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ON GENERALIZED C^v-REDUCIBLE FINSLER SPACES

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Abstract: A Finsler space is said to be generalized C^v-Reducible Finsler Space whose $C_{ijk}|_h$ can be written as product of tensor of type (0, 3) and (0, 1) such as,

$$LC_{ijk}|_h = A_{ijk}B_h + A_{ijh}B_k + A_{ihk}B_j + A_{hjk}B_i, \qquad A_{ijk} \neq \lambda C_{ijk}$$

where, $A_{i00} = 0$, $A_{ij0} \neq 0$ and $B_0 = 0$; $A_{i00} = A_{ijk}y^jy^k$, $B_0 = B_iy^i$.

In the present paper, we shall find out the value of A_{ijk}, taking the value of B_i as m_i and n_i, where m_i is C_i/C and n_i is unit vector perpendicular to the plane l_i and m_i.

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Key Words: two and three-dimensional Finsler spaces, main scalar, C^v-reducible Finsler space

1. Introduction

Let C_{ijk} be the hv-torsion tensor of a n-dimensional Finsler space F^n. In view of well know identity $C_{ijk}|_h - C_{ijh}|_k = 0$, $C_{ijk}|_h$ is symmetric in all of its indices. The symbol ' $|$ ' denotes v-covariant differentiation with respect to the Cartan connection $C\Gamma$. In the paper [1], first author has studied the Finsler

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spaces whose $C_{ijk}|h$ can be written as product of tensor of type (0, 3) and (0, 1) such as,

$$LC_{ijk}|_h = A_{ijk}B_h + A_{ijh}B_k + A_{ihk}B_j + A_{hjk}B_i, \qquad A_{ijk} \neq \lambda C_{ijk}$$

where, $A_{i00} = 0$, $A_{ij0} \neq 0$ and $B_0 = 0$; $A_{i00} = A_{ijk}y^j y^k$, $B_0 = B_i y^i$.

In the paper [1] first author has defined has defined C^v-Reducible Finsler Spaces taking $A_{ijk} = h_{ijk} = -L^2 L|_i|_j|_k$ and found value of B_i.

In the present paper, we shall find out the value of A_{ijk}, taking the value of B_i as m_i and n_i, where m_i is C_i/C and n_i is unit vector perpendicular to the plane l_i and m_i.

2. Preliminaries

Let us consider a Finsler space F^n of dimension n, whose $C_{ijk}|_h$ can be written as,

$$LC_{ijk}|_h = A_{ijk}B_h + A_{ijh}B_k + A_{ihk}B_j + A_{hjk}B_i \tag{1}$$

where, A_{ijk} is assumed to be a symmetric tensor. Contracting (1) by y^h, we have

$$-LC_{ijk} = A_{ijk}B_0 + A_{ij0}B_k + A_{i0k}B_j + A_{0jk}B_i \tag{2}$$

where, '0' denotes the contraction by y^h, for instance, $B_0 = B_h y^h$.

Again contracting (2) by y^k, y^j and y^i, in order we obtain,

$$0 = A_{ij0}B_0 + A_{ij0}B_0 + A_{i00}B_j + A_{0j0}B_i \tag{3}$$

$$0 = A_{i00}B_0 + A_{i00}B_0 + A_{i00}B_0 + A_{000}B_i \tag{4}$$

$$0 = A_{000}B_0 \tag{5}$$

If $B_0 \neq 0$, then we have $A_{000} = 0$, $A_{i00} = 0$, $A_{ij0} = 0$ and $A_{ijk} = -(L/B_0)C_{ijk}$ which contradicts to our assumption, therefore $B_0 = 0$. As B_0 (4) gives $A_{000} = 0$, and form (3), we have $A_{j00}B_i + A_{i00}B_j = 0$ which leads to $A_{j00}B_iB_k = -A_{i00}B_jB_k = A_{k00}B_iB_j = -A_{j00}B_iB_k$ that is $A_{j00}B_iB_k = 0$.

Thus we have two cases: $A_{i00} = 0$ or $B_i = 0$ leads to $C_{ijk} = 0$ by (2).

Consequently we have, $A_{i00} = 0$. Since if F^n is assumed to be non-Riemannian, we have $A_{ij0} \neq 0$ from (2).

Proposition 2.1. (see [1]) *If the v-covariant derivative of C_{ijk} of the non-Riemannian Finsler space F^n is written in the form (1), then A_{ijk} and B_i, satisfy $A_{i00} = 0$, $A_{ijk} \neq 0$ and $B_0 = 0$.*

3. Generalized C^v-Reducible Finsler Space of type I

A Finsler space F^n [4] (n$\geq$ 2) is called generalzed C^v-reducible Finsler Space of type I if $B_i = m_i$, i.e.,

$$L^2 C_{ijk}|_h = A_{ijk}m_h + A_{ijh}m_k + A_{ihk}m_j + A_{hjk}m_i, \tag{6}$$

contracting (6) by m^h, we have

$$L^2C^2\bar{C}_{ijk}|_0 = A_{ijk} + \bar{A}_{ij0}m_k + \bar{A}_{i0k}m_j + \bar{A}_{0jk}m_i \tag{7}$$

where, $C^2\bar{C}_{ijk}|_0 = C_{ijk}|_h m^h$, $\quad m_h m^h = 1$, $\quad A_{ijk}m^k = \bar{A}_{ij0}$.
contracting (7) by m^k, m^j and m^i in order, we obtain,

$$L^2C^2\bar{C}_{ij0}|_0 = \bar{A}_{ij0} + \bar{A}_{ij0} + \bar{A}_{i00}m_j + \bar{A}_{0j0}m_i \tag{8}$$

$$L^2C^2\bar{C}_{i00}|_0 = 3\bar{A}_{i00} + \bar{A}_{000}m_i \tag{9}$$

$$L^2C^2\bar{C}_{000}|_0 = 4\bar{A}_{000} \tag{10}$$

using (10) and (9), we have

$$\bar{A}_{i00} = \frac{L^2C^2}{3}[\bar{C}_{i00}|_0 - \frac{1}{4}\bar{C}_{000}|_0 m_i] \tag{11}$$

From (11) and (8), we get

$$\bar{A}_{ij0} = \frac{L^2C^2}{2}[\bar{C}_{ij0}|_0 - \frac{1}{3}(\bar{C}_{i00}|_0 m_j + \bar{C}_{j00}|_0 m_i) + \frac{1}{6}\bar{C}_{000}|_0 m_i m_j] \tag{12}$$

Again using (12) and (7), we get,

$$A_{ijk} = L^2C^2[\bar{C}_{ijk}|_0 - \frac{1}{2}\pi_{(ijk)}(\bar{C}_{ij0}|_0 m_k) + \frac{2}{3}\pi_{(ijk)}(\bar{C}_{i00}|_0 m_j m_k) - \frac{1}{2}\bar{C}_{000}|_0 m_i m_j m_k] \tag{13}$$

where, $\pi_{(ijk)}$ represents sum of cylic permutation of the indices i, j, k.

Proposition 3.1. *If $B_i = m_i$ in (1), the tensor A_{ijk} must be given by (13).*

Again from (5) and (13), we have

$$C_{ijk}|_h = C^2[\pi_{(ijkh)}(\bar{C}_{ijk}|_0 m_h) - \pi_{(ijkh)}(\bar{C}_{ij0}|_0 m_k m_h) + \tag{14}$$

$$2\pi_{(ijkh)}(\bar{C}_{i00}|_0 m_j m_k m_h) - 2\bar{C}_{000}|_0 m_i m_j m_k m_h]$$

where, $\pi_{(ijkh)}$ represents sum of cylic permutation of the indices i, j, k, h.
Definition 1 Let F^n be a Finsler space of dimension $n \geq 2$. Then F^n is said to be generalized C^v-reducible Finsler space of type I if the tensor $C_{ijk}|_h$ can be written as (14).

In two-dimensional Finsler space, [2, 3] Berwald frame considered which consists of two mutually perpendicular vector. The first vector $e^i_{1)}$ is the normalized supporting element $l^i = y^i \text{Ł}(x, y)$ and the second $e^i_{2)} = m^i = \pm\frac{C^i}{C}$ is the unit vector orthogonal to l^i relative to y^i. In two-dimensional Finsler space, the hv-torsion tensor C_{ijk} can be written as,

$$LC_{ijk} = Im_i m_j m_k \tag{15}$$

where, $I = C_{222}$.

Taking v-covariant derivative of (15) and using $L|_i = l_i, \quad Lm_i|_j = -l_i m_j$, we get,

$$LC_{ijk}|_h + l_h C_{ijk} = I|_h m_i m_j m_k - \frac{I}{L}(l_i m_j m_k m_h + m_i l_j m_k m_h + m_i m_j l_k m_h) \tag{16}$$

contracting above by m^h, we have

$$LC^2\bar{C}_{ijk}|_0 = \bar{I}|_0 m_i m_j m_k - \frac{I}{L}(l_i m_j m_k + m_i l_j m_k + m_i m_j l_k) \tag{17}$$

contracting (17) by m^k, m^j and m^i in order, we obtain,

$$LC^2\bar{C}_{ij0}|_0 = \bar{I}|_0 m_i m_j - \frac{I}{L}(l_i m_j + m_i l_j) \tag{18}$$

$$LC^2\bar{C}_{i00}|_0 = \bar{I}|_0 m_i - \frac{I}{L} l_i \tag{19}$$

$$LC^2\bar{C}_{000}|_0 = \bar{I}|_0 \tag{20}$$

Using above relation in (13), we get

$$A_{ijk} = L[\bar{I}|_0 m_i m_j m_k - \frac{2}{3}\frac{I}{L}\pi_{(ijk)}(l_i m_j m_k)] \tag{21}$$

Corollary 3.1. *If $B_i = m_i$ in (1), then in two-dimensional Finsler space, the tensor A_{ijk} must be given by (21).*

Using (21) and (5), we have

$$LC_{ijk}|_h = 4\bar{I}|_0 m_i m_j m_k m_h - \frac{2I}{L}\pi_{(ijkh)}(l_i m_j m_k m_h) \tag{22}$$

Thus, from (16) and (22), we have

$$I|_h = 4\bar{I}|_0 m_h$$

Contracting above by m^h, we have

$$\bar{I}|_0 = 0$$

Theorem 3.1. *A two-dimensional Finsler space is said to be generalized C^v-reducible Finsler space of type (I) if $\bar{I}|_0$ vanishes identically.*

The Miron frame of a three-dimensional Finsler space is called the Moor frame (l_i, m_i, n_i). The first vector is the normalized supporting element l_i, the second vector is the normalized torsion vector $m^i = C^i/C$ and the third n_i is unit vector perpendicular to the plane l_i and m_i. The (v)hv-torsion tensor C_{ijk} for three-dimensional Finsler space is given by,

$$LC_{ijk} = Hm_i m_j m_k - J\pi_{(ijk)}(m_i m_j n_k) + I\pi_{(ijk)}(m_i n_j n_k) + Jn_i n_j n_k \tag{23}$$

where, $H = C_{222}$, $I = C_{233}$, $J = C_{333} = -C_{233}$, $H + I = LC$

Taking v-covariant derivative of (23), and using $L|_i = l_i$, $Lm_i|_j = -l_i m_j + n_i v_j$, $Ln_i|_j = -l_i n_j - m_i v_j$, we get

$$\begin{aligned} LC_{ijk}|_h = {} & H|_h m_i m_j m_k - J|_h \pi_{(ijk)}(m_i m_j n_k) + \\ & I|_h \pi_{(ijk)}(m_i n_j n_k) + J|_h n_i n_j n_k + \frac{H}{L}\pi_{(ijkh)}(l_i m_j m_k m_h) \\ & + \frac{(H-I)}{L}\pi_{(ijk)}(m_i m_j n_k)v_h - \frac{J}{L}\pi_{(ijkh)}(l_i m_j m_k n_h) - \\ & \frac{I}{L}\pi_{(ijkh)}(l_i m_j n_k n_h) + \frac{3I}{L}n_i n_j n_k v_h - \frac{J}{L}\pi_{(ijkh)}(l_i n_j n_k n_h) \end{aligned} \tag{24}$$

contracting (24) by m^h, we have

$$\begin{aligned} LC^2 C_{ijk}|_0 = {} & \bar{H}|_0 m_i m_j m_k - \bar{J}|_0 \pi_{(ijk)}(m_i m_j n_k) + \\ & \bar{I}|_0 \pi_{(ijk)}(m_i n_j n_k) + \bar{J}|_0 n_i n_j n_k + \frac{H}{L}\pi_{(ijk)}(l_i m_j m_k) \end{aligned} \tag{25}$$

$$+\frac{(H-I)}{L}\pi_{(ijk)}(m_i m_j n_k)v_0 - \frac{J}{L}(l_i m_j n_k + l_k m_j n_i + l_j m_i n_k + l_k m_i n_j + l_i m_k n_j + l_j m_k n_i) - \frac{I}{L}\pi_{(ijk)}(l_i n_j n_k) + \frac{3I}{L} n_i n_j n_k v_0$$

contracting (25) by m^k, m^j and m^i in order, we obtain,

$$LC^2 C_{ij0}|_0 = \bar{H}|_0 m_i m_j - \bar{J}|_0(m_i n_j + m_j n_i) + \bar{I}|_0 n_i n_j + \frac{H}{L}(l_i m_j + l_j m_i) + \frac{(H-I)}{L}(m_i n_j + m_j n_i)v_0 - \frac{J}{L}(l_i n_j + l_j n_i) \tag{26}$$

$$LC^2 C_{i00}|_0 = \bar{H}|_0 m_i - (\bar{J}|_0 - \frac{(H-I)}{L})n_i + \frac{H}{L} l_i \tag{27}$$

$$LC^2 C_{000}|_0 = \bar{H}|_0 \tag{28}$$

Using equation (25), (26), (27) and (28) in (13), we get

$$A_{ijk} = L[\bar{H}|_0 m_i m_j m_k - \frac{2}{3}(\bar{J}|_0 - \frac{(H-I)v_0}{L})\pi_{(ijk)}(m_i m_j n_k) + \frac{\bar{I}|_0}{2}\pi_{(ijk)}(m_i n_j n_k) + \frac{2}{3}\frac{H}{L}\pi_{(ijk)}(l_i m_j m_k) - \frac{3}{2}\frac{J}{L}(l_i n_j m_k + l_j n_i m_k + l_i n_k m_j + l_k n_i m_j + l_j n_k m_i + l_k n_j m_i) - \frac{I}{L}\pi_{(ijk)}(l_i n_j n_k) + (\bar{J}|_0 + \frac{3Iv_0}{L})n_i n_j n_k] \tag{29}$$

Corollary 3.2. *If $B_i = m_i$ in (1), then in three-dimensional Finsler space, the tensor A_{ijk} must be given by (28).*

Using equation (29) and (6), we have

$$LC_{ijk}|_h = 4\bar{H}|_0 m_i m_j m_k m_h - 2(\bar{J}|_0 - \frac{(H-I)v_0}{L})\pi_{(ijkh)}(m_i m_j m_k n_h) + \bar{I}|_0 \pi_{(ijkh)}(m_i m_j n_k n_h) + \frac{2H}{L}\pi_{(ijkh)}(l_i m_j m_k m_h) - \frac{3J}{L}\pi_{(ijkh)}((l_i n_j + l_j n_i)m_k m_h) - \frac{I}{L}\pi_{(ijkh)}((l_i m_j + l_j m_i)n_k n_h) + (\bar{J}|_0 + \frac{3Iv_0}{L})\pi_{(ijkh)}(n_i n_j n_k m_h) \tag{30}$$

Thus, using relation (30) and (24), we have

Theorem 3.2. *A three-dimensional Finsler space is said to be generalized C^v-reducible Finsler space of type (I) if the relations*

$$H|_h = 4\bar{H}|_0 m_h, \qquad \bar{J}|_0 - \tfrac{(H-I)v_0}{L} = 0, \qquad \bar{J}|_0 + \tfrac{3Iv_0}{L} = 0, \qquad \bar{I}|_0 = 0$$

holds good.

4. Generalized C^v-Reducible Finsler Space of type II

A Finsler space F^n $(n \geq 3)$ is called generalized C^v-reducible Finsler space of type II, if $B_i = n_i$, where n_i is component of any vector perpendicular to y^i and m^i, i.e.,

$$L^2 C_{ijk}|_h = A_{ijk} n_h + A_{jkh} n_i + A_{khi} n_j + A_{hij} n_k \tag{31}$$

contracting equation (31) by n^h, we have,

$$L^2 C_{ijk}|_0 = A_{ijk} + \bar{A}_{jk0} n_i + \bar{A}_{koi} n_j + \bar{A}_{0ij} n_k \tag{32}$$

where, $A_{ijk} n^k = \bar{A}_{ij0}$, $\quad n_i n^i = 1$, $\quad L^2 C_{ijk}|_h n^h = C_{ijk}|_0$.

Again contracting (32) by n^k, n^j and n^i in order, we obtain

$$L^2 C_{ij0}|_0 = \bar{A}_{ij0} + \bar{A}_{j00} n_i + \bar{A}_{00i} n_j + \bar{A}_{0ij} \tag{33}$$

$$L^2 C_{i00}|_0 = 3\bar{A}_{i00} + \bar{A}_{000} n_i \tag{34}$$

$$L^2 C_{000}|_0 = 4\bar{A}_{000} \tag{35}$$

Using relation (34) and (35), we have

$$\bar{A}_{i00} = \frac{L^2}{3}[C_{i00}|_0 - \frac{1}{4} C_{000}|_0 n_i] \tag{36}$$

using (33) and (36), we get

$$\bar{A}_{ij0} = \frac{L^2}{2}[C_{ij0}|_0 - \frac{1}{3}(C_{j00}|_0 n_i + C_{i00}|_0 n_j) + \frac{1}{6} C_{000}|_0 n_i n_j] \tag{37}$$

using (32) and (37), we get

$$A_{ijk} = L^2[C_{ijk}|_0 - \frac{1}{2}\pi_{(ijk)}(C_{ij0}|_0 n_k) + \frac{2}{3}\pi_{(ijk)}(C_{i00}|_0 n_j n_k) - \tag{38}$$

$$\frac{1}{2}C_{000}|_0 n_i n_j n_k]$$

using (31) and (38), we have

$$C_{ijk}|_h = \pi_{(ijkh)}(C_{ijk}|_0 n_h) - \pi_{(ijkh)}(C_{ij0}|_0 n_k n_h) + 2\pi_{(ijkh)}(C_{i00}|_0 n_j n_k n_h) - 2C_{000}|_0 n_i n_j n_k n_h) \tag{39}$$

Proposition 4.1. *If $B_i = n_i$ in (1), the tensor A_{ijk} must be given by (38).*

Definition 2 Let F^n be a Finsler space of dimension $n \geq 3$, then F^n is said to be generalized C^v-reducible Finsler space of type II if the tensor $C_{ijk}|_h$ can be written as (39).

Now, contracting equation (24) by n^h, we get

$$LC_{ijk}|_0 = H|_0 m_i m_j m_k - J|_0 \pi_{(ijk)}(m_i m_j n_k) + I|_0 \pi_{(ijk)}(m_i n_j n_k) + J|_0 n_i n_j n_k + \frac{(H-I)}{L}\pi_{(ijk)}(m_i m_j n_k)\bar{v}_0 - \frac{J}{L}\pi_{(ijk)}(l_i m_j m_k) - \frac{I}{L}(l_i m_j n_k + l_j m_i n_k + l_j m_k n_i + l_k m_j n_i + l_i m_k n_j + l_k m_i n_j) + \frac{3I}{L}n_i n_j n_k \bar{v}_0 - \frac{J}{L}\pi_{(ijk)}(l_i n_j n_k) \tag{40}$$

where, $v_h n^h = \bar{v}_0$, $\quad n_i n^i = 1$, contracting (40) by n^k, n^j and n^i in order, we obtain,

$$LC_{ij0}|_0 = -J|_0 m_i m_j + I|_0(m_i n_j + m_j n_i) + J|_0 n_i n_j + \frac{(H-I)}{L}m_i m_j \bar{v}_0 - \frac{I}{L}(l_i m_j + l_j m_i) + \frac{3I}{L}n_i n_j \bar{v}_0 - \frac{J}{L}(l_i n_j + l_j n_i) \tag{41}$$

$$LC_{i00}|_0 = I|_0 m_i + (J|_0 + \frac{3I\bar{v}_0}{L})n_i - \frac{J}{L}l_i \tag{42}$$

$$LC_{000}|_0 = J|_0 + \frac{3I\bar{v}_0}{L} \tag{43}$$

using (43), (42), (41) and (40) in (38), we have

$$A_{ijk} = L[H|_0 m_i m_j m_k - \frac{1}{2}J|_0 \pi_{(ijk)}(m_i m_j n_k) + \frac{2}{3}I|_0 \pi_{(ijk)}(m_i n_j n_k) \tag{44}$$

$$+(J|_0 + \frac{3I\bar{v}_0}{L})n_i n_j n_k - \frac{J}{L}\pi_{(ijk)}(l_i m_j m_k) - \frac{I}{2L}(l_i m_j n_k + l_j m_i n_k + l_j m_k n_i + l_k m_j n_i + l_i m_k n_j + l_k m_i n_j) - \frac{1}{3}\frac{I}{L}\pi_{(ijk)}(l_i n_j n_k)]$$

using (44) in (31), we have

$$LC_{ijk}|_h = H|_0 \pi_{(ijkh)}(m_i m_j m_k n_h) - J|_0 \pi_{(ijkh)}(m_i m_j n_k n_h) + 2I|_0 \pi_{(ijkh)}(m_i n_j n_k n_h) + 4(J|_0 + \frac{3I\bar{v}_0}{L})n_i n_j n_k n_h - \frac{J}{L}\pi_{(ijkh)}(l_i m_j m_k n_h) - \frac{I}{L}\pi_{(ijkh)}(l_i m_j n_k n_h) - \frac{J}{L}\pi_{(ijkh)}(l_i n_j n_k n_h) \tag{45}$$

Thus, using (45) and (24), we have

Theorem 4.1. *A three-dimensional Finsler space is said to be generalized C^v-reducible Finsler space of type (II) if the relations*

$$J|_0 + \tfrac{3I\bar{v}_0}{L} = 0, \qquad J|_0 - \tfrac{(H-I)\bar{v}_0}{L} = 0, \qquad H|_0 = 0$$

holds good.

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