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THE UPPER HULL NUMBER OF A GRAPH

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Abstract: For a connected graph $G = (V, E)$, the hull number $h(G)$ of a graph G is the minimum cardinality of a set of vertices whose convex hull contains all vertices of G. A hull set S in a connected graph G is called a minimal hull set of G if no proper subset of S is a hull set of G. The upper hull number $h^+(G)$ of G is the maximum cardinality of a minimal hull set of G. Connected graphs of order p with upper hull number p or $p - 1$ are characterized. It is shown that for every integer $a \geq 2$, there exists a connected graph G with $h(G) = a$ and $h^+(G) = 2a$. A graph G is an extreme hull graph if $h(G) = ex(G)$, that is if G has a unique minimum hull set consisting of the extreme vertices of G. It is shown that for every pair a, b of integers with $2 \leq a \leq b$, there exists a connected extreme hull graph G such that $h(G) = a = ex(G)$ and $g(G) = b$, where $g(G)$ is the geodetic number of a graph.

AMS Subject Classification: 05C12
Key Words: hull number, upper hull number, geodetic number

1. Introduction

By a graph $G = (V, E)$, we mean a finite undirected connected graph without loops or multiple edges. The order and size of G are denoted by p and q

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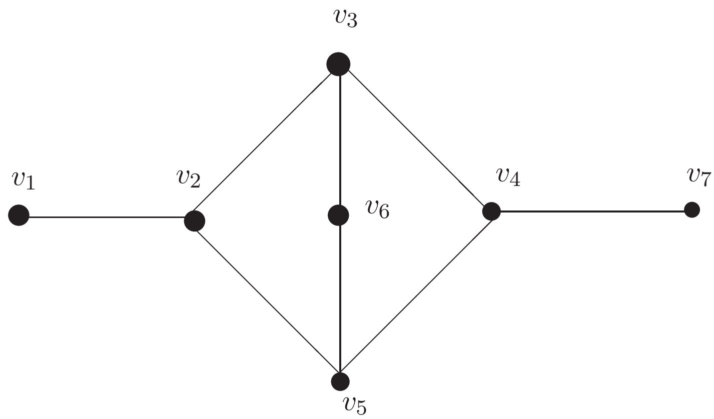
respectively. For basic graph theoretic terminology, we refer to Harary [1] and [7]. A convexity on a finite set V is a family C of subsets of V, convex sets which are closed under intersection and which contains both V and the empty set. The pair(V, E) is called a convexity space. A finite graph convexity space is a pair (V, E), formed by a finite connected graph $G = (V, E)$ and a convexity C on V such that (V, E) is a convexity space satisfying that every member of C induces a connected subgraph of G. Thus, classical convexity can be extended to graphs in a natural way. We know that a set X of R^n is convex if every segment joining two points of X is entirely contained in it. Similarly a vertex set W of a finite connected graph is said to be convex set of G if it contains all the vertices lying in a certain kind of path connecting vertices of W[8]. The distance $d(u, v)$ between two vertices u and v in a connected graph G is the length of a shortest $u-v$ path in G. An $u-v$ path of length $d(u, v)$ is called an $u-v$ geodesic. A vertex x is said to lie on a $u-v$ geodesic P if x is a vertex of P including the vertices u and v. The eccentricity $e(v)$ of a vertex v in G is the maximum distance from v and a vertex of G. The minimum eccentricity among the vertices of G is the radius, rad G or $r(G)$ and the maximum eccentricity is its diameter, diam G of G. For two vertices u and v, let $I[u, v]$ denotes the set of all vertices which lie on $u-v$ geodesic. For a set S of vertices, let $I[S] = \bigcup_{u,v \in S} I[u, v]$. The set S is convex if $I[S] = S$. Clearly if $S = \{v\}$or $S = V$, then S is convex. The convexity number, denoted by $C(G)$, is the cardinality of a maximum proper convex subset of V. The smallest convex set containing S is denoted by $I_h[S]$ and called the convex hull of S. Since the intersection of two convex sets is convex, the convex hull is well defined. Note that $S \subseteq I[S] \subseteq I_h[S] \subseteq V$. A subset $S \subseteq V$ is called a geodetic set if $I[S] = V$ and a hull set if $I_h[S] = V$. The geodetic number $g(G)$ of G is the minimum order of its geodetic sets and any geodetic set of order $g(G)$ is a minimum geodetic set or simply a g- set of G. Similarly, the hull number $h(G)$ of G is the minimum order of its hull sets and any hull set of order $h(G)$ is a minimum hull set or simply a h- set of G. The geodetic number of a graph is studied in [1],[2],[3],[5],[9]and[11] and the hull number of a graph is studied in [1],[4],[6]and[11]. For the graph G given in Figure 1.1, $S = \{v_1, v_7\}$ is a h- set of G so that $h(G) = 2$ and also $S_1 = \{v_1, v_6, v_7\}$ is a g-set of G so that $g(G) = 3$. A vertex v is an extreme vertex of a graph G if the subgraph induced by its neighbors is complete. Throughout the following G denotes a connected graph with at least two vertices.



G
Figurer 1.1

The following theorems are used in the sequel.

Theorem 1.1. (see [5]) *Each extreme vertex of a connected graph G belongs to every geodetic set of G.*

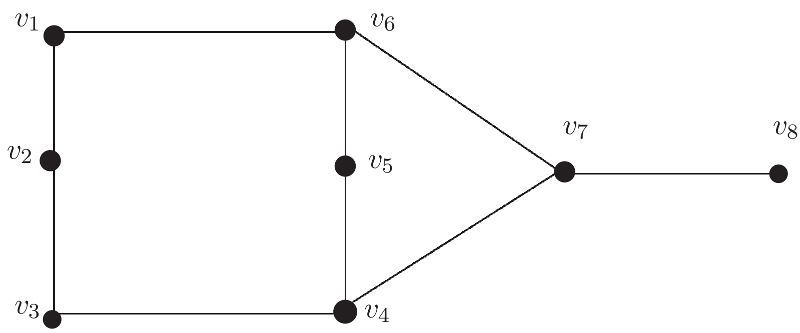
Theorem 1.2. (see [1]) *For a connected graph $G, h(G) = p$ if and only if $G = K_p$*

Theorem 1.3. (see [1]) *For a connected graph $G, h(G) = p-1$ if and only if $G = K_1 \bigcup m_j K_j$, where $\sum m_j \geq 2$.*

2. The Upper Hull Number of a Graph

Definition 2.1. A hull set S in a connected graph G is called a minimal hull set of G if no proper subset of S is a hull set of G. The upper hull number $h^+(G)$ of G is the maximum cardinality of a minimal hull set of G.

Example 2.2. For the graph G given in Figure 2.1, $S_1 = \{v_2, v_8\}$ is a h-set of G so that $h(G) = 2$. The sets $S_2 = \{v_1, v_3, v_8\}, S_3 = \{v_1, v_4, v_8\}, S_4 = \{v_3, v_6, v_8\}, S_5 = \{v_1, v_5, v_8\}$ and $S_6 = \{v_3, v_5, v_8\}$ are also hull sets of G. Since no proper subsets of $S_i (2 \leq i \leq 6)$ is a hull set of $G, S_i (2 \leq i \leq 6)$ is a minimal hull set of S so that $h^+(G) \geq 3$. It is easily verified that there is no minimal hull set S of G with $|S| \geq 4$.Hence $h^+(G) = 3$.



G
Figure 2.1

Remark 2.3. Every minimum hull set of G is a minimal hull set of G and the converse is not true. For the graph G given in Figure 2.1, $S_2 = \{v_1, v_3, v_8\}$ is a minimal hull set but not a minimum hull set of G.

Theorem 2.4. *Each extreme vertex of G belongs to every hull set of G.*

Proof. Let S be a hull set of G and v an extreme vertex of G. Suppose that $v \notin S$. Then v is an internal vertex of a geodesic, which is a contradiction to v an extreme vertex of G. □

The proof of the following theorem is straight forward so we omit it.

Theorem 2.5. *For any connected graph G, no cut-vertex of G belongs to any minimal hull set of G.*

Corollary 2.6. For any non-trivial tree, upper hull number $h^+(T) = k$, where k is the number of end vertices of T.

Proof. This follows from Theorem 2.4 and 2.5. □

Corollary 2.7. For a complete graph $K_p, p \geq 2, h^+(K_p) = p$.

Proof. This follows from Theorem 2.4. □

Theorem 2.8. *For a complete bipartite graph $G = K_{m,n}(m, n \geq 2), S = \{u, v\}$ is a minimum hull set of G if and only if u and v are independent*

Proof. Let $S = \{u, v\}$, be a minimum hull set of G. Suppose that u and v are adjacent. Then $I_h[S] = S$ and so S is not a hull set of G, which is a contradiction. Conversely, let $S = \{u, v\}$, where u and v are independent.It is clear that $I_h[S] = V$ so that S is a hull set of G. Since $|S| = 2, S$ is a minimum hull set of G. □

Theorem 2.9. *For a complete bipartite graph* $G = K_{m,n}(m, n \geq 2), h^+(G) = 2$.

Proof. Let $U = \{x_1, x_2, ..., x_m\}$ and $W = \{y_1, y_2, ..., y_n\}$ be a bipartite set of G. Let $S = \{x, y\}$, where x and y are two independent vertices of G. Then S is a hull set of G, so that $h(G) = 2$. We prove that $h^+(G) = 2$. If not, let S_1 be a minimal hull set of G. with $|S_1| \geq 3$. Then S_1 consists of at least two independent vertices of G say, u, v. By Theorem 2.8, $\{u, v\}$ is a hull set of G, which is a contradiction to S_1 a minimal hull set of G. Hence $h^+(G) = 2$. □

Theorem 2.10. *For a connected graph* $G, 2 \leq h(G) \leq h^+(G) \leq p$.

Proof. Any hull set needs at least two vertices and so $h(G) \geq 2$. Since every minimal hull set is a hull set, $h(G) \leq h^+(G)$. Also, since V is a hull set of G, it is clear that $h^+(G) \leq p$. Thus $2 \leq h(G) \leq h^+(G) \leq p$. □

Remark 2.11. The bounds in Theorem 2.10 are sharp. For the graph G given in Figure 2.1, $h(G) = 2$. For any non-trivial tree $T, h(T) = h^+(T)$ and for the complete graph $G = K_p, h^+(G) = p$. Also, all the inequalities in Theorem 2.10 are strict. For the graph G given in Figure 2.2, $S_1 = \{v_1, v_4, v_5\}, S_2 = \{v_2, v_4, v_7\}$ and $S_3 = \{v_3, v_5, v_7\}$ are the only three h-sets of G so that $h(G) = 3$. Also $S_4 = \{v_1, v_2, v_3, v_6\}$ is a minimal hull set of G and so $h^+(G) \geq 4$. It is easily verified that there is no minimal hull set S of G with $|S| \geq 5$ and hence $h^+(G) = 4$. Thus $2 < h(G) < h^+(G) < p$.

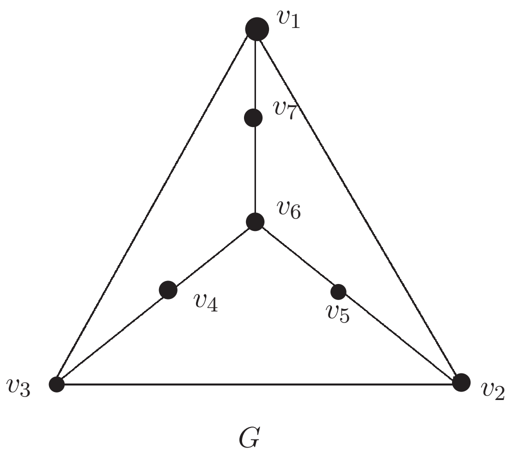


Figure 2.2

Theorem 2.12. *For a connected graph $G, h(G) = p$ if and only if $h^+(G) = p$.*

Proof. Let $h^+(G) = p$. Then $S = V$ is the unique minimal hull set of G. Since no proper subset of S is a hull set, it is clear that S is the unique minimum hull set of G and so $h(G) = p$. The converse follows from Theorem 2.10. □

Corollary 2.13. For a connected graph G of order p, the following are equivalent:
(i) $h(G) = p$
(ii) $h^+(G) = p$
(iii) $G = K_p$.

Theorem 2.14. *Let G be a non complete connected graph without cut vertices. Then $h^+(G) \leq p - 2$.*

Proof. Suppose that $h^+(G) \geq p-1$. Then by Corollary 2.13, $h^+(G) = p-1$. Let v be a vertex of G and let $S = V - \{v\}$ be a minimal hull set of G. By Theorem 2.4, v is not an extreme vertex of G. Then there exist $x, y \in N(v)$ such that $xy \notin E(G)$. Since v is not a cut vertex of $G, < G - v >$ is connected and also $< G - v >$ contains a geodesic of length at least two. Let $x, x_1, x_2, ..., x_n, y$ be a geodesic in $< G - v >$ of length at least two. Then $S_1 = S - \{x_1\}$ is a hull set of G. Since $S_1 \subseteq S, S_1$ is not a minimal hull set of G, which is a contradiction. Therefore $h^+(G) \leq p - 2$. □

Theorem 2.15. *For a connected graph G, $h(G) = p - 1$ if and only if $h^+(G) = p - 1$.*

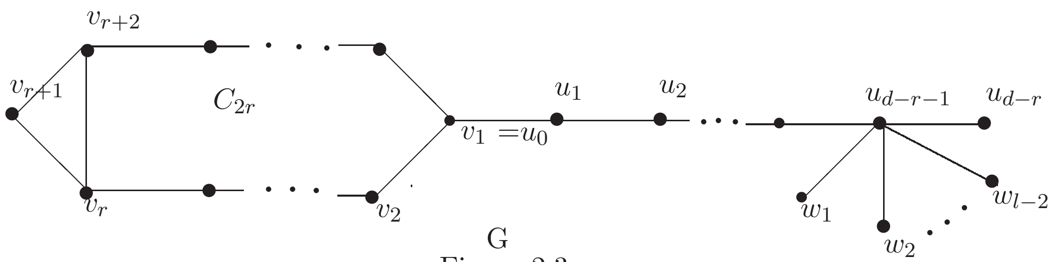
Proof. Let $h(G) = p-1$. Then it follows from Theorem 2.10, $h^+(G) = p$ or $p-1$. If $h^+(G) = p$, then by Theorem 2.12, $h(G) = p$, which is a contradiction. Hence $h^+(G) = p - 1$. Conversely, let $h^+(G) = p - 1$, then it follows from Corollary 2.13 that G is non-complete. Hence by Theorem 2.14, G contains a cut vertex, say v. Since $h^+(G) = p - 1$, it follows from Theorem 2.5 that $S = V - \{v\}$ is the unique minimal hull set of G. Therefore $h(G) = p - 1$. □

Corollary 2.16. For a connected graph G of order p, the following are equivalent: (i) $h(G) = p - 1$
(ii) $h^+(G) = p - 1$
(iii) $G = K_1 + \bigcup m_j K_j$.

For every connected graph $G, radG \leq diamG \leq 2radG$. Ostrand[10] showed that every two positive integers a and b with $a \leq b \leq 2b$ are realizable as the radius and diameter, respectively, of some connected graph. Now, Ostrand's theorem can be extended so that the upper hull number can also be prescribed.

Theorem 2.17. *For positive integers r, d and $l \geq 2$ with $r < d \leq 2r$, there exists a connected graph G with $radG = r, diamG = d, h^+(G) = l$.*

Proof. When $r = 1$, let $G = K_{1,l}$.Then $d = 2$ and by Corollary 2.6, $h^+(G) = l$. Now, let $r \geq 2$. Suppose $l \geq 2$. Construct a graph G as follows. Let $C_{2r} : v_1, v_2, ..., v_{2r}, v_1$ be a cycle of order $2r$ and let $P_{d-r+1} : u_0, u_1, u_2, ..., u_{d-r}$ be a path of order $d - r + 1$. Let H be the graph obtained from C_{2r} and P_{d-r+1} by identifying v_1 in C_{2r} and u_0 in P_{d-r+1}. Add $(l - 2)$ new vertices $w_1, w_2, ..., w_{l-2}$ to H and join each vertex $w_i (1 \leq i \leq l-2)$ to the vertex u_{d-r-1} and join the vertices v_r and v_{r+2} and obtain the graph G of Figure 2.3. Then $radG = r$ and $diamG = d$. Let $W = \{v_{r+1}, w_1, w_2, ..., w_{l-2}, u_{d-r}\}$ be the set of all extreme vertices of G. By Theorem 2.4, W is contained in every hull set of G. It is clear that W is a minimal hull set of G so that $h^+(G) = l$. □



G
Figure 2.3

In view of Theorem 2.10, we have the following realization result.

Theorem 2.18. *For any positive integer $2 \leq a$, there exists a connected graph G such that $h(G) = a$ and $h^+(G) = 2a$.*

Proof. Let $R_i : w_i, x_i, y_i, z_i, w_i (1 \leq i \leq a)$ be a copy of cycle C_4. Let G be the graph given in Figure 2.4 is obtained from R_i by adding new vertex v and the edges $vw_i, vy_i (1 \leq i \leq a)$. Let $H_i = \{x_i, z_i\}$. Then it is easily observed that every minimum hull set contains exactly one vertex from each $H_i (1 \leq i \leq a)$ and so $h(G) \geq a$. Let $W = \{x_1, x_2, ..., x_a\}$. Then $I_h[W] = V$and so W is a hull set of G. Hence $h(G) = a$. Now, $S = \{w_1, w_2, ..., w_a, y_1, y_2, ..., y_a\}$ is a hull set of G. We show that S is a minimal hull set of G. Let M be any proper subset of S. Then there exist at least one vertex, say $u \in S$ such that $u \notin M$. First assume that $u = w_i$ for some $i(1 \leq i \leq a)$. Then $I_h[M] \neq V$ and so M is not a hull set of G. Next assume that $u = y_j$ for some $j(1 \leq j \leq a)$. Then also $I_h[M] \neq V$ and so M is not a hull set of G. Hence S is a minimal hull set of G so that $h^+(G) \geq 2a$. Since every minimum hull set contains exactly one vertex from each $H_i(1 \leq i \leq a)$, it follows that there is no minimal hull set $X$of G with $|X| \geq 2a + 1$. Thus $h^+(G) = 2a$. □

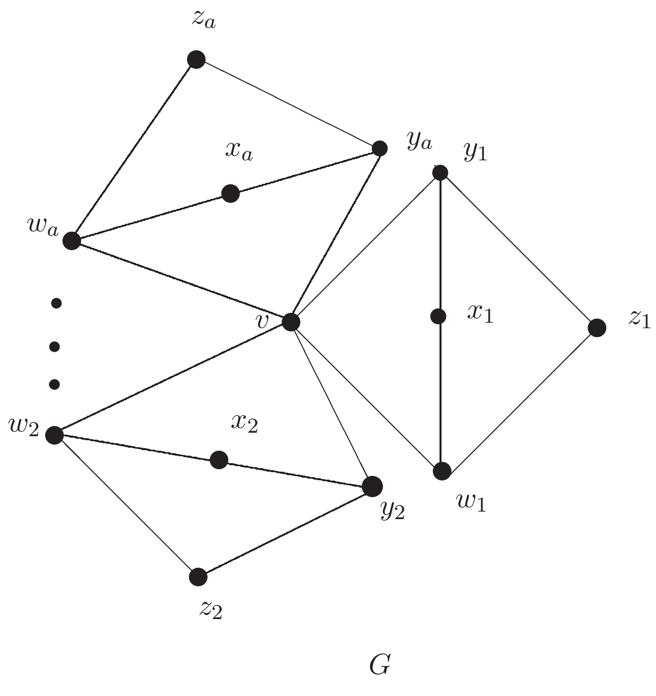


Figure 2.4

We leave the following as an open problem.

Problem 2.19. For any pair of integers a, b with $2 \leq a \leq b$, does there exists a connected graph G such that $h(G) = a$ and $h^+(G) = b$?

3. Extreme Hull Graphs

By Theorem 2.4, $0 \leq ex(G) \leq h(G)$,where $ex(G)$ = number of extreme vertices of G. A graph G is an extreme hull graph if $h(G) = ex(G)$, that is if G has a unique minimum hull set, consisting of the extreme vertices of G. The graph G given in Figure 1.1 has two extreme vertices namely $v_1$and v_8. Since $h(G) = 2 = ex(G), G$ is an extreme hull graph. On the other hand the graph given in Figure 3.1 contains two extreme vertices namely v_1 and v_7. But $\{v_1, v_8\}$ is not a hull set of G. However $\{v_1, v_5, v_8\}$ is a hull set of G so that $h(G) = 3$. Since $h(G) \neq ex(G), G$ is not a extreme hull graph.

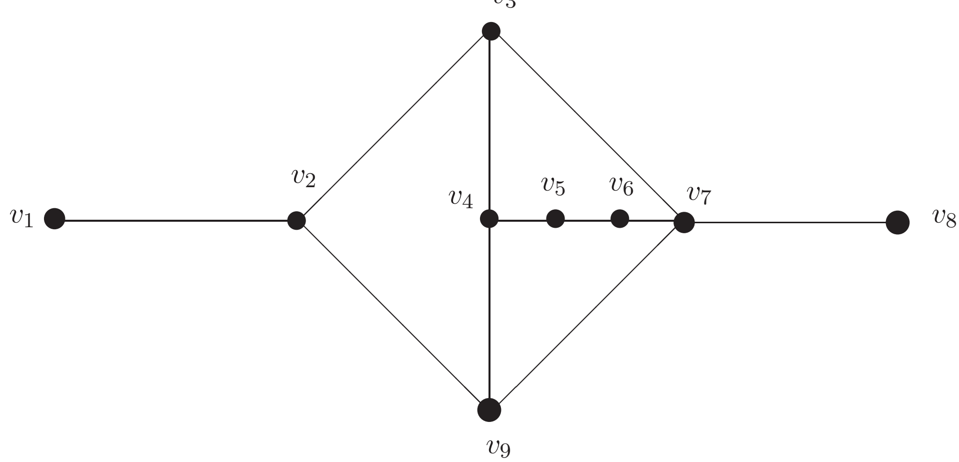


Figure 3.1

For $p \geq 2$, the complete graph K_p is the only connected graph of order p having the largest possible hull number, namely p. Since every vertex of K_p is an extreme vertex, $ex(K_p) = h(K_p) = p$. Hence K_p is an extreme hull graph. Every tree of order p is an extreme hull graph. Obviously, a cycle $C_p(p \geq 4)$ contains no extreme vertices and so C_p is not an extreme hull graph. Similarly, no complete bipartite graph $K_{m,n}$ with $2 \leq m \leq n$ is an extreme hull graph.

Theorem 3.1. *If every vertex of G is either a cut vertex or an extreme vertex, then G is an extreme hull graph.*

Proof. This follows from Theorems 2.4 and 2.5 □

Theorem 3.2. *Let G be a connected graph of order $p \geq 2$. Then G is an* *extreme hull graph with hull number p if and only if $G = K_p$.*

Proof. This follows from Theorem 1.2. □

Theorem 3.3. *Let G be a connected graph of order $p \geq 3$. Then G is an extreme hull graph with hull number $p-1$ if and only if $G = K_1 + \cup m_j K_j$, where $\sum m_j \geq 2$.*

Proof. This follows from Theorem 1.3. □

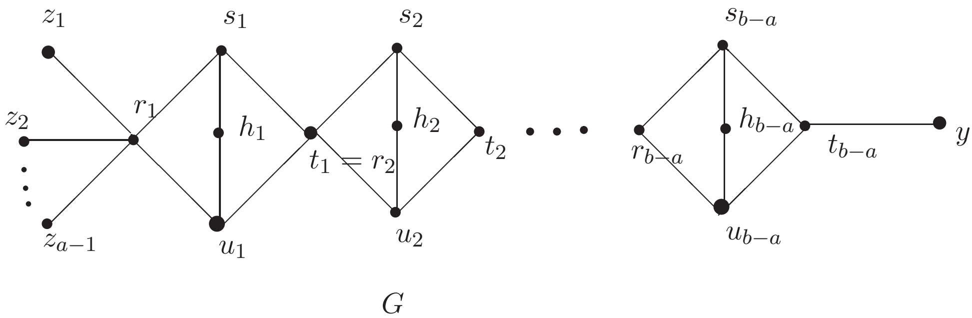


Figure 3.2

We have the following realization result.

Theorem 3.4. *For every pair a, b of integers with $2 \leq a \leq b$, there exists a connected extreme hull graph G such that $h(G) = a = ex(G)$ and $g(G) = b$.*

Proof. If $a = b$, then $a \geq 2$ and $G = K_a$ has the desired properties. Thus we assume that $a < b$. Let $R_i : r_i, s_i, t_i, u_i, r_i (1 \leq i \leq b-a)$ be a copy of cycle C_4. Let Q_i be the graph obtained from R_i by adding new vertex h_i and the edges $h_i u_i, h_i s_i (1 \leq i \leq b-a)$. Let W_{b-a} be the graph obtained from R_i's by identifying t_{i-1} of Q_{i-1} and r_i of $Q_i (2 \leq i \leq b-a)$. Let G be the graph given in Figure 3.2 obtained from W_{b-a} by adding new vertices $z_1, z_2, ..., z_{a-1}, y$ and joining the edges $r_1 z_1, r_1 z_2, ..., r_1 z_{a-1}$ and $t_{b-a} y$. Let $Z = \{z_1, z_2, ..., z_{a-1}, y\}$ be the set of extreme vertices of G. Then it follows from Theorem 2.4 that Z is a hull set of G so that $h(G) = a = ex(G)$. Therefore G is an extreme hull graph. Next we show that $g(G) = b$. Let S be any geodetic set of G. Then by Theorem 1.1, $Z \subseteq S$. It is clear that Z is not a geodetic set of G. We observe that every g-set of G must contain $h_i (1 \leq i \leq b-a)$ and so that $g(G) \geq a - 1 + 1 + b - a = b$. Now $W = Z \cup \{h_1, h_2, ..., h_{b-a}\}$ is a geodetic set of G so that $g(G) \leq a + b - a = b$. Thus $g(G) = b$. □

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