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ON THE ARITHMETIC COMPENSATION METHOD

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Abstract: The arithmetic compensation method is a power tool which can be used to prove many symmetric inequalities of the form $F(x_1, x_2, \cdots, x_n) \geq 0$, where $x_1, x_2, \cdots, x_n$ are nonnegative real variables satisfying $x_1 + x_2 + \cdots + x_n = s$, $s > 0$. Notice that the AC-method can be applied especially to those inequalities where equality holds when some variables are equal and the others are zero. Several applications are given to show the feasibility and effectiveness of the AC-method.

AMS Subject Classification: 26D05
Key Words: arithmetic compensation, contradiction method, global minimum

1. Introduction and Main Results

The arithmetic compensation theorem (AC-Theorem) was established for the first time in [1], but the proof given here is not detailed enough. Two years later, the AC-Theorem is exposed in [3] as undefined mixing variable theorem (UMV-Theorem), with a proof sketch based on the mixing variable method. In this paper, we give a detailed proof of the AC-Theorem, present two practical versions of them, and give four applications to emphasize the effectiveness of the method.

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Ac-Theorem. *Let $s > 0$ and let F be a symmetric continuous function on the compact set in $\mathbb{R}^n$*

$$S = \{(x_1, x_2, \cdots, x_n) : x_1 + x_2 + \cdots + x_n = s,\ x_i \ge 0,\ i = 1, 2, \cdots, n\}.$$

If

$$F(x_1, x_2, x_3, \cdots, x_n) \ge$$

$$\ge \min\left\{F\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \cdots, x_n\right), F(0, x_1+x_2, x_3, \cdots, x_n)\right\} \quad (1.1)$$

for all $(x_1, x_2, \cdots, x_n) \in S$, then

$$F(x_1, x_2, x_3, \cdots, x_n) \ge \min_{1\le k\le n} F\left(\frac{s}{k}, \cdots, \frac{s}{k}, 0 \cdots, 0\right) \quad (1.2)$$

for all $(x_1, x_2, \cdots, x_n) \in S$.

Remark 1.1. Let $x_1, x_2, \cdots, x_n$ be nonnegative real numbers. If

$$F(x_1, x_2, x_3, \cdots, x_n) < F\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \cdots, x_n\right)$$

involves

$$F(x_1, x_2, x_3, \cdots, x_n) \ge F(0, x_1+x_2, x_3, \cdots, x_n),$$

then the condition (1.1) in AC-Theorem is satisfied. To prove this, we only need to consider the following two possible cases

$$F(x_1, x_2, x_3, \cdots, x_n) \ge F\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \cdots, x_n\right)$$

or

$$F(x_1, x_2, x_3, \cdots, x_n) < F\left(\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2}, x_3, \cdots, x_n\right).$$

If the first case, the condition (1.1) is obviously satisfied. In the second case, according to the above hypothesis, we have

$$F(x_1, x_2, x_3, \cdots, x_n) \ge F(0, x_1+x_2, x_3, \cdots, x_n),$$

and the condition (1.1) is also obviously satisfied.

Remark 1.2. By replacing the condition (1.1) by the condition

$$F(x_1, x_2, x_3, \cdots, x_n) >$$

$$> \min\left\{F\left(\frac{x_1+x_2}{2},\frac{x_1+x_2}{2},x_3,\cdots,x_n\right),F(0,x_1+x_2,x_3,\cdots,x_n)\right\} \quad (1.3)$$

for all $(x_1,x_2,\cdots,x_n)\in S$ such that $x_1>x_2>0$, we get a weaker version of the AC-Theorem.

It is easy to prove this statement by the contradiction method. Since the function F is continuous on the compact set S, F attains its minimum value at one or more points of the set. Let $(y_1,y_2,\cdots,y_n)$ be a global minimum point of F over the set S. For the sake of contradiction, assume that there exist two numbers y_i and y_j such that $y_i>y_j>0$; for convenience, let $i=1$ and $j=2$. According to the hypothesis (1.3), we have

$$F(y_1,y_2,y_3,\cdots,y_n)>F\left(\frac{y_1+y_2}{2},\frac{y_1+y_2}{2},y_3,\cdots,y_n\right)$$

or

$$F(y_1,y_2,y_3,\cdots,y_n)>F(0,y_1+y_2,y_3,\cdots,y_n).$$

In both cases, we got a contradiction, because F is minimal at $(y_1,y_2,\cdots,y_n)$.

2. Proof of AC-Theorem

The AC-Theorem is clearly true for $n=2$. Consider further $n\geq 3$. Since the function F is continuous on the compact set S, F attains its minimum value F_0 at one or more points of the set. We need to prove that among these global minimum points there is a point having k coordinates equal to s/k and $n-k$ coordinates equal to zero, where $k\in\{1,2,\cdots,n\}$. Let

$$Y_0=(y_1,y_2,\cdots,y_n)$$

be a global minimum point of F over the set S, and let F_0 be the minimal value of F; that is,

$$F_0=F(y_1,y_2,\cdots,y_n).$$

If $y_1=y_2=\cdots=y_n$, then the proof is completed. If one of y_i is zero, then the desired result follows by induction. Otherwise, due to symmetry, assume that $y_1\leq y_2\leq\cdots\leq y_n$. From the hypothesis (1.1), we have

$$F_0=\min\left\{F\left(\frac{y_1+y_n}{2},y_2,\cdots,y_{n-1},\frac{y_1+y_n}{2},\right),F(0,y_2,\cdots,y_{n-1},y_1+y_n)\right\},$$

because the case

$$F_0 > \min\left\{F\left(\frac{y_1+y_n}{2}, y_2, \cdots, y_{n-1}, \frac{y_1+y_n}{2},\right), F(0, y_2, \cdots, y_{n-1}, y_1+y_n)\right\}$$

contradicts the fact that F_0 is the minimal value of F. Therefore,

$$F_0 = F(0, y_2, \cdots, y_{n-1}, y_1+y_n)$$

or

$$F_0 = F\left(\frac{y_1+y_n}{2}, y_2, \cdots, y_{n-1}, \frac{y_1+y_n}{2}\right).$$

In the first case, the desired result follows by induction. Consider further the second case, when

$$Y_0 = (y_1, y_2, \cdots, y_n)$$

and

$$Y_1 = \left(\frac{y_1+y_n}{2}, y_2, \cdots, y_{n-1}, \frac{y_1+y_n}{2}\right)$$

are global minimum points of F, and this process is repeated infinitely. Then by Lemma 2.3 below it follows that

$$Y_\infty = \left(\frac{s}{n}, \frac{s}{n}, \cdots, \frac{s}{n}\right)$$

is also a global minimum point of F. This completes the proof.

Lemma 2.1. (*see [2]*) *For $a < b$, consider the real intervals*

$$\mathbb{I}_1 = \left[a, \frac{2a+b}{3}\right), \quad \mathbb{I}_2 = \left[\frac{2a+b}{3}, \frac{a+2b}{3}\right], \quad \mathbb{I}_3 = \left(\frac{a+2b}{3}, b\right].$$

If

$$a_1, a_2, \cdots, a_n \in [a, b], \quad a_i \in \mathbb{I}_1, \quad a_j \in \mathbb{I}_3,$$

then

$$\frac{a_i+a_j}{2} \in \mathbb{I}_2.$$

Proof. We have

$$\frac{a_i+a_j}{2} \geq \frac{a+\frac{a+2b}{3}}{2} = \frac{2a+b}{3},$$

$$\frac{a_i+a_j}{2} \leq \frac{\frac{2a+b}{3}+b}{2} = \frac{a+2b}{3}.$$

Lemma 2.2. *Let $A_0, A_1, \cdots, A_k$ be a sequence of n-tuples*

$$A_i = (a_{i1}, a_{i2}, \cdots, a_{in}) \in \mathbb{R}^n.$$

For A_0, let us define

$$a = \min\{a_{01}, a_{02}, \cdots, a_{0n}\}, \quad b = \max\{a_{01}, a_{02}, \cdots, a_{0n}\}, \quad a < b,$$

$$\mathbb{J}_1 = [a, b], \quad \mathbb{I}_1 = \left[a, \frac{2a+b}{3}\right), \quad \mathbb{I}_2 = \left[\frac{2a+b}{3}, \frac{a+2b}{3}\right], \quad \mathbb{I}_3 = \left(\frac{a+2b}{3}, b\right].$$

In addition, assume that A_0 has k_1 elements in $\mathbb{I}_1$ and k_2 elements in $\mathbb{I}_3$. The n-tuple A_i is constructed from the preceding n-tuple A_{i-1} by replacing its smallest and largest elements by their arithmetic mean. Then, for $k = \min\{k_1, k_2\}$, all elements of A_k belong to the one of the intervals $\mathbb{I}_1 \cup \mathbb{I}_2$ or $\mathbb{I}_2 \cup \mathbb{I}_3$. Denoting this closed interval by $\mathbb{J}_2$, we have

$$|\mathbb{J}_2| = \frac{2}{3}|\mathbb{J}_1|,$$

where $|\mathbb{J}|$ is the length of the closed interval $\mathbb{J}$.

Proof. All elements of each n-tuple of the sequence $A_0, A_1, \cdots, A_k$ belong to the interval $[a, b]$. Then, by Lemma 2.1, the n-tuple A_i has $k_1 - i$ elements in $\mathbb{I}_1$ and $k_2 - i$ elements in $\mathbb{I}_3$. Therefore, the n-tuple A_k contains only elements in

$$\mathbb{I}_1 \cup \mathbb{I}_2 = \left[a, \frac{a+2b}{3}\right]$$

or in

$$\mathbb{I}_2 \cup \mathbb{I}_3 = \left[\frac{2a+b}{3}, b\right].$$

Since

$$|\mathbb{I}_1 \cup \mathbb{I}_2| = |\mathbb{I}_2 \cup \mathbb{I}_3| = \frac{2(b-a)}{3},$$

the conclusion follows.

Lemma 2.3. *Let $A_0, A_1, \cdots$ be a sequence of n-tuples $A_i = (a_{i1}, a_{i2}, \cdots, a_{in}) \in \mathbb{R}^n$, where the n-tuple A_i is constructed from the preceding n-tuple A_{i-1} by replacing its smallest and largest elements by their arithmetic mean. Then*

$$\lim_{i\to\infty} A_i = (a, a, \cdots, a),$$

where

$$a = \frac{a_{01} + a_{02} + \cdots + a_{0n}}{n}.$$

Proof. Repeating the procedure in Lemma 2.2 with $\mathbb{J}_2$ for $\mathbb{J}_1$ etc., we progressively get

$$\mathbb{J}_1 \supset \mathbb{J}_2 \supset \mathbb{J}_3 \supset \cdots ,$$

such that

$$|\mathbb{J}_i| = \frac{2}{3}|\mathbb{J}_{i-1}| = \cdots = \left(\frac{2}{3}\right)^{i-1} |\mathbb{J}_1|$$

and

$$a_{i1} + a_{i2} + \cdots + a_{in} = a_{01} + a_{02} + \cdots + a_{0n}.$$

Therefore, $|\mathbb{J}_i| \to 0$ for $i \to \infty$, and hence $\mathbb{J}_\infty$ contains a unique number, namely

$$\frac{a_{01} + a_{02} + \cdots + a_{0n}}{n}.$$

3. Applications

Application 3.1. *Let $x_1, x_2, ..., x_n$ be nonnegative real numbers such that*

$$x_1 + x_2 + \cdots + x_n = n.$$

If $n \le 9$, then

$$\frac{1}{3 - x_1x_2\cdots x_{n-1}} + \frac{1}{3 - x_2x_3\cdots x_n} + \cdots + \frac{1}{3 - x_nx_1\cdots x_{n-2}} \le \frac{n-1}{3} + \frac{1}{3 - e_{n-1}},$$

where

$$e_{n-1} = \left(1 + \frac{1}{n-1}\right)^{n-1}.$$

Proof. Write the inequality as

$$F(x_1, x_2, \cdots, x_n) + \frac{n-1}{m} + \frac{1}{m - e_{n-1}} \ge 0, \qquad (3.1)$$

where

$$F(x_1, x_2, \cdots, x_n) = -\left(\frac{1}{m - x_1x_2\cdots x_{n-1}} + \frac{1}{m - x_2x_3\cdots x_n} + \cdots + \frac{1}{m - x_nx_1\cdots x_{n-2}}\right)$$

and $m = 3$. We assert that

$$F(x_1, x_2, x_3, \cdots, x_n) < F(t, t, x_3, \cdots, x_n) \tag{3.2}$$

involves

$$F(x_1, x_2, x_3, \cdots, x_n) \geq F(0, 2t, x_3, \cdots, x_n), \tag{3.3}$$

where $t = (x_1 + x_2)/2$. Then, by the AC-Theorem and Remark 1.1 it follows that F is minimal when either all numbers x_i are equal to 1, or at least one of them is zero.

In the first case, the inequality (3.1) becomes

$$e_{n-1} \geq \frac{mn}{m+n-1} = \frac{3n}{n+2}, \tag{3.4}$$

and is true for $n \leq 13$, but is not true for $n \geq 14$.

In the second case, when at least one of x_i is zero, for example, $x_n = 0$, the inequality (3.1) becomes

$$\frac{1}{m - x_1 x_2 \cdots x_{n-1}} \leq \frac{1}{m - e_{n-1}}, \tag{3.5}$$

and follows immediately from the AM-GM inequality

$$x_1 x_2 \cdots x_{n-1} \leq \left(\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1} \right)^{n-1} = e_{n-1}.$$

To end the proof, we only need to show that (3.2) involves (3.3). Let us denote

$$y = x_3 x_4 \cdots x_n.$$

We first show that (3.2) implies $y > 0$. Assuming, for example, that $x_n = 0$, the inequality (3.2) reduces to

$$(x_1 x_2 - t^2) x_3 \cdots x_{n-1} > 0,$$

which is false. Consider now that $y > 0$. Since

$$-F(x_1, x_2, x_3, \cdots, x_n) = \frac{1}{m - x_1 y} + \frac{1}{m - x_2 y} + \sum_{i=3}^{n} \frac{x_i}{m x_i - x_1 x_2 y},$$

$$-F(t, t, x_3, \cdots, x_n) = \frac{2}{m - ty} + \sum_{i=3}^{n} \frac{x_i}{m x_i - t^2 y},$$

$$-F(0,2t,x_3,\cdots,x_n)=\frac{n-1}{m}+\frac{1}{m-2ty},$$

the hypothesis (3.2) is equivalent to

$$y(t^2-x_1x_2)\left[\frac{2y}{(m-ty)(m-x_1y)(m-x_2y)}-\sum_{i=3}^{n}\frac{1}{mx_i-x_1x_2y}\right]>0. \quad (3.6)$$

On the other hand, the desired inequality (3.3) becomes as follows

$$\left(\frac{1}{m-2ty}-\frac{1}{m}\right)-\left(\frac{1}{m-x_1y}-\frac{1}{m}\right)-\left(\frac{1}{m-x_2y}-\frac{1}{m}\right)$$
$$\geq\sum_{i=3}^{n}\left(\frac{x_i}{mx_i-x_1x_2y}-\frac{1}{m}\right),$$

$$\frac{2ty}{m-2ty}-\frac{2(mt-x_1x_2y)y}{(m-x_1y)(m-x_2y)}\geq\sum_{i=3}^{n}\frac{x_1x_2y}{mx_i-x_1x_2y},$$

$$x_1x_2y\left[\frac{2y(m-ty)}{(m-2ty)(m-x_1y)(m-x_2y)}-\sum_{i=3}^{n}\frac{1}{mx_i-x_1x_2y}\right]\geq 0. \quad (3.7)$$

According to (3.6), this inequality is true if

$$\frac{m-ty}{m-2ty}\geq\frac{m}{m-ty}.$$

Since this inequality is equivalent to the obvious inequality $t^2y^2\geq 0$, the proof is completed. Equality holds when one of x_i is zero, and the others are equal to $\frac{n}{n-1}$.

Similarly, we can prove the following more general result.

Proposition 3.1. *Let $x_1,x_2,...,x_n$ be nonnegative real numbers such that*

$$x_1+x_2+\cdots+x_n=n.$$

If

$$e_{n-1}<m\leq\frac{n-1}{\frac{n}{e_{n-1}}-1}, \quad e_{n-1}=\left(1+\frac{1}{n-1}\right)^{n-1},$$

then

$$\frac{1}{m - x_1 x_2 \cdots x_{n-1}} + \frac{1}{m - x_2 x_3 \cdots x_n} + \cdots + \frac{1}{m - x_n x_1 \cdots x_{n-2}} \le \frac{n-1}{m} + \frac{1}{m - e_{n-1}}.$$

Application 3.2. *Let $x_1, x_2, ..., x_n$ $(n \ge 3)$ be nonnegative real numbers such that*

$$x_1 + x_2 + \cdots + x_n = n.$$

If

$$m \ge \frac{n-1}{\frac{n}{e_{n-1}} - 1}, \quad e_{n-1} = \left(1 + \frac{1}{n-1}\right)^{n-1},$$

then

$$\frac{1}{m - x_1 x_2 \cdots x_{n-1}} + \frac{1}{m - x_2 x_3 \cdots x_n} + \cdots + \frac{1}{m - x_n x_1 \cdots x_{n-2}} \le \frac{n}{m-1}.$$

Proof. The proof is similar to to the one of the preceding Application 3.1. Here, the analogue of the desired inequality (3.1) has the form

$$F(x_1, x_2, \cdots, x_n) + \frac{n}{m-1} \ge 0,$$

the analogue of the inequality (3.4) is an identity, and the analogue of the inequality (3.5) has the form

$$\frac{1}{m - x_1 x_2 \cdots x_{n-1}} + \frac{n-1}{m} \le \frac{n}{m-1},$$

which is equivalent to

$$x_1 x_2 \cdots x_{n-1} \le \frac{mn}{m+n-1}.$$

To prove the last inequality, since

$$x_1 x_2 \cdots x_{n-1} \le \left(\frac{x_1 + x_2 + \cdots + x_{n-1}}{n-1}\right)^{n-1} = e_{n-1},$$

it suffices to show that

$$e_{n-1} \le \frac{mn}{m+n-1},$$

which is equivalent to the hypothesis

$$m \ge \frac{n-1}{\frac{n}{e_{n-1}} - 1}.$$

Equality holds for $x_1 = x_2 = \cdots = x_n = 1$. In addition, for the particular case

$$m = \frac{n-1}{\dfrac{n}{e_{n-1}} - 1},$$

equality holds again when one of x_i is zero, and the others are equal to $\dfrac{n}{n-1}$.

Application 3.3. (*see [4]*) *Let $x_1, x_2, ..., x_n$ be nonnegative real numbers such that*

$$x_1 + x_2 + \cdots + x_n = 1.$$

If $m \in \{1, 2, \cdots, n+1\}$, then

$$m(m-1)(x_1^3 + x_2^3 + \cdots + x_n^3) + 1 \geq (2m-1)(x_1^2 + x_2^2 + \cdots + x_n^2).$$

Proof. We need to prove that $F(x_1, x_2, \cdots, x_n) \geq 0$, where

$$F(x_1, x_2, \cdots, x_n) = m(m-1)(x_1^3+x_2^3+\cdots+x_n^3)+1-(2m-1)(x_1^2+x_2^2+\cdots+x_n^2).$$

We will show that

$$F(x_1, x_2, x_3, \cdots, x_n) < F(t, t, x_3, \cdots, x_n) \tag{3.8}$$

involves

$$F(x_1, x_2, x_3, \cdots, x_n) \geq F(0, 2t, x_3, \cdots, x_n) \tag{3.9}$$

where $t = (x_1 + x_2)/2$. Then, by AC-Theorem and Remark 1.1, we have

$$F(x_1, x_2, \cdots, x_n) \geq \min_{1 \leq k \leq n} f(k),$$

where

$$f(k) = \frac{(k-m)(k-m+1)}{k^2}$$

is the value of F for $x_1 = \cdots = x_k = 1/k$ and $x_{k+1} = \cdots = x_n = 0$. Since $f(k) \geq 0$ for $k \in \{1, 2, \cdots, n\}$, we get $F(x_1, x_2, \cdots, x_n) \geq 0$.

To prove that (3.8) implies (3.9), we write (3.8) as

$$m(m-1)(x_1^3 + x_2^3 - 2t^3) - (2m-1)(x_1^2 + x_2^2 - 2t^2) < 0,$$

$$(t^2 - x_1 x_2)\,[3m(m-1)(x_1 + x_2) - 2(2m-1)] < 0. \tag{3.10}$$

Similarly, we write (3.9) as

$$m(m-1)(x_1^3 + x_2^3 - 8t^3) - (2m-1)(x_1^2 + x_2^2 - 4t^2) \geq 0,$$

$$x_1x_2[2(2m-1)-3m(m-1)(x_1+x_2)] \geq 0. \tag{3.11}$$

Clearly, (3.10) implies (3.11), and hence (3.8) implies (3.9). This completes the proof. For $m = 1$, equality holds when $n-1$ of the numbers x_i are zero. For $m \in \{2, 3, \cdots, n\}$, equality holds when m or $m-1$ of the numbers x_i are equal and the others are zero. For $m = n+1$, equality holds when the numbers $x_1, x_2, ..., x_n$ are equal.

Remark 3.2. Actually, the inequality in Application 3.3 holds for all $m \in [0,1] \cup \{2, 3, \cdots, n\} \cup [n+1, \infty)$, and does not hold for $m \in (1,2) \cup (2,3) \cup \cdots \cup (n, n+1)$.

Application 3.4. *If $x_1, x_2, ..., x_n$ $(n \geq 3)$ are nonnegative real numbers such that*

$$x_1 + x_2 + \cdots + x_n = n,$$

then

$$x_1^3 + x_2^3 + \cdots + x_n^3 - n \geq \frac{n(2n-1)}{(n-1)^2}(1 - x_1x_2\cdots x_n).$$

Proof. Write the inequality as $F(x_1, x_2, \cdots, x_n) \geq 0$, where

$$F(x_1, x_2, \cdots, x_n) = x_1^3 + x_2^3 + \cdots + x_n^3 - n - \frac{n(2n-1)}{(n-1)^2}(1 - x_1x_2\cdots x_n).$$

We claim that

$$F(x_1, x_2, x_3, \cdots, x_n) < F(t, t, x_3, \cdots, x_n) \tag{3.12}$$

involves

$$F(x_1, x_2, x_3, \cdots, x_n) \geq F(0, 2t, x_3, \cdots, x_n) \tag{3.13}$$

where $t = (x_1 + x_2)/2$. On this assumption, by AC-Theorem and Remark 1.1, we have

$$F(x_1, x_2, \cdots, x_n) \geq \min_{1 \leq k \leq n} f(k),$$

where

$$f(n) = F(1, 1, \cdots, 1) = 0$$

and

$$f(k) = \frac{n^3}{k^2} - \frac{n^3}{(n-1)^2} \geq 0$$

for $k \in \{1, 2, \cdots, n-1\}$. Since $f(k) \geq 0$ for $k \in \{1, 2, \cdots, n\}$, we have

$$F(x_1, x_2, \cdots, x_n) \geq 0.$$

To prove that (3.12) implies (3.13), we write these inequalities as

$$(t^2 - x_1x_2)\left[6 - \frac{n(2n-1)}{(n-1)^2}x_3x_4\cdots x_n\right] < 0 \tag{3.14}$$

and

$$x_1x_2\left[\frac{n(2n-1)}{(n-1)^2}x_3x_4\cdots x_n - 6\right] \geq 0, \tag{3.15}$$

respectively. Clearly, (3.14) implies (3.15), and hence (3.12) implies (3.13). This completes the proof. Equality holds when $x_1 = x_2 = \cdots = x_n = 1$, and also when one of the numbers x_i is zero and the others are equal to $\frac{n}{n-1}$.

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