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MILD SOLUTIONS FOR A SYSTEM OF FRACTIONAL ORDER DIFFERENTIAL EQUATIONS WITH NONLOCAL CONDITIONS

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Abstract: A system of fractional order differential equations is studied in this article, and some sufficient conditions for existence of unique mild solutions for the system is established by Krasnoselskii fixed point theorem.

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1. Introduction and Preliminaries

This article is concerned with the existence and uniqueness of mild solutions of the following system of fractional order differential equations with nonlocal conditions:

$$\begin{aligned} \frac{d^q x_i(t)}{dt^q} &= -A_i x_i(t) + f_i\big(t, x_1(t), \cdots, x_n(t), G_1 x_1(t), \cdots, G_n x_n(t)\big), \\ & x_i(0) + g_i(x_i) = x_{i0}, \quad t \in [0,T], i \in \{1, 2, \cdots, n\}, \end{aligned} \tag{1.1}$$

where $0 < q < 1, T > 0$, and $-A_i$ generate analytic compact semigroups $\{S_i(t)\}_{t\geq 0}$ of uniformly bounded linear operators on a Banach space X with norm $\|\cdot\|$, that is, there exist $M_i > 1$ such that $\|S_i(t)\| \leq M_i$, and without loss of generality, assume $0 \in \rho(A)$. f_i are continuous mappings defined on $[0,T] \times X_{1\alpha} \times \cdots \times X_{n\alpha} \times X_{1\alpha} \times \cdots \times X_{n\alpha}$ and g_i are defined on $C([0,T], X_{i\alpha})$,

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where $X_{i\alpha} = D(A_i^\alpha)$, for $0 < \alpha < 1$, the domain of the fractional power of A_i. $G_i x_i(t)$ that may be interpreted as controls on the system are defined by

$$G_i x_i(t) = \int_0^t K_i(t,s) x_i(s) ds, \tag{1.2}$$

where $K_i(t,s)$ are all positive continuous functions defined on $D = \{(t,s) \in \mathbb{R}^2 : 0 \le s \le t \le T\}$ and

$$G_i^* = \sup_{t\in[0,T]} \int_0^t K_i(t,s) ds < +\infty. \tag{1.3}$$

Recently, fractional differential equations and systems have been of great interest. For detailed discussion on this topic, refer to the monographs of Kilbas et al [7], Lakshmikantham et al [8,9], Miller and Ross [12], Pazy [13], Podlubny [14], Smart [15], and the papers by Ahmad et al [1,2], Benchohra et al.[3], Guo and Liu [4-6], Lakshmikantham et al [10], Li [11], Su [16] and the references therein.

Applying Krasnoselskii fixed point theorem, we obtain a result of existence of mild solutions for system (1.1).

The following notations, definitions, and preliminary facts will be used throughout this paper.

Let $i \in \{1, 2, \cdots, n\}$. Define a space

$$\Omega = \{x = (x_1, \cdots, x_n) : [0,T] \to X_{1\alpha} \times \cdots \times X_{n\alpha} \big| x_i : [0,T] \to X_{i\alpha}\}. \tag{1.4}$$

We use $\|f_i\|_p$ to denote the L^p norm of f_i whenever $f_i \in L^p(0,T)$ for some p with $1 \le p < \infty$ and $C_{i\alpha}$ to denote the Banach space $C([0,T], X_{i\alpha})$ endowed with the sup norm given by

$$\|x\|_\infty := \sup_{t\in[0,T]} \|x\|_{i\alpha} \tag{1.5}$$

for $x \in C_{i\alpha}$.

Lemma 1.1. (see [13]) *(1) $X_{i\alpha} = D(A_i^\alpha)$ is a Banach space with the norm $\|x\|_\alpha := \|A_i^\alpha x\|$ for $x \in D(A_i^\alpha)$.*

(2) $S_i(t) : X \to X_{i\alpha}$ for each $t > 0$ and $\alpha > 0$.

(3) For each $u \in D(A_i^\alpha)$ and $t \ge 0, S_i(t) A_i^\alpha u = A_i^\alpha S_i(t) u$.

(4) For each $t > 0, A_i^\alpha S_i(t)$ are bounded on X and there exist $M_{i\alpha} > 0$ such that

$$\|A_i^\alpha S_i(t)\| \le M_{i\alpha} t^{-\alpha}. \tag{1.6}$$

Definition 1.2. A continuous vector function $x \in \Omega$ is called a mild solution of (1.1) if

$$\begin{aligned} x_i(t) =& S_i(t)\big(x_{i0} - g_i(x_i)\big) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} S_i(t-s) f_i\big(s, \\ & x_1(s), \cdots, x_n(s), G_1 x_1(s), \cdots, G_n x_n(s)\big) ds \end{aligned} \tag{1.7}$$

for $t \in [0,T]$.

Theorem 1.3. (Krasnoselskii Fixed Point Theorem, see [15]) *Let D be a closed convex and nonempty subset of a Banach space X, and A, B be two operators such that:*

(i) $Ax + By \in D$ whenever $x, y \in D$,

(ii) A is compact and continuous,

(iii) B is a contraction mapping.

Then there exists $z \in D$ such that $z = Az + Bz$.

Now list the following hypotheses for convenience.

(H1) Functions $f_i : [0,T] \times X_{1\alpha} \times \cdots \times X_{n\alpha} \times X_{1\alpha} \times \cdots \times X_{n\alpha} \to X$ are continuous and there exist positive functions $h_i(\cdot) : [0,T] \to \mathbb{R}^+$ such that

$$\|f_i(t, x_1, \cdots, x_n, y_1, \cdots, y_n)\| \le h_i(t), \tag{1.8}$$

$$\text{functions } s \mapsto \frac{h_i(s)}{(t-s)^\alpha} \text{ belong to } L^p([0,t], \mathbb{R}^+), \tag{1.9}$$

$$\gamma(t) := \Big(\int_0^t \big(\frac{h_i(s)}{(t-s)^\alpha}\big)^p ds \Big)^{1/p} \le M_{iT} < \infty \text{ for } t \in [0,T], \tag{1.10}$$

where $p > 1/q > 1$.

(H2) Functions $g_i : C_{i\alpha} \to X_{i\alpha}$ are continuous and there exist $k_i > 0$ such that

$$\|g_i(x_i) - g_i(y_i)\|_{i\alpha} \le k_i \|x_i - y_i\|_\infty, \ \forall x_i, y_i \in C_{i\alpha}. \tag{1.11}$$

2. Existence of a Mild Solution

In this section, a few sufficient conditions of existence of mild solutions for system (1.1) will be given.

Theorem 2.1. *Assume $-A_i$ are the infinitesimal generators of analytic compact semigroups $\{S_i(t)\}_{t\geq 0}$ with $\|S_i(t)\| \leq M_i, t \geq 0$, and $0 \in \rho(A_i)$. If the mappings f_i and g_i satisfy (H1), (H2), respectively, and $M_i k_i < 1$, then system (1.1) has a mild solution for every $x_0 = (x_{10}, \cdots, x_{n0}) \in \Omega$.*

Proof. Let $i \in \{1, 2, \cdots, n\}$, $\lambda_i = \sup_{x_i \in C_{i\alpha}} \|g_i(x_i)\|_\alpha$ and choose r_i such that

$$r_i \geq \frac{M_{i\alpha} M_{iT}}{\Gamma(q)} M_{p,q} T^{q-1/p} + M_i(\|x_{i0}\|_\alpha + \lambda_i), \tag{2.1}$$

where $M_{p,q} = \big((p-1)/(pq-1)\big)^{(p-1)/q}$.

Define two mappings $A, B : \Omega \to \Omega$ by

$$\begin{aligned} (Ax)(t) &= \big((A_1 x_1)(t), \cdots, (A_n x_n)(t)\big), \\ (Bx)(t) &= \big((B_1 x_1)(t), \cdots, (B_n x_n)(t)\big), \end{aligned} \tag{2.2}$$

for $x = (x_1, \cdots, x_n) \in \Omega$ and $t \in [0, T]$, where

$$\begin{aligned} (A_i x_i)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} S_i(t-s) f_i\big(s, x_1(s), \cdots, \\ x_n(s), G_1 x_1(s), \cdots, G_n x_n(s)\big) ds, \\ (B_i x_i)(t) = S_i(t)\big(x_{i0} - g_i(x_i)\big). \end{aligned} \tag{2.3}$$

Set $D_{r_i} = \{x \in C([0, T], X_{i\alpha}) : \|x_i\|_\infty \leq r_i\}$.

(i) It is claimed that $A_i x_i + B_i y_i \in D_{r_i}, \quad \forall x_i, y_i \in D_{r_i}$.

In fact, for every $x_i, y_i \in D_{r_i}$ and $t \in [0, T]$, we educe that

$$\begin{aligned} &\|(A_i x_i)(t) + (B_i y_i)(t)\|_\alpha \\ \leq &\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \|A^\alpha S_i(t-s) f_i\big(s, x_1(s), \cdots, x_n(s), \\ &G_1 x_1(s), \cdots, G_n x_n(s)\big)\| ds + \|S_i(t)\|(\|x_{i0}\|_\alpha + \lambda_i) \\ \leq &\frac{M_{i\alpha}}{\Gamma(q)} \int_0^t (t-s)^{q-1} \frac{h_i(s)}{(t-s)^\alpha} ds + M_i(\|x_{i0}\|_\alpha + \lambda_i) \\ \leq &\frac{M_{i\alpha}}{\Gamma(q)} \Big(\int_0^t (t-s)^{(q-1)p/(p-1)} ds\Big)^{(p-1)/p} \Big(\int_0^t \big(\frac{h_i(s)}{(t-s)^\alpha}\big)^p ds\Big)^{1/p} \\ &+ M_i(\|x_{i0}\|_\alpha + \lambda_i) \\ \leq &\frac{M_{i\alpha} M_{iT}}{\Gamma(q)} M_{ipq} T^{q-1/p} + M_i(\|x_{i0}\|_\alpha + \lambda_i) \\ \leq &r_i. \end{aligned} \tag{2.4}$$

That is, $A_i x_i + B_i y_i \in D_{r_i}$.

(ii) It is declared that A_i are continuous. Let $\{x_{ik}\}$ be a sequence of D_{r_i} such that $x_{ik} \to x_i$ in D_{r_i}. Then the continuity of f_i ensures that

$$\begin{aligned} &f_i\big(s, x_{1k}(s), \cdots, x_{nk}(s), G_1 x_{1k}(s), \cdots, G_n x_{nk}(s)\big) \\ &\quad \to f_i\big(s, x_1(s), \cdots, x_n(s), G_1 x_1(s), \cdots, G_n x_n(s)\big). \end{aligned} \tag{2.5}$$

For $t \in [0, T]$, we obtain

$$\begin{aligned} &\|(A_i x_{ik})(t) - (A_i x_i)(t)\|_\alpha \\ =&\frac{1}{\Gamma(q)}\Big\| \int_0^t (t-s)^{q-1} S_i(t-s)\big[f_i\big(s, x_{1k}(s), \cdots, x_{nk}(s), G_1 x_{1k}(s), \\ &\cdots, G_n x_{nk}(s)\big) - f_i\big(s, x_1(s), \cdots, x_n(s), G_1 x_1(s), \cdots, G_n x_n(s)\big)\big] ds \Big\|_\alpha \\ \le&\frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} \big\| A_i^\alpha S_i(t-s)\big[f_i\big(s, x_{1k}(s), \cdots, x_{nk}(s), G_1 x_{1k}(s), \\ &\cdots, G_n x_{nk}(s)\big) - f_i\big(s, x_1(s), \cdots, x_n(s), G_1 x_1(s), \cdots, G_n x_n(s)\big)\big] \big\| ds \\ \le&\frac{M_{i\alpha}}{\Gamma(q)} \int_0^t (t-s)^{q-1} \big\| f_i\big(s, x_{1k}(s), \cdots, x_{nk}(s), G_1 x_{1k}(s), \cdots, G_n x_{nk}(s)\big) \\ &- f_i\big(s, x_1(s), \cdots, x_n(s), G_1 x_1(s), \cdots, G_n x_n(s)\big) \big\| (t-s)^{-\alpha} ds. \end{aligned} \tag{2.6}$$

According to the fact that

$$\begin{aligned} &\big\| f_i\big(s, x_{1k}(s), \cdots, x_{nk}(s), G_1 x_{1k}(s), \cdots, G_n x_{nk}(s)\big) \\ &\quad - f_i\big(s, x_1(s), \cdots, x_n(s), G_1 x_1(s), \cdots, G_n x_n(s)\big) \big\| \le 2h_i(s), \end{aligned} \tag{2.7}$$

for any $s \in [0, T]$ and the functions $s \to 2h_i(s)(t-s)^{-\alpha}$ are integrable on $[0, t]$, the Lebesgue Dominated Convergence Theorem guarantees that

$$\begin{aligned} &\int_0^t (t-s)^{q-1} \big\| f_i\big(s, x_{1k}(s), \cdots, x_{nk}(s), G_1 x_{1k}(s), \cdots, G_n x_{nk}(s)\big) \\ &- f_i\big(s, x_1(s), \cdots, x_n(s), G_1 x_1(s), \cdots, G_n x_n(s)\big) \big\| (t-s)^{-\alpha} ds \to 0 \end{aligned} \tag{2.8}$$

as $k \to \infty$. Therefore,

$$\lim_{k\to\infty} \|(A_i x_{ik})(t) - (A_i x_i)(t)\|_\infty = 0. \tag{2.9}$$

(iii) It can be asserted that A_i are compact.

First to show that A_i are uniformly bounded on D_{r_i}.

$$\begin{aligned}\|(A_ix_i)(t)\|_\alpha =&\frac{1}{\Gamma(q)}\Big\|\int_0^t (t-s)^{q-1}S_i(t-s)f_i\big(s,x_1(s),\\ &\cdots,x_n(s),G_1x_1(s),\cdots,G_nx_n(s)\big)ds\Big\|_\alpha\\ \le&\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}\|A_i^\alpha S_i(t-s)f_i\big(s,x_1(s),\\ &\cdots,x_n(s),G_1x_1(s),\cdots,G_nx_n(s)\big)\|ds\\ \le&\frac{M_{i\alpha}}{\Gamma(q)}\int_0^t (t-s)^{q-1}\frac{h_i(s)}{(t-s)^\alpha}ds\\ \le&\frac{M_{i\alpha}M_{iT}}{\Gamma(q)}M_{p,q}T^{q-1/p}.\end{aligned} \tag{2.10}$$

Next to prove that $(A_ix_i)(t)$ are equicontinuous. Let $0<t_1<t_2<T$ and $\epsilon>0$ be small enough, then we have

$$\begin{aligned}&\|(A_ix_i)(t_2)-(A_ix_i)(t_1)\|_\alpha\\ \le&\frac{1}{\Gamma(q)}\Big\|\int_0^{t_1}\big[(t_2-s)^{q-1}-(t_1-s)^{q-1}\big]S_i(t_1-s)f_i\big(s,x_1(s),\\ &\cdots,x_n(s),G_1x_1(s),\cdots,G_nx_n(s)\big)ds\Big\|_\alpha\\ &+\frac{1}{\Gamma(q)}\Big\|\int_{t_1}^{t_2}(t_2-s)^{q-1}S_i(t_2-s)f_i\big(s,x_1(s),\\ &\cdots,x_n(s),G_1x_1(s),\cdots,G_nx_n(s)\big)ds\Big\|_\alpha\\ &+\frac{1}{\Gamma(q)}\Big\|\int_0^{t_1}(t_2-s)^{q-1}\big[S_i(t_2-s)-S_i(t_1-s)\big]f_i\big(s,x_1(s),\\ &\cdots,x_n(s),G_1x_1(s),\cdots,G_nx_n(s)\big)ds\Big\|_\alpha\\ =&I_1+I_2+I_3.\end{aligned} \tag{2.11}$$

By (1.6) and (H1), we get

$$\begin{aligned}I_1=&\frac{1}{\Gamma(q)}\Big\|\int_0^{t_1}\big[(t_2-s)^{q-1}-(t_1-s)^{q-1}\big]S_i(t_1-s)\\ &f_i\big(s,x_1(s),\cdots,x_n(s),G_1x_1(s),\cdots,G_nx_n(s)\big)ds\Big\|_\alpha\\ \le&\frac{1}{\Gamma(q)}\int_0^{t_1}\big[(t_2-s)^{q-1}-(t_1-s)^{q-1}\big]\|A_i^\alpha S_i(t_1-s)\end{aligned}$$

$$
\begin{aligned}
&f_i\big(s, x_1(s), \cdots, x_n(s), G_1x_1(s), \cdots, G_nx_n(s)\big)\|ds \\
&\le \frac{M_{i\alpha}}{\Gamma(q)}\int_0^{t_1}\big[(t_2-s)^{q-1}-(t_1-s)^{q-1}\big]\frac{h_i(s)}{(t_1-s)^{\alpha}}ds \qquad (2.12)\\
&\le \frac{M_{i\alpha}}{\Gamma(q)}\int_0^{t_1-\epsilon}\big[(t_2-s)^{q-1}-(t_1-s)^{q-1}\big]\frac{h_i(s)}{(t_1-s)^{\alpha}}ds \\
&\quad + \frac{M_{i\alpha}}{\Gamma(q)}\int_{t_1-\epsilon}^{t_1}(t_2-s)^{q-1}\frac{h_i(s)}{(t_1-s)^{\alpha}}ds \\
&= I_1' + I_1''.
\end{aligned}
$$

It follows from the assumption of $h_i(s)$ that I_1' tends to 0 as $t_1 \to t_2$. For I_1'', we can see that I_1'' tends to 0 as $t_1 \to t_2$ and $\epsilon \to 0$.

It can be seen from (1.6) and (H1) that

$$
\begin{aligned}
I_2 &= \frac{1}{\Gamma(q)}\Big\|\int_{t_1}^{t_2}(t_2-s)^{q-1}S_i(t_2-s)f_i\big(s, x_1(s), \\
&\quad \cdots, x_n(s), G_1x_1(s), \cdots, G_nx_n(s)\big)ds\Big\|_{\alpha} \\
&\le \frac{1}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\big\|A_i^{\alpha}S_i(t_2-s)f_i\big(s, x_1(s), \qquad (2.13)\\
&\quad \cdots, x_n(s), G_1x_1(s), \cdots, G_nx_n(s)\big)\big\|ds \\
&\le \frac{M_{i\alpha}}{\Gamma(q)}\int_{t_1}^{t_2}(t_2-s)^{q-1}\frac{h_i(s)}{(t_2-s)^{\alpha}}ds \to 0 \quad \text{as} \quad t_1 \to t_2.
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
I_3 \le &\frac{1}{\Gamma(q)}\Big\|\int_0^{t_1-\epsilon}(t_2-s)^{q-1}\big[S_i(t_2-s)-S_i(t_1-s)\big]f_i\big(s, x_1(s), \\
&\cdots, x_n(s), G_1x_1(s), \cdots, G_nx_n(s)\big)ds\Big\|_{\alpha} \\
&+ \frac{1}{\Gamma(q)}\Big\|\int_{t_1-\epsilon}^{t_1}(t_2-s)^{q-1}\big[S_i(t_2-s)-S_i(t_1-s)\big]f_i\big(s, x_1(s), \\
&\cdots, x_n(s), G_1x_1(s), \cdots, G_nx_n(s)\big)ds\Big\|_{\alpha} \\
\le &\frac{1}{\Gamma(q)}\int_0^{t_1-\epsilon}(t_2-s)^{q-1}\Big\|S_i\big(\frac{t_2-t_1}{2}-\frac{t_2-s}{2}\big)-S_i\big(\frac{t_1-s}{2}\big)\Big\| \qquad (2.14)\\
&\cdot\Big\|A_i^{\alpha}S_i\big(\frac{t_1-s}{2}\big)f_i\big(s, x_1(s), \cdots, x_n(s), G_1x_1(s), \cdots, G_nx_n(s)\big)\Big\|ds \\
&+ \frac{M_{i\alpha}}{\Gamma(q)}\int_{t_1-\epsilon}^{t_1}(t_2-s)^{q-1}\big[\frac{h_i(s)}{(t_2-s)^{\alpha}}+\frac{h_i(s)}{(t_1-s)^{\alpha}}\big]ds
\end{aligned}
$$

$$
\begin{aligned}
\leq & \frac{2^{\alpha} M_{i\alpha}}{\Gamma(q)} \int_0^{t_1-\epsilon} (t_2-s)^{q-1} \Big\| S_i\big(\frac{t_2-t_1}{2} - \frac{t_2-s}{2}\big) - S_i\big(\frac{t_1-s}{2}\big) \Big\| \\
& \cdot \frac{h_i(s)}{(t_1-s)^{\alpha}} ds + \frac{M_{i\alpha}}{\Gamma(q)} \int_{t_1-\epsilon}^{t_1} (t_2-s)^{q-1} \big[\frac{h_i(s)}{(t_2-s)^{\alpha}} + \frac{h_i(s)}{(t_1-s)^{\alpha}} \big] ds \\
= & I_3' + I_3''.
\end{aligned}
$$

Applying the compactness of $S_i(t)$ in X implies the continuity of $t \mapsto \|S_i(t)\|$ for $t \in [0,T]$; integrating with $s \mapsto h_i(s)/(t_1-s)^{\alpha} \in L^1_{\text{loc}}([0,t_1],\mathbb{R}^+)$, we see that I_3' tends to 0, as $t_1 \to t_2$. For I_3'', it follows from the assumption of $h_i(s)$ that I_3'' tends to 0 as $t_1 \to t_2$ and $\epsilon \to 0$.

Therefore, $\|(A_i x_i)(t_2) - (A_i x_i)(t_1)\|_{\alpha} \to 0$ as $t_1 \to t_2$, which do not depend on x_i. Thus, $A_i(D_{r_i})$ are relatively compact. In virtue of the Arzela-Ascoli Theorem, A_i are compact.

(iv) Obviously, B_i are contraction mappings. It follows from

$$
\begin{aligned}
\|(B_i x_i)(t) - (B_i y_i)(t)\|_{i\alpha} &\leq \|S_i(t)\| \|g_i(x_i) - g_i(y_i)\|_{i\alpha} \\
&\leq M_i k_i \|x_i - y_i\|_{\infty} < \|x_i - y_i\|_{\infty}
\end{aligned}
\tag{2.15}
$$

that

$$
\|(B_i x_i)(t) - (B_i y_i)(t)\|_{\infty} < \|x_i - y_i\|_{\infty}. \tag{2.16}
$$

Now the proof is completed by Krasnoselskii fixed point theorem.

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