ON GENERALIZED $C^v$-REDUCIBLE FINSLER SPACES

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Abstract: A Finsler space is said to be generalized $C^v$-Reducible Finsler Space whose $C_{ijk}|_h$ can be written as product of tensor of type $(0, 3)$ and $(0, 1)$ such as,

$$LC_{ijk}|_h = A_{ijk}B_h + A_{ijh}B_k + A_{ihk}B_j + A_{hjk}B_i; \quad A_{ijk} \neq \lambda C_{ijk}$$

where, $A_{i00} = 0$, $A_{ij0} \neq 0$ and $B_0 = 0$; $A_{i00} = A_{ijk}y^j y^k$, $B_0 = B_i y^i$.

In the present paper, we shall find out the value of $A_{ijk}$, taking the value of $B_i$ as $m_i$ and $n_i$, where $m_i$ is $C_i/C$ and $n_i$ is unit vector perpendicular to the plane $l_i$ and $m_i$.

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1. Introduction

Let $C_{ijk}$ be the hv-torsion tensor of a n-dimensional Finsler space $F^n$. In view of well know identity $C_{ijk}|_h - C_{ijh}|_k = 0$, $C_{ijk}|_h$ is symmetric in all of its indices. The symbol $'|'$ denotes v-covariant differentiation with respect to the Cartan connection $C_T$. In the paper [1], first author has studied the Finsler
spaces whose $C_{ijk}|h$ can be written as product of tensor of type $(0, 3)$ and $(0, 1)$ such as,

$$L C_{ijk}|h = A_{ijk} B_h + A_{i j h} B_k + A_{i h k} B_j + A_{h j k} B_i, \quad A_{ijk} \neq \lambda C_{ijk}$$

where, $A_{i00} = 0$, $A_{ij0} \neq 0$ and $B_0 = 0$; $A_{i00} = A_{ijk} y^j y^k$, $B_0 = B_i y^i$.

In the paper [1] first author has defined $C''$-Reducible Finsler Spaces taking $A_{ijk} = h_{ijk} = -L^2 L_{i|j|k}$ and found value of $B_i$.

In the present paper, we shall find out the value of $A_{ijk}$, taking the value of $B_i$ as $m_i$ and $n_i$, where $m_i$ is $C_i/C$ and $n_i$ is unit vector perpendicular to the plane $l_i$ and $m_i$.

2. Preliminaries

Let us consider a Finsler space $F^n$ of dimension $n$, whose $C_{ijk}|h$ can be written as,

$$L C_{ijk}|h = A_{ijk} B_h + A_{i j h} B_k + A_{i h k} B_j + A_{h j k} B_i$$

where, $A_{ijk}$ is assumed to be a symmetric tensor. Contracting (1) by $y^h$, we have

$$-L C_{ijk} = A_{ijk} B_0 + A_{ij0} B_k + A_{i0k} B_j + A_{0jk} B_i$$

where, '0' denotes the contraction by $y^h$, for instance, $B_0 = B_h y^h$.

Again contracting (2) by $y^k$, $y^j$ and $y^i$, in order we obtain,

$$0 = A_{ij0} B_0 + A_{ij0} B_0 + A_{i00} B_j + A_{0j0} B_i$$

$$0 = A_{i00} B_0 + A_{i00} B_0 + A_{i00} B_0 + A_{000} B_i$$

$$0 = A_{000} B_0$$

If $B_0 \neq 0$, then we have $A_{000} = 0$, $A_{i00} = 0$, $A_{ij0} = 0$ and $A_{ijk} = -(L/B_0)C_{ijk}$ which contradicts to our assumption, therefore $B_0 = 0$. As $B_0$ (4) gives $A_{000} = 0$, and form (3), we have $A_{j00} B_i + A_{i00} B_j = 0$ which leads to $A_{j00} B_i B_k = -A_{i00} B_j B_k = A_{k00} B_i B_j = -A_{j00} B_i B_k$ that is $A_{j00} B_i B_k = 0$.

Thus we have two cases: $A_{i00} = 0$ or $B_i = 0$ leads to $C_{ijk} = 0$ by (2).

Consequently we have, $A_{i00} = 0$. Since if $F^n$ is assumed to be non-Riemannian, we have $A_{ij0} \neq 0$ from (2).

**Proposition 2.1.** (see [1]) If the v-covariant derivative of $C_{ijk}$ of the non-Riemannian Finsler space $F^n$ is written in the form (1), then $A_{ijk}$ and $B_i$, satisfy $A_{i00} = 0$, $A_{ijk} \neq 0$ and $B_0 = 0$. 
3. Generalized $C^v$-Reducible Finsler Space of type I

A Finsler space $F^n$ [4] ($n \geq 2$) is called generalized $C^v$-reducible Finsler Space of type I if $B_i = m_i$, i.e.,

$$L^2 C_{ijk|h} = A_{ijk}m_h + A_{ijh}m_k + A_{ihk}m_j + A_{hjk}m_i,$$

contracting (6) by $m^h$, we have

$$L^2 C^2 C_{ijk|0} = A_{ijk} + \bar{A}_{ij0}m_k + \bar{A}_{i0k}m_j + \bar{A}_{0jk}m_i$$

(7)

where, $C^2 C_{ijk|0} = C_{ijk|m^h}$, $m^h m^h = 1$, $A_{ijk} m^k = \bar{A}_{ij0}$.

contracting (7) by $m^k$, $m^i$ and $m^j$ in order, we obtain,

$$L^2 C^2 C_{i0j|0} = \bar{A}_{ij0} + \bar{A}_{i0j} + \bar{A}_{i00}m_j + \bar{A}_{0j0}m_i$$

(8)

$$L^2 C^2 C_{i00|0} = 3\bar{A}_{i00} + \bar{A}_{000}m_i$$

(9)

$$L^2 C^2 C_{000|0} = 4\bar{A}_{000}$$

(10)

using (10) and (9), we have

$$\bar{A}_{i00} = \frac{L^2 C^2}{3} [\bar{C}_{i00|0} - \frac{1}{4} \bar{C}_{000|0} m_i]$$

(11)

From (11) and (8), we get

$$\bar{A}_{ij0} = \frac{L^2 C^2}{2} [\bar{C}_{ij0|0} - \frac{1}{3} (\bar{C}_{i00|0} m_j + \bar{C}_{j00|0} m_i) + \frac{1}{6} \bar{C}_{000|0} m_i m_j]$$

(12)

Again using (12) and (7), we get,

$$A_{ijk} = L^2 C^2 [\bar{C}_{ijk|0} - \frac{1}{2} \pi_{(ijk)} (\bar{C}_{ij0|0} m_k) + \frac{2}{3} \pi_{(ijk)} (\bar{C}_{i00|0} m_j m_k)$$

$$- \frac{1}{2} \bar{C}_{000|0} m_i m_j m_k]$$

(13)

where, $\pi_{(ijk)}$ represents sum of cyclic permutation of the indices i, j, k.

**Proposition 3.1.** If $B_i = m_i$ in (1), the tensor $A_{ijk}$ must be given by (13).

Again from (5) and (13), we have

$$C_{ijk|h} = C^2 [\pi_{(ijk)} (\bar{C}_{ijk|0} m_h) - \pi_{(ijk)} (\bar{C}_{ij0|0} m_k m_h)$$

(14)
\[ 2\pi_{ijkl}(\bar{C}_{ik00}m_jm_km_h) - 2\bar{C}_{000}m_im_jm_km_h \]

where, \( \pi_{ijkl} \) represents sum of cyclic permutation of the indices i, j, k, h.

**Definition 1** Let \( F^n \) be a Finsler space of dimension \( n \geq 2 \). Then \( F^n \) is said to be generalized \( C^n \)-reducible Finsler space of type I if the tensor \( C_{ijk|h} \) can be written as (14).

In two-dimensional Finsler space, [2, 3] Berwald frame considered which consists of two mutually perpendicular vector. The first vector \( e_1^i \) is the normalized supporting element \( l^i = y^i L(x, y) \) and the second \( e_2^i = m^i = \pm \frac{C^i}{C} \) is the unit vector orthogonal to \( l^i \) relative to \( y^i \). In two-dimensional Finsler space, the hv-torsion tensor \( C_{ijk} \) can be written as,

\[ LC_{ijk} = Im_im_jm_k \]  \hspace{1cm} (15)

where, \( I = C_{222} \).

Taking \( \nu \)-covariant derivative of (15) and using \( L|_i = l_i \), \( Lm_i|_j = -l_i m_j \), we get,

\[ LC_{ijk|h} + l_h C_{ijk} = I|_h m_im_jm_k - I \frac{1}{L}(l_im_jm_km_h + m_il_jm_km_h + m_im_jl_km_h) \]  \hspace{1cm} (16)

contracting above by \( m^h \), we have

\[ LC^2_{ijk|0} = I|_0 m_im_jm_k - I \frac{1}{L}(l_im_jm_k + m_il_jm_k + m_im_jl_k) \]  \hspace{1cm} (17)

contracting (17) by \( m^k, m^j \) and \( m^i \) in order, we obtain,

\[ LC^2_{ij0|0} = I|_0 m_im_j - I \frac{1}{L}(l_i m_j + m_i l_j) \]  \hspace{1cm} (18)

\[ LC^2_{i00|0} = I|_0 m_i - I \frac{1}{L} l_i \]  \hspace{1cm} (19)

\[ LC^2_{000|0} = I|_0 \]  \hspace{1cm} (20)

Using above relation in (13), we get

\[ A_{ijk} = L[I|_0 m_im_jm_k - \frac{2}{3} \frac{I}{L} \pi_{ijkl}(l_im_jm_k)] \]  \hspace{1cm} (21)

**Corollary 3.1.** If \( B_i = m_i \) in (1), then in two-dimensional Finsler space, the tensor \( A_{ijk} \) must be given by (21).
Using (21) and (5), we have
\[ LC_{ijk|h} = 4\bar{I}_0 m_i m_j m_k m_h - \frac{2L}{L} \pi_{(ijkh)}(l_i m_j m_k m_h) \] (22)

Thus, from (16) and (22), we have
\[ I|h = 4\bar{I}_0 m_h \]

Contracting above by \( m^h \), we have
\[ \bar{I}_0 = 0 \]

**Theorem 3.1.** A two-dimensional Finsler space is said to be generalized \( C^\nu \)-reducible Finsler space of type (I) if \( \bar{I}_0 \) vanishes identically.

The Miron frame of a three-dimensional Finsler space is called the Moor frame \((l_i, m_i, n_i)\). The first vector is the normalized supporting element \( l_i \), the second vector is the normalized torsion vector \( m^i = C^i / C \) and the third \( n_i \) is unit vector perpendicular to the plane \( l_i \) and \( m_i \). The \((v)hv\)-torsion tensor \( C_{ijk} \) for three-dimensional Finsler space is given by,
\[ LC_{ijk} = H m_i m_j m_k - J \pi_{(ijk)}(m_i m_j n_k) + I \pi_{(ijk)}(m_i n_j n_k) + J n_i n_j n_k \] (23)

where, \( H = C_{222} \), \( I = C_{233} \), \( J = C_{333} = -C_{233} \), \( H + I = LC \)

Taking \( v \)-covariant derivative of (23), and using \( L|i = l_i \), \( Lm_i|j = -l_i m_j + n_i v_j \), \( Ln_i|j = -l_i n_j - m_i v_j \), we get
\[ LC_{ijk|h} = H|_h m_i m_j m_k - J|_h \pi_{(ijk)}(m_i m_j n_k) + \]
\[ I|_h \pi_{(ijk)}(m_i n_j n_k) + J|_h n_i n_j n_k + \frac{H}{L} \pi_{(ijkh)}(l_i m_j m_k m_h) \]
\[ + \frac{(H-I)}{L} \pi_{(ijk)}(m_i m_j n_k) v_h - \frac{J}{L} \pi_{(ijkh)}(l_i m_j m_k n_h) - \]
\[ \frac{I}{L} \pi_{(ijkh)}(l_i m_j n_k n_h) + \frac{3I}{L} n_i n_j n_k v_h - \frac{J}{L} \pi_{(ijkh)}(l_i n_j n_k n_h) \]

contracting (24) by \( m^h \), we have
\[ LC^2 C_{ijk}|_0 = \bar{H}|_0 m_i m_j m_k - \bar{J}|_0 \pi_{(ijk)}(m_i m_j n_k) + \]
\[ \bar{I}|_0 \pi_{(ijk)}(m_i n_j n_k) + \bar{J}|_0 n_i n_j n_k + \frac{H}{L} \pi_{(ijk)}(l_i m_j m_k) \] (25)
Using equation (25), (26), (27) and (28) in (13), we get
\[ m \text{ contracting (25) by } m^k, \ m^j \text{ and } m^i \text{ in order, we obtain,} \]
\[
LC^2 C_{ij0}|_0 = \bar{H}|_0 m_i m_j - J|_0 (m_i n_j + m_j n_i) + \bar{I}|_0 n_i n_j + \\
\frac{H}{L} (l_i m_j + l_j m_i) + \frac{(H - I)}{L} (m_i n_j + m_j n_i) v_0 - \\
\frac{J}{L} (l_i n_j + l_j n_i) - \\
\frac{H}{L} (l_i m_j + l_j m_i) \pi_{(ijk)}(m_i m_j m_k) + \\
\frac{3 I}{L} \pi_{(ijk)}(l_i n_j n_k) + (\bar{J}|_0 + \frac{3 I v_0}{L}) n_i n_j n_k \]
contracting (25) by \( m^k, \ m^j \) and \( m^i \) in order, we obtain,
\[
LC^2 C_{i00}|_0 = \bar{H}|_0 m_i - (\bar{J}|_0 - \frac{(H - I)}{L} v_0) n_i + \frac{H}{L} l_i \\
LC^2 C_{000}|_0 = \bar{H}|_0 \\
\]
Using equation (25), (26), (27) and (28) in (13), we get
\[
A_{ijk} = L[\bar{H}|_0 m_i m_j m_k - \frac{2}{3} (\bar{J}|_0 - \frac{(H - I)v_0}{L}) \pi_{(ijk)}(m_i m_j m_k) + \\
\frac{\bar{I}|_0}{2} \pi_{(ijk)}(m_i n_j n_k) + \frac{2 H}{3 L} \pi_{(ijk)}(l_i m_j m_k) - \frac{3 J}{2 L} (l_i n_j m_k + \\
l_j n_i m_k + l_i n_k m_j + l_k n_i m_j + l_j n_k m_i + l_k n_j m_i) - \\
\frac{I}{L} \pi_{(ijk)}(l_i n_j n_k) + (\bar{J}|_0 + \frac{3 I v_0}{L}) n_i n_j n_k] \]

**Corollary 3.2.** If \( B_i = m_i \) in (1), then in three-dimensional Finsler space, the tensor \( A_{ijk} \) must be given by (28).

Using equation (29) and (6), we have
\[
LC_{ijk}|_h = 4 \bar{H}|_0 m_i m_j m_k m_h - 2(\bar{J}|_0 - \\
\frac{(H - I)v_0}{L}) \pi_{(ijkh)}(m_i m_j m_k m_h) + \bar{I}|_0 \pi_{(ijkh)}(m_i m_j m_k n_h) + \\
\frac{2 H}{L} \pi_{(ijkh)}(l_i m_j m_k m_h) - \frac{3 J}{L} \pi_{(ijkh)}((l_i n_j + l_j n_i) m_k m_h) - \\
\frac{I}{L} \pi_{(ijkh)}((l_i m_j + l_j m_i) n_k n_h) + (\bar{J}|_0 + \frac{3 I v_0}{L}) \pi_{(ijkh)}(n_i n_j n_k n_h) \]

Thus, using relation (30) and (24), we have
Theorem 3.2. A three-dimensional Finsler space is said to be generalized $C^v$-reducible Finsler space of type (I) if the relations

$$H|_h = 4\bar{H}|_0 m_h, \quad \bar{J}|_0 - \frac{(H-I)^0}{L} = 0, \quad \bar{J}|_0 + \frac{3I^0}{L} = 0, \quad \bar{I}|_0 = 0$$

holds good.

4. Generalized $C^v$-Reducible Finsler Space of type II

A Finsler space $F^n$ ($n \geq 3$) is called generalized $C^v$-reducible Finsler space of type II, if $B_i = n_i$, where $n_i$ is component of any vector perpendicular to $y^i$ and $m^i$, i.e.,

$$L^2C_{ijk}|_h = A_{ijk}n_h + A_{jkh}n_i + A_{khi}n_j + A_{hij}n_k$$  \hspace{1cm} (31)

contracting equation (31) by $n^h$, we have,

$$L^2C_{ijk}|_0 = A_{ijk} + \bar{A}_{jko}n_i + \bar{A}_{ko}n_j + \bar{A}_{0ij}n_k$$  \hspace{1cm} (32)

where, $A_{ijk}n^k = \bar{A}_{ij0}$, $n_in^i = 1$, $L^2C_{ijk}|_hn^h = C_{ijk}|_0$.

Again contracting (32) by $n^k$, $n^j$ and $n^i$ in order, we obtain

$$L^2C_{ij0}|_0 = \bar{A}_{ij0} + \bar{A}_{j00}n_i + \bar{A}_{i00}n_j + \bar{A}_{0ij}$$  \hspace{1cm} (33)

$$L^2C_{i00}|_0 = 3\bar{A}_{i00} + \bar{A}_{000}n_i$$  \hspace{1cm} (34)

$$L^2C_{000}|_0 = 4\bar{A}_{000}$$  \hspace{1cm} (35)

Using relation (34) and (35), we have

$$\bar{A}_{i00} = \frac{L^2}{3} [C_{i00}|_0 - \frac{1}{4}C_{000}|_0 n_i]$$  \hspace{1cm} (36)

using (33) and (36), we get

$$\bar{A}_{ij0} = \frac{L^2}{2} [C_{ij0}|_0 - \frac{1}{3}(C_{j00}|_0 n_i + C_{i00}|_0 n_j) + \frac{1}{6}C_{000}|_0 n_i n_j]$$  \hspace{1cm} (37)

using (32) and (37), we get

$$A_{ijk} = L^2[C_{ijk}|_0 - \frac{1}{2}\pi_{(ijk)}(C_{ij0}|_0 n_k) + \frac{2}{3}\pi_{(ijk)}(C_{i00}|_0 n_j n_k) - (38)$$
using (31) and (38), we have

$$C_{ijk|h} = \pi_{(ijkh)}(C_{ijk|0n_h}) - \pi_{(ijkh)}(C_{ij0|0n_kn_h}) + 2\pi_{(ijkh)}(C_{i00|0n_jn_kn_h}) - 2C_{000|0n_i n_j n_k n_h}$$

**Proposition 4.1.** If \( B_i = n_i \) in (1), the tensor \( A_{ijk} \) must be given by (38).

**Definition 2** Let \( F^n \) be a Finsler space of dimension \( n \geq 3 \), then \( F^n \) is said to be generalized \( C^v \)-reducible Finsler space of type II if the tensor \( C_{ijk|h} \) can be written as (39).

Now, contracting equation (24) by \( n^h \), we get

$$LC_{ijk|0} = H|0m_i m_j m_k - J|0\pi_{(ijk)}(m_i m_j n_k) + I|0\pi_{(ijk)}(m_i n_j n_k) + \frac{(H - I)}{L} \pi_{(ijk)}(m_i m_j n_k)\bar{v}\_0 - \frac{J}{L} \pi_{(ijk)}(l_i m_j m_k) - \frac{I}{L}(l_i m_j n_k + l_j m_i n_i + l_k m_j n_i + l_k m_i n_j + l_k m_i n_j) + \frac{3I}{L} n_i n_j n_k \bar{v}\_0 - \frac{J}{L} \pi_{(ijk)}(l_i n_j n_k)$$

where, \( v_h n^h = \bar{v}_0 \), \( n_i n^i = 1 \), contracting (40) by \( n^k \), \( n^j \) and \( n^i \) in order, we obtain,

$$LC_{ij0|0} = -J|0m_i m_j + I|0(m_i n_j + m_j n_i) + J|0n_i n_j + \frac{(H - I)}{L} m_i m_j \bar{v}\_0 - \frac{I}{L}(l_i m_j + l_j m_i) + \frac{3I}{L} n_i n_j \bar{v}\_0 - \frac{J}{L}(l_i n_j + l_j n_i)$$

$$LC_{i00|0} = I|0m_i + (J|0 + \frac{3I}{L} \bar{v}_0)n_i - \frac{J}{L} l_i$$

$$LC_{000|0} = J|0 + \frac{3I}{L} \bar{v}_0$$

using (43), (42), (41) and (40) in (38), we have

$$A_{ijk} = L[H|0m_i m_j m_k - \frac{1}{2}J|0\pi_{(ijk)}(m_i m_j n_k) + \frac{2}{3}I|0\pi_{(ijk)}(m_i n_j n_k)$$

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\[ + (J|_0 + \frac{3Iv_0}{L})n_i n_j n_k - \frac{J}{L} \pi_{(ijk)}(l_i m_j m_k) - \frac{J}{2L} (l_i m_j n_k + l_j m_i n_k + l_j m_j n_i + l_k m_i n_j) + \frac{1}{3} L \pi_{(ijk)}(l_i n_j n_k) \]

using (44) in (31), we have

\[ LC_{ijk|h} = H|_0 \pi_{(ijkh)}(m_i m_j m_k n_h) - J|_0 \pi_{(ijkh)}(m_i m_j n_k n_h) - 2J|_0 \pi_{(ijkh)}(m_i n_j n_k n_h) + 4(J|_0 + \frac{3Iv_0}{L})n_i n_j n_k n_h - \frac{J}{L} \pi_{(ijkh)}(l_i m_j m_k n_h) - \frac{I}{L} \pi_{(ijkh)}(l_i m_j n_k n_h) - \frac{J}{L} \pi_{(ijkh)}(l_i n_j n_k n_h) \]

Thus, using (45) and (24), we have

**Theorem 4.1.** A three-dimensional Finsler space is said to be generalized $C^v$-reducible Finsler space of type (II) if the relations

\[ J|_0 + \frac{3Iv_0}{L} = 0, \quad J|_0 - \frac{(H-L)v_0}{L} = 0, \quad H|_0 = 0 \]

holds good.

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**References**

