1.3.** *Suppose that the series $\sum_{n=1}^{\infty} |f(n)n^{-s}|$ does not converge for all $s$ or diverge for all $s$. Then there exists a real number $x_a$, called the abscissa of absolute convergence, such that the series $\sum_{n=1}^{\infty} f(n)n^{-s}$ converges absolutely if $\Re(s) > x_a$ but does not converge absolutely if $\Re(s) < x_a$.*

Theorem 1.3 is taken from [1].

## 2. Main Results

**Theorem 2.1.** *Let $J \subset \mathcal{P} = \{p_1, p_2, \cdots, p_r\}$ and the theoretic function $f : \mathbb{N} \to \mathbb{R}$ is defined by*

$$f(n) = \begin{cases} 0 & \text{if } p \mid n \text{ for some } p \in \mathcal{P}; \\ 1 & \text{otherwise,} \end{cases}$$

*then*

$$d_T(s) = \left( \sum_{J \subset \mathcal{P}} (-1)^{|J|} \prod_{p_i \in J} \frac{1}{p_i^s} \right) \zeta(s)$$

*and the abscissa of convergence of $d_T$ is 1.*

*Proof.* We will prove this by induction on the number of $\mathcal{P}$. If $\mathcal{P} = \emptyset$ then $f(n) = 1$, hence $d_T(s) = \zeta(s)$. The inductive hypothesis in Theorem 2.1 holds if $\mathcal{P} = \{p_1, p_2, \cdots, p_r\}$. We consider by adding $\mathcal{P}$ by a prime $q$.

$$\begin{aligned}
d_T(s) &= \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \sum_{p_1 \nmid n, \cdots, p_r \nmid n, q \nmid n} \frac{1}{n^s} \\
&= \sum_{q \nmid n} \left( \sum_{J \subset \mathcal{P}} (-1)^{|J|} \prod_{p_i \in J} \frac{1}{p_i^s} \right) \frac{1}{n^s} \\
&= \sum_{n=1}^{\infty} \left( \sum_{J \subset \mathcal{P}} (-1)^{|J|} \prod_{p_i \in J} \frac{1}{p_i^s} \right) \frac{1}{n^s} - \sum_{n=1}^{\infty} \left( \sum_{J \subset \mathcal{P}} (-1)^{|J|} \prod_{p_i \in J} \frac{1}{p_i^s} \right) \frac{1}{(qn)^s} \\
&= \sum_{n=1}^{\infty} \left( \sum_{J \subset \mathcal{P}} (-1)^{|J|} \prod_{p_i \in J} \frac{1}{p_i^s} \right) \frac{1}{n^s} - \sum_{n=1}^{\infty} \left( \sum_{J \subset \mathcal{P}} (-1)^{|J|} \prod_{p_i \in J} \frac{1}{(qp_i)^s} \right) \frac{1}{n^s} \\
&= \left( \sum_{J \subset \mathcal{P}} (-1)^{|J|} \prod_{p_i \in J} \frac{1}{p_i^s} + \sum_{J \cup \{q\} \subset \mathcal{P} \cup \{q\}} (-1)^{|J \cup \{q\}|} \prod_{p_i \in J \cup \{q\}} \frac{1}{p_i^s} \right) \zeta(s) \\
&= \sum_{J' \subset \mathcal{P}'} (-1)^{|J'|} \prod_{p_i \in J'} \frac{1}{p_i^s} \zeta(s)
\end{aligned}$$

which is $J' \subset \mathcal{P}' = \mathcal{P} \cup \{q\}$.

As known, the abscissa of convergence of the Riemann zeta function is 1 and

$$\sum_{J \subset \mathcal{P}} (-1)^{|J|} \prod_{p_i \in J} \frac{1}{p_i^s}$$

converges for any $s \in \mathbb{C}$. Hence, the abscissa of convergence of $d_T$ is 1. $\square$

**Example 2.2.** Suppose that

$$f(n) = \begin{cases} 0 & \text{if } p \mid n \text{ for some } p \in \{3, 5\}; \\ 1 & \text{otherwise,} \end{cases}$$

then by theorem 2.1, we have

$$\begin{aligned}
d_T(s) &= \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \sum_{3 \nmid n, 5 \nmid n} \frac{1}{n^s} \\
&= \sum_{5 \nmid n} \frac{1}{n^s} - \sum_{5 \nmid n} \frac{1}{(3n)^s} \\
&= \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{5 \mid n} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{(3n)^s} + \sum_{5 \mid n} \frac{1}{(3n)^s}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{(5n)^s} - \frac{1}{3^s} \sum_{n=1}^{\infty} \frac{1}{n^s} + \sum_{n=1}^{\infty} \frac{1}{(15n)^s} \\
&= \left(1 - \frac{1}{3^s} - \frac{1}{5^s} + \frac{1}{15^s}\right)\zeta(s),
\end{aligned}
$$

it is obvious, the abscissa of convergence of $d_T$ is 1.

The next two theorems will make another function to find the Dirichlet series and see what happen with the abscissa of convergence.

**Theorem 2.3.** *Let $a, b, m \in \mathbb{N}$, $p_i$ is prime number and $s_i, t_i$ are not negative integer for all $i = 1, 2, 3, \cdots m$. If*

$$
f(n) = \begin{cases} a \prod_{i=1}^{m} (p_i^{s_i})^{t_i} & \text{if } n = b \prod_{i=1}^{m} p_i^{s_i} \quad \text{when} \quad s_i \in \mathbb{N}_0; \\ 0 & \text{otherwise,} \end{cases}
$$

*then*

$$
d_T(s) = \frac{a}{b^s \prod_{i=1}^{m} (1 - p_i^{t_i - s})}
$$

*and the abscissa of convergence of $d_T$ is* $\max\{t_1, t_2, \cdots, t_m\}$.

*Proof.*

$$
\begin{aligned}
d_T(s) &= \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \sum_{s_i \geq 0} \frac{a \prod_{i=1}^{m} (p_i^{s_i})^{t_i}}{(b \prod_{i=1}^{m} p_i^{s_i})^s} \\
&= \sum_{s_1 \geq 0} \sum_{s_2 \geq 0} \cdots \sum_{s_m \geq 0} \frac{a (p_1^{s_1})^{t_1} (p_2^{s_2})^{t_2} \cdots (p_m^{s_m})^{t_m}}{(b p_1^{s_1} p_2^{s_2} \cdots p_m^{s_m})^s} \\
&= \frac{a}{b^s} \sum_{s_1 \geq 0} \frac{(p_1^{s_1})^{t_1}}{(p_1^{s_1})^s} \cdot \sum_{s_2 \geq 0} \frac{(p_2^{s_2})^{t_2}}{(p_2^{s_2})^s} \cdots \sum_{s_m \geq 0} \frac{(p_m^{s_m})^{t_m}}{(p_m^{s_m})^s} \\
&= \frac{a}{b^s} \sum_{s_1 \geq 0} \frac{1}{(p_1^{s_1})^{s - t_1}} \sum_{s_2 \geq 0} \frac{1}{(p_2^{s_2})^{s - t_2}} \cdots \sum_{s_m \geq 0} \frac{1}{(p_m^{s_m})^{s - t_m}} \\
&= \frac{a}{b^s} \frac{1}{1 - p_1^{t_1 - s}} \frac{1}{1 - p_2^{t_2 - s}} \cdots \frac{1}{1 - p_m^{t_m - s}} \\
&= \frac{a}{b^s \prod_{i=1}^{m} (1 - p_i^{t_i - s})},
\end{aligned}
$$

it is obvious that the abscissa of convergence of $d_T$ is $\max\{t_1, t_2, \cdots, t_m\}$. □

**Example 2.4.** Let

$$f(n) = \begin{cases} 3 & \text{if } n = 2\cdot 3^k, \qquad k \in \mathbb{N}_0; \\ 0 & \text{otherwise,} \end{cases}$$

then

$$\begin{aligned} d_T(s) &= \sum_{i=1}^{\infty} \frac{f(n)}{n^s} = \sum_{k\geq 0} \frac{3}{(2\cdot 3^k)^s} \\ &= \frac{3}{2^s} \sum_{k\geq 0} \frac{1}{(3^k)^s} = \frac{3}{2^s(1-3^{-s})}. \end{aligned}$$

Thus $d_T$ has abscissa of convergence at $s = 0$.

**Example 2.5.** Let

$$f(n) = \begin{cases} 3^k & \text{if } n = 2\cdot 3^k, \qquad k \in \mathbb{N}_0; \\ 0 & \text{otherwise,} \end{cases}$$

then

$$\begin{aligned} d_T(s) &= \sum_{i=1}^{\infty} \frac{f(n)}{n^s} = \sum_{k\geq 0} \frac{3^k}{(2\cdot 3^k)^s} \\ &= \frac{1}{2^s} \sum_{k\geq 0} \frac{1}{(3^k)^{s-1}} = \frac{1}{2^s(1-3^{1-s})}. \end{aligned}$$

Thus $d_T$ has abscissa of convergence at $s = 1$.

**Theorem 2.6.** *Let* $\mathcal{P} = \{p_1, p_2, \cdots, p_r\}$ *be subset of primes and*

$$f(n) = \begin{cases} 0 & \text{if } p \mid n \text{ for some } p \in \mathbb{P} \setminus \mathcal{P}; \\ 1 & \text{otherwise,} \end{cases}$$

*then*

$$d_T(s) = \prod_{i=1}^{r} \frac{1}{1 - {p_i}^{-s}}$$

*and the abscssa of convergence of* $d_T$ *is* $0$.

*Proof.* We know that

$$\begin{aligned} f(n) &= \begin{cases} 0 & \text{if } p \mid n \text{ for some } p \in \mathbb{P} \setminus \mathcal{P}; \\ 1 & \text{otherwise,} \end{cases} \\ &= \begin{cases} 1 & \text{if } n = p_1^{k_1} p_2^{k_2} \cdots p_r^{k_r}\ , k_i \in \mathbb{N}_0; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

So,

$$\begin{aligned} d_T(s) &= \sum_{n=1}^{\infty} \frac{f(n)}{n^s} = \sum_{n=p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r}} \frac{1}{n^s} \\ &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_r=0}^{\infty} \frac{1}{(p_1^{k_1}p_2^{k_2}\cdots p_r^{k_r})^s} \\ &= \sum_{k_1=0}^{\infty}\frac{1}{(p_1{}^{k_1})^s}\sum_{k_2=0}^{\infty}\frac{1}{(p_2{}^{k_2})^s}\cdots\sum_{k_r=0}^{\infty}\frac{1}{(p_r{}^{k_r})^s}\ ,\Re(s)>0 \\ &= \frac{1}{1-p_1{}^{-s}}\cdot\frac{1}{1-p_2{}^{-s}}\cdots\frac{1}{1-p_r{}^{-s}} = \prod_{i=1}^{r}\frac{1}{1-p_i{}^{-s}}, \end{aligned}$$

it is obvious, the abscissa of convergence of $d_T$ is 0. □

**Example 2.7.** Let

$$\begin{aligned} f(n) &= \begin{cases} 0 & \text{if } p \mid n \text{ for some } p \in \mathbb{P}\setminus\{3,5\}; \\ 1 & \text{otherwise,} \end{cases} \\ &= \begin{cases} 1 & \text{if } n = 3^{k_1}\cdot 5^{k_2}\ , k_1,k_2\in\mathbb{N}_0; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Then

$$\begin{aligned} d_T(s) &= \sum_{n=1}^{\infty}\frac{f(n)}{n^s} = \sum_{n=3^{k_1}\cdot 5^{k_2}}\frac{1}{n^s} \\ &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{1}{(3^{k_1}\cdot 5^{k_2})^s} \\ &= \sum_{k_1=0}^{\infty}\frac{1}{(3^{k_1})^s}\cdot\sum_{k_2=0}^{\infty}\frac{1}{(3^{k_2})^s}\ ,\Re(s)>0 \\ &= \frac{1}{1-3^{-s}}\cdot\frac{1}{1-5^{-s}} \end{aligned}$$

and the abscissa of convergence of $d_T$ is 0.

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