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EXISTENCE AND ITERATIVE ALGORITHMS OF POSITIVE SOLUTIONS FOR A THIRD ORDER NEUTRAL DELAY DIFFERENTIAL EQUATION

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Abstract: This paper deals with a third order nonlinear neutral delay differential equation. By using the Banach fixed point theorem, we establish three existence results of uncountably many bounded positive solutions for the equation, construct the Mann iterative schemes for approximating these bounded positive solutions and discuss error estimates between the approximate solutions and the bounded positive solutions. Three examples are given to explain the results in this paper.

AMS Subject Classification: 34K40
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1. Introduction and Preliminaries

Recently, the existence problems of solutions for first, second and third order neutral delay differential equations have been investigated by many researchers,

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see [1-5] and the references therein.

Using the Banach fixed point theorems, Zhang et al. [4] studied the existence of nonoscillatory solutions for the first order neutral delay differential equation with variable coefficients and delays

$$\begin{aligned}&[x(t)+P(t)x(t-\tau)]'+Q_1(t)x(t-\tau_1)-Q_2(t)x(t-\tau_2)\\&=0,\quad \forall t\geq t_0,\end{aligned}\tag{1.1}$$

where $\tau>0$, $\sigma_1,\sigma_2\in\mathbb{R}^+$, $P\in C([t_0,+\infty),\mathbb{R}\setminus\{\pm 1\})$ and $Q_1,Q_2\in C([t_0,+\infty),\mathbb{R}^+)$. Kulenović and Hadžiomerspahić [1] considered the second order neutral delay differential equation

$$\begin{aligned}&[x(t)+px(t-\tau)]''+Q_1(t)x(t-\sigma_1)-Q_2(t)x(t-\sigma_2)\\&=0,\quad \forall t\geq t_0,\end{aligned}\tag{1.2}$$

where $p\in\mathbb{R}\setminus\{\pm 1\}$, $\tau>0$, $\sigma_1,\sigma_2\in\mathbb{R}^+$ and $Q_1,Q_2\in C([t_0,+\infty),\mathbb{R}^+)$. Making use of Krasnosel'skiĭ's fixed point theorem, Zhang, Yang and Di [5] studied the existence of bounded positive solutions for the second order nonlinear neutral differential equation with positive and negative terms

$$\begin{aligned}&[x(t)-p(t)x(\tau(t))]''+f_1(t,x(\sigma_1(t)))-f_2(t,x(\sigma_2(t)))\\&=0,\quad \forall t\geq t_0,\end{aligned}\tag{1.3}$$

and its corresponding equation with forced term

$$\begin{aligned}&[x(t)-p(t)x(\tau(t))]''+f_1(t,x(\sigma_1(t)))-f_2(t,x(\sigma_2(t)))\\&=g(t),\quad \forall t\geq t_0,\end{aligned}\tag{1.4}$$

where $p\in C([t_0,+\infty),(-1,+\infty)\setminus\{1\})$, τ, $\sigma_1,\sigma_2\in C([t_0,+\infty),\mathbb{R})$ and $f_1,f_2\in C([t_0,\infty)\times\mathbb{R},\mathbb{R})$. Liu et al. [3] established the existence of uncountably many bounded positive solutions for the third order nonlinear neutral delay differential equation

$$\begin{aligned}&[x(t)+p(t)x(t-\tau)]'''+f(t,x(t-\tau_1),\cdots,x(t-\tau_k))\\&=0,\quad \forall t\geq t_0,\end{aligned}\tag{1.5}$$

where $\tau>0$, $\tau_i\in\mathbb{R}^+$ for $i\in\{1,2,\cdots,k\}$, $p\in C([t_0,+\infty),(-1,1))$ and $f\in C([t_0,+\infty)\times\mathbb{R}^k,\mathbb{R})$.

The aim of this paper is to investigate the following third order neutral delay differential equation

$$\begin{aligned}&[x(t)+c(t)x(t-\tau)]'''+[h(t,x(t-\alpha_1),\cdots,x(t-\alpha_l))]'\\&\quad+f(t,x(t-\beta_1),\cdots,x(t-\beta_l))=g(t),\quad \forall t\geq t_0,\end{aligned}\tag{1.6}$$

where $\tau > 0$, $\alpha_i, \beta_i \in \mathbb{R}^+$ for $i \in \{1, 2, \cdots, l\}$, $c, g \in C([t_0, +\infty), \mathbb{R})$, $h \in C^1([t_0, +\infty) \times \mathbb{R}^l, \mathbb{R})$ and $f \in C([t_0, +\infty) \times \mathbb{R}^l, \mathbb{R})$. Using the Banach fixed point theorem, we present three existence results of uncountably many bounded positive solutions for Eq.(1.6), construct the Mann iterative schemes and discuss the error estimate between the sequences generated by the Mann iterative schemes and the positive solutions. Three nontrivial examples are included to illustrate the importance of our results in this paper.

Throughout this paper, we assume that $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$,

$$\gamma = t_0 - \max\big\{\tau, \alpha_i, \beta_i : i \in \{1, 2, \cdots, l\}\big\},$$

$BC([\gamma, +\infty), \mathbb{R})$ stands for the set of all continuous and bounded functions on $[\gamma, +\infty)$ with norm $\|x\| = \sup_{t\ge\gamma} |x(t)|$ and for $M > N > 0$

$$A(N, M) = \big\{x \in BC([\gamma, +\infty), \mathbb{R}) : N \le x(t) \le M, \ \forall t \ge \gamma\big\}.$$

Obviously, $A(N, M)$ is a nonempty bounded closed subset of $BC([\gamma, +\infty), \mathbb{R})$.

By a solution of Eq.(1.6), we mean a function $x \in C([\gamma, +\infty), \mathbb{R})$ for some $T \ge \tau + |t_0| + |\gamma|$, such that $x(t) + c(t)x(t-\tau)$ is thrice continuously differentiable on $[T, +\infty)$ and Eq.(1.6) is satisfied for $t \ge T$.

2. Main Results

Now we prove the existence of uncountably many bounded positive solutions for Eq.(1.6) by using the Banach fixed point theorem and discuss the convergence of the Mann iterative schemes to these positive solutions.

Theorem 2.1. *Assume that there exist five constants M, N, a, b, T_0 and four functions $P, Q, R, W \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying*

$$\min\{a, b\} \ge 0, \quad a + b < 1, \quad 0 < N < (1 - a - b)M; \tag{2.1}$$

$$-a \le c(t) \le b, \quad \forall t \ge T_0 > t_0; \tag{2.2}$$

$$\begin{gathered} |h(t, u_1, \cdots, u_l)| \le P(t), \quad |f(t, u_1, \cdots, u_l)| \le Q(t), \\ \forall (t, u_i) \in [t_0, +\infty) \times [N, M], \ i \in \{1, 2, \cdots, l\}; \end{gathered} \tag{2.3}$$

$$\begin{aligned} &|h(t, u_1, \cdots, u_l) - h(t, \bar{u}_1, \cdots, \bar{u}_l)| \\ &\le R(t) \max\{|u_j - \bar{u}_j| : 1 \le j \le l\}, \\ &|f(t, u_1, \cdots, u_l) - f(t, \bar{u}_1, \cdots, \bar{u}_l)| \\ &\le W(t) \max\{|u_j - \bar{u}_j| : 1 \le j \le l\}, \\ &\quad \forall (t, u_i, \bar{u}_i) \in [t_0, +\infty) \times [N, M]^2, \ i \in \{1, 2, \cdots, l\}; \end{aligned} \tag{2.4}$$

$$\begin{aligned}&\int_{t_0}^{+\infty} s\max\{P(s), R(s)\}ds < +\infty,\\&\int_{t_0}^{+\infty} s^2\max\{Q(s), W(s), |g(s)|\}ds < +\infty.\end{aligned} \tag{2.5}$$

Then (a) *for any* $L \in (bM + N, (1-a)M)$, *there exist* $\theta \in (0,1)$ *and* $T \geq \tau + |T_0| + |t_0| + |\gamma|$ *such that for each* $x_0 \in A(N, M)$, *the Mann iterative sequence* $\{x_k\}_{k\in\mathbb{N}_0}$ *generated by the following scheme:*

$$x_{k+1}(t) = \begin{cases} (1-\alpha_k)x_k(t) + \alpha_k\big\{L - c(t)x_k(t-\tau) \\ \ -\int_t^{+\infty}(s-t)h(s, x_k(s-\alpha_1), \cdots, x_k(s-\alpha_l))ds \\ \ +\frac{1}{2}\int_t^{+\infty}(s-t)^2[f(t, x_k(s-\beta_1), \cdots, x_k(s-\beta_l)) \\ \ -g(s)]ds\big\},\ t \geq T,\ k \in \mathbb{N}_0, \\ (1-\alpha_k)x_k(T) + \alpha_k\big\{L - c(T)x_k(T-\tau) \\ \ -\int_T^{+\infty}(s-T)h(s, x_k(s-\alpha_1), \cdots, x_k(s-\alpha_l))ds \\ \ +\frac{1}{2}\int_T^{+\infty}(s-T)^2[f(T, x_k(s-\beta_1), \cdots, x_k(s-\beta_l)) \\ \ -g(s)]ds\big\},\ t_0 \leq t < T,\ k \in \mathbb{N}_0 \end{cases} \tag{2.6}$$

converges to a bounded positive solution $x \in A(N, M)$ *of Eq.*(1.6) *and has the following error estimate:*

$$\|x_{k+1} - x\| \leq e^{-(1-\theta)\sum_{p=0}^{k}\alpha_p}\|x_0 - x\|, \quad \forall k \in \mathbb{N}_0, \tag{2.7}$$

where $\{\alpha_k\}_{k\in\mathbb{N}_0}$ *is an arbitrary sequence in* $[0,1]$ *such that*

$$\sum_{k=0}^{\infty}\alpha_k = +\infty; \tag{2.8}$$

(b) *Eq.*(1.6) *has uncountably many bounded positive solutions in* $A(N, M)$.

Proof. Let $L \in (bM + N, (1-a)M)$. In view of (2.1) and (2.5), there exist $\theta \in (0,1)$ and $T \geq \tau + |T_0| + |t_0| + |\gamma|$ sufficiently large satisfying

$$\theta = a + b + \int_T^{+\infty} sR(s)ds + \frac{1}{2}\int_T^{+\infty} s^2W(s)ds \tag{2.9}$$

and

$$\begin{aligned}&\int_T^{+\infty} sP(s)ds + \frac{1}{2}\int_T^{+\infty} s^2[Q(s) + |g(s)|]ds\\&< \min\{(1-a)M - L, L - bM - N\}.\end{aligned} \tag{2.10}$$

Define a mapping $S_L : A(N,M) \to BC([\gamma,+\infty),\mathbb{R})$ by

$$(S_Lx)(t) = \begin{cases} L - c(t)x(t-\tau) \\ \ -\int_t^{+\infty}(s-t)h(s,x(s-\alpha_1),\cdots,x(s-\alpha_l))ds \\ \ +\frac{1}{2}\int_t^{+\infty}(s-t)^2[f(s,x(s-\beta_1),\cdots,x(s-\beta_l)) \\ \ \ -g(s)]ds,\ t \geq T,\ x \in A(N,M), \\ (S_Lx)(T),\ \gamma \leq t < T,\ x \in A(N,M). \end{cases} \tag{2.11}$$

From (2.2), (2.3), (2.10) and (2.11), we get that for every $x \in A(N,M)$ and $t \geq T$

$$\begin{aligned}(S_Lx)(t) &= L - c(t)x(t-\tau) - \int_t^{+\infty}(s-t)h(s,x(s-\alpha_1),\cdots,x(s-\alpha_l))ds \\ &\quad + \frac{1}{2}\int_t^{+\infty}(s-t)^2[f(s,x(s-\beta_1),\cdots,x(s-\beta_l)) - g(s)]ds \\ &\leq L + aM + \int_t^{+\infty}(s-t)|h(s,x(s-\alpha_1),\cdots,x(s-\alpha_l))|ds \\ &\quad + \frac{1}{2}\int_t^{+\infty}(s-t)^2[|f(s,x(s-\beta_1),\cdots,x(s-\beta_l))| + |g(s)|]ds \\ &\leq L + aM + \int_T^{+\infty} sP(s)ds + \frac{1}{2}\int_T^{+\infty} s^2[Q(s) + |g(s)|]ds \\ &< L + aM + \min\{(1-a)M - L, L - bM - N\} \\ &\leq M\end{aligned}$$

and

$$\begin{aligned}S_Lx(t) &\geq L - bM - \int_t^{+\infty}(s-t)|h(s,x(s-\alpha_1),\cdots,x(s-\alpha_l))|ds \\ &\quad - \frac{1}{2}\int_t^{+\infty}(s-t)^2[|f(s,x(s-\beta_1),\cdots,x(s-\beta_l))| + |g(s)|]ds \\ &\geq L - bM - \int_T^{+\infty} sP(s)ds - \frac{1}{2}\int_T^{+\infty} s^2[Q(s) + |g(s)|]ds \\ &> L - bM - \min\{(1-a)M - L, L - bM - N\} \\ &\geq N,\end{aligned}$$

which imply that $S_L(A(N,M)) \subseteq A(N,M)$. It follows from (2.1), (2.4), (2.9) and (2.11) that for every $x, y \in A(N, M)$ and $t \geq T$

$$\begin{aligned}
|(S_Lx)(t) - (S_Ly)(t)| &= \Bigg| - c(t)[x(t-\tau) - y(t-\tau)] \\
&\quad - \int_t^{+\infty} (s-t)[h(s, x(s-\alpha_1), \cdots, x(s-\alpha_l)) \\
&\qquad - h(s, y(s-\alpha_1), \cdots, y(s-\alpha_l))]ds \\
&\quad + \frac{1}{2}\int_t^{+\infty} (s-t)^2[f(s, x(s-\beta_1), \cdots, x(s-\beta_l)) \\
&\qquad - f(s, y(s-\beta_1), \cdots, y(s-\beta_l))]ds\Bigg| \\
&\leq |c(t)||x(t-\tau) - y(t-\tau)| \\
&\quad + \int_t^{+\infty} s|h(s, x(s-\alpha_1), \cdots, x(s-\alpha_l)) \\
&\qquad - h(s, y(s-\alpha_1), \cdots, y(s-\alpha_l))|ds \\
&\quad + \frac{1}{2}\int_t^{+\infty} s^2|f(s, x(s-\beta_1), \cdots, x(s-\beta_l)) \\
&\qquad - f(s, y(s-\beta_1), \cdots, y(s-\beta_l))|ds \\
&\leq (a+b)\|x-y\| \\
&\quad + \|x-y\| \int_t^{+\infty} sR(s)ds + \frac{1}{2}\|x_1 - y\| \int_t^{+\infty} s^2W(s)ds \\
&\leq (a+b)\|x-y\| \\
&\quad + \left(\int_T^{+\infty} sR(s)ds + \frac{1}{2}\int_T^{+\infty} s^2W(s)ds\right)\|x-y\| \\
&= \theta\|x-y\|,
\end{aligned}$$

which means that

$$\|S_Lx - S_Ly\| \leq \theta\|x-y\|, \quad \forall x, y \in A(N, M), \tag{2.12}$$

which yields that S_L is a contraction mapping and it has a unique fixed point $x \in A(N, M)$, which is a bounded positive solution of Eq.(1.6).

By (2.6), (2.11) and (2.12), we deduce that

$$\begin{aligned}|x_{k+1}(t)-x(t)| &= \Big|(1-\alpha_k)x_k(t)+\alpha_k\big\{L-c(t)x_k(t-\tau)\\ &\quad -\int_t^{+\infty}(s-t)h(s,x_k(s-\alpha_1),\cdots,x_k(s-\alpha_l))ds\\ &\quad +\frac{1}{2}\int_t^{+\infty}(s-t)^2[f(t,x_k(s-\beta_1),\cdots,x_k(s-\beta_l))\\ &\quad - g(s)]ds\big\}-x(t)\Big|\\ &\le (1-\alpha_k)|x_k(t)-x(t)|+\alpha_k|S_Lx_k(t)-S_Lx(t)|\\ &\le (1-\alpha_k)|x_k(t)-x(t)|+\alpha_k\theta|x_k(t)-x(t)|\\ &= (1-(1-\theta)\alpha_k)|x_k(t)-x(t)|\\ &\le e^{-(1-\theta)\alpha_k}\|x_k-x\|, \quad \forall k\in\mathbb{N}_0,\ t\ge T,\end{aligned}$$

which gives that

$$\|x_{k+1}-x\|\le e^{-(1-\theta)\alpha_k}\|x_k-x\|\le e^{-(1-\theta)\sum_{p=0}^{k}\alpha_p}\|x_0-x\|, \quad \forall k\in\mathbb{N}_0,$$

that is, (2.7) holds. It follows from (2.7) and (2.8) that $\lim_{k\to\infty}x_k=x$.

Now we show that (b) holds. Let $L_1, L_2\in(bM+N,(1-a)M)$ with $L_1\ne L_2$. As in the proof of (a), for each $i\in\{1,2\}$, we conclude that there exist $\theta_i\in(0,1)$, $T_i\ge\tau+|T_0|+|t_0|+|\gamma|$ and $S_{L_i}: A(N,M)\to BC([t_0,+\infty),\mathbb{R})$ satisfying (2.9)-(2.11), where θ, T, L and S_L are replaced by θ_i, T_i, L_i and S_{L_i}, respectively, and S_{L_i} has a unique fixed point $z_i\in A(N,M)$. Obviously, z_i is a bounded positive solution of Eq.(1.6) in $A(N,M)$, that is, for each $t\ge T_i$ and $i\in\{1,2\}$, we have

$$\begin{aligned}z_i(t) &= (S_{L_i}z_i)(t)\\ &= L_i-c(t)z_i(t-\tau)\\ &\quad -\int_t^{+\infty}(s-t)h(s,z_i(s-\alpha_1),\cdots,z_i(s-\alpha_l))]ds\\ &\quad +\frac{1}{2}\int_t^{+\infty}(s-t)^2[f(s,z_i(s-\beta_1),\cdots,z_i(s-\beta_l))\\ &\quad -g(s)]ds.\end{aligned} \tag{2.13}$$

Next we need to show that $z_1\ne z_2$. (2.1), (2.2), (2.4) and (2.13) guarantee that for any $t \ge \max\{T_1, T_2\}$

$$
\begin{aligned}
|z_1(t) - z_2(t)| = \bigg| & L_1 - L_2 - c(t)\big[z_1(t-\tau) - z_2(t-\tau)\big] \\
& - \int_t^{+\infty} (s-t)\big[h(s, z_1(s-\alpha_1), \cdots, z_1(s-\alpha_l)) \\
& \quad - h(s, z_2(s-\alpha_1), \cdots, z_2(s-\alpha_l))\big]ds \\
& + \frac{1}{2}\int_t^{+\infty} (s-t)^2\big[f(s, z_1(s-\beta_1), \cdots, z_1(s-\beta_l)) \\
& \quad - f(s, z_2(s-\beta_1), \cdots, z_2(s-\beta_l))\big]ds\bigg| \\
\ge{} & |L_1 - L_2| - |c(t)|\big|z_1(t-\tau) - z_2(t-\tau)\big| \\
& - \int_t^{+\infty} s\big|h(s, z_1(s-\alpha_1), \cdots, z_1(s-\alpha_l)) \\
& \quad - h(s, z_2(s-\alpha_1), \cdots, z_2(s-\alpha_l))\big|ds \\
& - \frac{1}{2}\int_t^{+\infty} s^2\big|f(s, z_1(s-\beta_1), \cdots, z_1(s-\beta_l)) \\
& \quad - f(s, z_2(s-\beta_1), \cdots, z_2(s-\beta_l))\big|ds \\
\ge{} & |L_1 - L_2| - (a+b)\|z_1 - z_2\| \\
& - \left(\int_{\max\{T_1,T_2\}}^{+\infty} sR(s)ds + \frac{1}{2}\int_{\max\{T_1,T_2\}}^{+\infty} s^2W(s)ds\right)\|z_1 - z_2\| \\
\ge{} & |L_1 - L_2| - \max\{\theta_1, \theta_2\}\|z_1 - z_2\|,
\end{aligned}
$$

which means that

$$
\|z_1 - z_2\| \ge \frac{|L_1 - L_2|}{1 + \max\{\theta_1, \theta_2\}} > 0,
$$

that is, $z_1 \ne z_2$. This completes the proof. □

Theorem 2.2. *Assume that there exist four constants M, N, b, T_0 and four functions $P, Q, R, W \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying* (2.3)-(2.5) *and*

$$
0 < N < (1-b)M, \quad 0 \le c(t) \le b < 1, \quad \forall t \ge T_0 > t_0. \tag{2.14}
$$

Then (a) *for any $L \in (bM + N, M)$, there exist $\theta \in (0,1)$ and $T \ge \tau + |T_0| + |t_0| + |\gamma|$ such that for each $x_0 \in A(N, M)$, the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by* (2.6) *converges to a positive solution $x \in A(N, M)$ of Eq.*(1.6) *and has the error estimate* (2.7)*, where $\{\alpha_k\}_{k\in\mathbb{N}_0}$ is an arbitrary sequence in* $[0, 1]$ *satisfying* (2.8);

(b) *Eq.*(1.6) *has uncountably many positive solutions in $A(N, M)$.*

Proof. Let $L \in (bM + N, M)$. It follows from (2.5) and (2.14) that there exist $\theta \in (0,1)$ and $T \geq \tau + |T_0| + |t_0| + |\gamma|$ sufficiently large satisfying

$$\theta = b + \int_T^{+\infty} sR(s)ds + \frac{1}{2}\int_T^{+\infty} s^2 W(s)ds \tag{2.15}$$

and

$$\begin{aligned}&\int_T^{+\infty} sP(s)ds + \frac{1}{2}\int_T^{+\infty} s^2[Q(s) + |g(s)|]ds \\ &< \min\{M - L, L - bM - N\}.\end{aligned} \tag{2.16}$$

Define a mapping $S_L : A(N,M) \to BC([\gamma, +\infty), \mathbb{R})$ by (2.11). By virtue of (2.3), (2.11), (2.14) and (2.16), we obtain that for every $x \in A(N,M)$ and $t \geq T$

$$\begin{aligned}(S_L x)(t) &= L - c(t)x(t-\tau) - \int_t^{+\infty} (s-t)h(s, x(s-\alpha_1), \cdots, x(s-\alpha_l))ds \\ &\quad + \frac{1}{2}\int_t^{+\infty} (s-t)^2\big[f(s, x(s-\beta_1), \cdots, x(s-\beta_l)) - g(s)\big]ds \\ &\leq L + \int_t^{+\infty} s\big|h(s, x(s-\alpha_1), \cdots, x(s-\alpha_l))\big|ds \\ &\quad + \frac{1}{2}\int_t^{+\infty} s^2\big[|f(s, x(s-\beta_1), \cdots, x(s-\beta_l))| + |g(s)|\big]ds \\ &\leq L + \int_T^{+\infty} sP(s)ds + \frac{1}{2}\int_T^{+\infty} s^2[Q(s) + |g(s)|]ds \\ &< L + \min\{M - L, L - bM - N\} \\ &\leq M\end{aligned}$$

and

$$\begin{aligned}(S_L x)(t) &\geq L - bM - \int_t^{+\infty} s\big|h(s, x(s-\alpha_1), \cdots, x(s-\alpha_l))\big|ds \\ &\quad - \frac{1}{2}\int_t^{+\infty} s^2\big[|f(s, x(s-\beta_1), \cdots, x(s-\beta_l))| + |g(s)|\big]ds \\ &\geq L - bM - \int_T^{+\infty} sP(s)ds - \frac{1}{2}\int_T^{+\infty} s^2[Q(s) + |g(s)|]ds \\ &> L - bM - \min\{M - L, L - bM - N\} \\ &\geq N,\end{aligned}$$

which imply that $S_L(A(N,M)) \subseteq A(N,M)$. Using (2.4), (2.11), (2.14) and (2.15), we infer easily that (2.12) holds. Thus S_L is a contraction mapping and it owns a unique fixed point $x \in A(N, M)$, which is a bounded positive solution of Eq.(1.6). The rest of the proof is similar to that of Theorem 2.1, and is omitted. This completes the proof. □

Theorem 2.3. *Assume that there exist four constants M, N, a, T_0 and four functions $P, Q, R, W \in C([t_0, +\infty), \mathbb{R}^+)$ satisfying* (2.3)-(2.5) *and*

$$0 < N < (1-a)M, \quad -1 < -a \le c(t) \le 0, \quad \forall t \ge T_0 > t_0. \tag{2.17}$$

Then (a) *for any $L \in (N, (1-a)M)$, there exist $\theta \in (0,1)$ and $T \ge \tau + |T_0| + |t_0| + |\gamma|$ such that for each $x_0 \in A(N, M)$, the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by* (2.6) *converges to a positive solution $x \in A(N, M)$ of Eq.*(1.6) *and has the error estimate* (2.7), *where $\{\alpha_k\}_{k\in\mathbb{N}_0}$ is an arbitrary sequence in* $[0, 1]$ *satisfying* (2.8);

(b) *Eq.*(1.6) *has uncountably many positive solutions in $A(N, M)$.*

Proof. Let $L \in (N, (1-a)M)$. Following from (2.5) and (2.17), we get that there exist $\theta \in (0, 1)$ and $T \ge \tau + |T_0| + |t_0| + |\gamma|$ sufficiently large satisfying

$$\theta = a + \int_T^{+\infty} sR(s)ds + \frac{1}{2}\int_T^{+\infty} s^2 W(s)ds \tag{2.18}$$

and

$$\begin{aligned}&\int_T^{+\infty} sP(s)ds + \frac{1}{2}\int_T^{+\infty} s^2[Q(s) + |g(s)|]ds \\ &< \min\{(1-a)M - L, L - N\}.\end{aligned} \tag{2.19}$$

Define a mapping $S_L : A(N, M) \to BC([\gamma, +\infty), \mathbb{R})$ by (2.11). On account of (2.3), (2.11), (2.17) and (2.19), we acquire that for every $x \in A(N, M)$ and $t \ge T$

$$\begin{aligned}(S_L x)(t) &= L - c(t)x(t-\tau) - \int_t^{+\infty} (s-t)h(s, x(s-\alpha_1), \cdots, x(s-\alpha_l))ds \\ &\quad + \frac{1}{2}\int_t^{+\infty} (s-t)^2\big[f(s, x(s-\beta_1), \cdots, x(s-\beta_l)) - g(s)\big]ds \\ &\le L + aM + \int_t^{+\infty} s\big|h(s, x(s-\alpha_1), \cdots, x(s-\alpha_l))\big|ds \\ &\quad + \frac{1}{2}\int_t^{+\infty} s^2\big[|f(s, x(s-\beta_1), \cdots, x(s-\beta_l))| + |g(s)|\big]ds \\ &\le L + aM + \int_T^{+\infty} sP(s)ds + \frac{1}{2}\int_T^{+\infty} s^2[Q(s) + |g(s)|]ds \\ &< L + aM + \min\{(1-a)M - L, L - N\} \\ &\le M\end{aligned}$$

and

$$
\begin{aligned}
(S_L x)(t) \geq L &- \int_t^{+\infty} s\big|h(s, x(s-\alpha_1), \cdots, x(s-\alpha_l))\big| ds \\
&- \frac{1}{2}\int_t^{+\infty} s^2\big[|f(s, x(s-\beta_1), \cdots, x(s-\beta_l))| + |g(s)|\big] ds \\
&\geq L - \int_T^{+\infty} sP(s)ds - \frac{1}{2}\int_T^{+\infty} s^2[Q(s) + |g(s)|]ds \\
&> L - \min\{(1-a)M - L, L - N\} \\
&\geq N,
\end{aligned}
$$

which yield that $S_L(A(N, M)) \subseteq A(N, M)$. From (2.4), (2.11), (2.17) and (2.18), it is easy to prove that (2.12) holds. Therefore S_L is a contraction mapping and it possesses a unique fixed point $x \in A(N, M)$, which is a bounded positive solution of Eq.(1.6). The rest of the proof is analogous to that of Theorem 2.1, and is omitted. This completes the proof. □

3. Examples

In this section, we construct three examples to show the applications of the results in Section 2.

Example 3.1. Consider the third order neutral delay differential equation

$$
\begin{aligned}
&\left(x(t) + \frac{1 - 3\sin(t^2+1)}{7 + 2\cos^2(t-1)} x(t-\tau)\right)''' \\
&\quad + \left(\frac{\sqrt{5t+1}\,x^2(t-3) - tx^3(t-4)}{t^4 + t^2 + 1}\right)' + \frac{t^2 x(t-2) x^2(t-13)}{3(t+1)^6 + 2t^3 + 1} \\
&= \frac{\sqrt{t^2+3t+1}\sin(5t^3+3t+2)}{t^5 + 4t^3 + 2}, \quad \forall t \geq 0,
\end{aligned}
\tag{3.1}
$$

where $\tau > 0$ is a constant. Let $l = 2$, $t_0 = 0$, $M = 10$, $N = 1$, $a = \frac{2}{7}$, $b = \frac{4}{7}$, $T_0 = 1$, $\gamma = -\max\{\tau, 13\}$ and $\forall (t, u, v) \in [t_0, +\infty) \times [N, M]^2$

$$
\begin{aligned}
c(t) &= \frac{1-3\sin(t^2+1)}{7+2\cos^2(t-1)}, \quad \alpha_1 = 3, \quad \alpha_2 = 4, \\
\beta_1 &= 2, \quad \beta_2 = 13, \quad h(t,u,v) = \frac{\sqrt{5t+1}u^2 - tv^3}{t^4+t^2+1}, \\
f(t,u,v) &= \frac{t^2uv^2}{3(t+1)^6+2t^3+1}, \\
g(t) &= \frac{\sqrt{t^2+3t+1}\sin(5t^3+3t+2)}{t^5+4t^3+2}, \\
P(t) &= \frac{M^2\sqrt{5t+1}+M^3t}{t^4+t^2+1}, \quad Q(t) = \frac{M^3t^2}{3(t+1)^6+2t^3+1}, \\
R(t) &= \frac{2M\sqrt{5t+1}+3M^2t}{t^4+t^2+1}, \quad W(t) = \frac{3M^2t^2}{3(t+1)^6+2t^3+1}.
\end{aligned}
$$

Obviously, (2.1)-(2.5) are fulfilled. It follows from Theorem 2.1 that Eq.(3.1) has uncountably many bounded positive solutions in $A(N,M)$, and for every $L \in (bM+N,(1-a)M)$, there exist $\theta \in (0,1)$ and $T \geq \tau + |T_0| + |t_0| + |\gamma|$ such that the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by (2.6) and (2.8) converges to a bounded positive solution $x \in A(N,M)$ of Eq.(3.1) and has the error estimate (2.7).

Example 3.2. Consider the third order neutral delay differential equation

$$
\begin{aligned}
&\left(x(t) + \frac{2t\arctan(t^2\ln t+2t+1)}{3t^2+3}x(t-\tau)\right)''' \\
&\quad + \left(\frac{(t+1)x^3(t-7)x(t-1)}{6t^5+5t^4+3x^2(t-1)}\right)' \\
&\quad + \frac{\sin(1+tx^2(t-5))+t^2x(t-6)}{t^6+(t+2)^4+\sqrt{3t^2+1}+2} \\
&= \frac{t^3+t\ln(t^2+1)+1}{7t^8+3t^3+2}, \quad \forall t \geq 1,
\end{aligned}
\tag{3.2}
$$

where $\tau > 0$ is a constant. Let $l = 2$, $t_0 = 1$, $M = 7$, $N = 2$, $b = \frac{\pi}{6}$, $T_0 = 2$, $\gamma = 1 - \max\{\tau, 7\}$ and $\forall(t,u,v) \in [t_0,+\infty) \times [N,M]^2$

$$\begin{aligned}
c(t) &= \frac{2t\arctan(t^2\ln t + 2t + 1)}{3t^2 + 3}, \quad \alpha_1 = 7, \quad \alpha_2 = 1,\\
\beta_1 &= 5, \quad \beta_2 = 6, \quad h(t,u,v) = \frac{(t+1)u^3 v}{6t^5 + 5t^4 + 3v^2},\\
f(t,u,v) &= \frac{\sin(1+tu^2) + t^2 v}{t^6 + (t+2)^4 + \sqrt{3t^2+1} + 2},\\
g(t) &= \frac{t^3 + t\ln(t^2+1) + 1}{7t^8 + 3t^3 + 2}, \quad P(t) = \frac{M^4(t+1)}{6t^5 + 5t^4 + 3N^2},\\
Q(t) &= \frac{1 + Mt^2}{t^6 + (t+2)^4 + \sqrt{3t^2+1} + 2},\\
R(t) &= \frac{4M^3(t+1)(6t^5 + 5t^4 + 3M^2)}{(6t^5 + 5t^4 + 3N^2)^2},\\
W(t) &= \frac{2Mt + t^2}{t^6 + (t+2)^4 + \sqrt{3t^2+1} + 2}.
\end{aligned}$$

It is easy to verify that (2.3)-(2.5) and (2.14) hold. Thus Theorem 2.2 ensures that Eq.(3.2) has uncountably many bounded positive solutions in $A(N,M)$, and for every $L \in (bM+N, M)$, there exist $\theta \in (0,1)$ and $T \geq \tau + |T_0| + |t_0| + |\gamma|$ such that the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by (2.6) and (2.8) converges to a bounded positive solution $x \in A(N,M)$ of Eq.(3.2) and has the error estimate (2.7).

Example 3.3. Consider the third order neutral delay differential equation

$$\begin{aligned}
&\left(x(t) - \frac{t^2\sin^2(5t^3-1)}{1+3t^2}x(t-\tau)\right)'''\\
&\quad + \left(\frac{t\ln(1+2tx^2(t-9)|x(t-10)|) - x^4(t-8)}{5t^5 + 4t^2 + 1}\right)'\\
&\quad + \frac{t^3\sqrt{t+3}\,x^2(t-100)}{(t+2)^9 + (t+1)^2x^2(t-\frac{1}{2})|x^3(t-5)|}\\
&= \frac{(3t^4 + 2t^3 + t^2 + 1)\sin(2t)}{t^{11} + t^2\cos(5t+1) + 3}, \quad \forall t \geq 0,
\end{aligned} \tag{3.3}$$

where $\tau > 0$ is a constant. Let $l = 3$, $t_0 = 0$, $M = 5$, $N = 1$, $a = \frac{1}{3}$, $T_0 = 1$,

$\gamma = -\max\{\tau, 100\}$ and $\forall (t,u,v,w) \in [t_0, +\infty) \times [N, M]^3$

$$c(t) = -\frac{t^2 \sin^2(5t^3-1)}{1+3t^2}, \quad \alpha_1 = 9, \quad \alpha_2 = 10, \quad \alpha_3 = 8,$$
$$\beta_1 = 100, \quad \beta_2 = \frac{1}{2}, \quad \beta_3 = 5,$$
$$h(t,u,v,w) = \frac{t \ln(1+2tu^2|v|) - w^4}{5t^5+4t^2+1},$$
$$f(t,u,v,w) = \frac{t^3\sqrt{t+3}u^2}{(t+2)^9 + (t+1)^2 v^2 |w|^3},$$
$$g(t) = \frac{(3t^4+2t^3+t^2+1)\sin(2t)}{t^{11}+t^2\cos(5t+1)+3},$$
$$P(t) = \frac{t\ln(1+2M^3t)+M^4}{5t^5+4t^2+1},$$
$$Q(t) = \frac{M^2t^3\sqrt{t+3}}{(t+2)^9+N^5(t+1)^2}, \quad R(t) = \frac{\frac{6M^2t^2}{1+2N^3t}+4M^3}{5t^5+4t^2+1},$$
$$W(t) = \frac{t^3\sqrt{t+3}[2M(t+2)^9+7M^6(t+1)^2]}{[(t+2)^9+N^5(t+1)^2]^2}.$$

Clearly, (2.3)-(2.5) and (2.17) hold. Hence Theorem 2.3 ensures that Eq.(3.3) has uncountably many bounded positive solutions in $A(N, M)$, and for every $L \in (N, (1-a)M)$, there exist $\theta \in (0, 1)$ and $T \geq \tau + |T_0| + |t_0| + |\gamma|$ such that the Mann iterative sequence $\{x_k\}_{k\in\mathbb{N}_0}$ generated by (2.6) and (2.8) converges to a bounded positive solution $x \in A(N, M)$ of Eq.(3.3) and has the error estimate (2.7).

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