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New INTEGRAL INEQUALITIES FOR PRODUCTS OF SIMILAR s-CONVEX FUNCTIONS IN THE FIRST SENSE

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Abstract: In this article we define the class of s^*-convex functions in the first sense and establish new integral inequalities for products of differentiable s^*-convex functions in the first sense.

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1. Introduction

Let us denote by $A(f;a,b)$, $A(a,b)$ and $B(f;a,b;s)$ the integral arithematic mean of f on $[a,b]$, the arithematic mean of a and b and the generalized mean of f on $[a,b]$, respectively, given by

$$A(f;a,b) = \frac{1}{b-a}\int_a^b f(z)dz, \quad A(a,b) = \frac{a+b}{2},$$
$$B(f;a,b;s) = \frac{f(a)+sf(b)}{s+1}.$$

The following double inequality is well-known in the literature as Hadamard's

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inequality for convex mappings [8, 10]: Let $f : I \subseteq R \to R$ be a convex mapping defined on the interval I in R and $a, b \in I$ with $a < b$. Then the following inequality holds:

$$f(A(a,b)) \leq A(f;a,b) \leq A(f(a), f(b)).$$

Here let us define the class of nonnegative similar s-convex mappings in the first sense:

Definition 1.1. A mapping $f : I \subseteq R^+ = [0, \infty) \to R$ is said to be similar s-convex in the first sense if the inequality

$$f(tx + (1-t)y) \leq t^s f(x) + (1 - t^s) f(y)$$

holds for all $x, y \in I$ and for some fixed $s \in (0, 1]$.

We denote the class of these similar s-convex mappings in the first sense by $K^1_{s^*}(I)$.

Definition 1.2. A mapping $f : I \subseteq R^+ \to R$ is said to be s-convex in the second sense if the inequality

$$f(tx + (1-t)y) \leq t^s f(x) + (1 - t)^s f(y)$$

holds for all $x, y \in I$ and for some fixed $s \in (0, 1]$.

We denote the class of these s-convex mappings in the second sense by $K^2_s(I)$. For the elementary properties, refinements and generalizations for these class see [1, 2, 5, 7, 9].

In [4], Orlicz introduced two definitions of s-convexity of real valued mappings. These definitions of s-convexity in the second sense was used in the theory of Orlicz spaces.

In [2], Dragomir and Fitzpatrick proved a variant of Hadamard's inequality which holds for s-convex mapping in the first and in the second sense:

Theorem 1.1. *Suppose that $f : I \subseteq R^+ \to R$ is in $K^1_s(I)$, where $s \in (0, 1)$ and let $a, b \in I$ with $a < b$. If f is in $L^1([0, 1])$, then the following double inequality holds:*

$$f(A(a,b)) \leq A(f;a,b) \leq B(f;a,b;s).$$

Theorem 1.2. *Suppose that $f : I \subseteq R^+ \to R$ is in $K^2_s(I)$, where $s \in (0, 1)$ and let $a, b \in I$ with $a < b$. If f is in $L^1([0, 1])$, then the following double inequality holds:*

$$2^{s-1} f(A(a,b)) \leq A(f;a,b) \leq \frac{2}{s+1} A(f(a), f(b)).$$

In [9], Mevlüt Tunç proved variants of Hadamard's inequality for s-convex mappings in the second sense.

Theorem 1.3. *Let $f, g : I \subset [0, b^*] \to R$ be nonnegative and s-convex mappings in the second sense, where $b^* > 0$ and $a, b \in I$ with $a < b$. If $f \in K^2_{s_1}([a, b])$ and $g \in K^2_{s_2}([a, b])$ for all $x, y \in [a, b]$ and some $t \in [0, 1]$ and some fixed $s_1, s_2 \in (0, 1]$, then the following inequality holds:*

$$\int_a^b \int_a^b A(fg; x, y)dydx \\ \leq \frac{2(b-a)}{s_1+s_2+1} A(fg; a, b) + \frac{\Gamma(s_1+1)\Gamma(s_2+1)}{2\Gamma(s_1+s_2+1)}[M(a, b) + N(a, b)],$$

where $M(a, b) = f(a)g(a) + f(b)g(b)$ and $N(a, b) = f(a)g(b) + f(b)g(a)$.

Theorem 1.4. *Let $f, g : I \subset [0, b^*] \to R$ be nonnegative and s-convex mappings in the second sense, where $b^* > 0$ and $a, b \in I$ with $a < b$. If $f \in K^2_{s_1}([a, b])$ and $g \in K^2_{s_2}([a, b])$ for all $y \in [a, b]$ and some fixed $s_1, s_2 \in (0, 1]$, then the following inequality holds:*

$$\frac{1}{b-a}\int_a^b A(fg; A(a, b), y)dy \\ \leq \frac{1}{s_1+s_2+1} A(fg; a, b) \\ + \left\{\frac{1}{4(s_1+s_2+1)} + \frac{\Gamma(s_1+1)\Gamma(s_2+1)}{2\Gamma(s_1+s_2+1)}\right\}[M(a, b) + N(a, b)],$$

where $M(a, b)$ and $N(a, b)$ are as defined in Theorem 1.3.

The main purpose of this article is to establish new integral inequalities like those given in Theorem 1.3 and Theorem 1.4 but now for the class of s-convex mappings in the first sense.

2. Main results

We need the following lemma which deals with the simple characterization of s-convex mappings:

Lemma 1. *For a mapping $f : [a, b] \to R$ the following statements are equivalent;*

(i) f is similar s-convex in the first sense on $[a, b]$,

(ii) For all x, y in $[a, b]$, the mapping $g : [0, 1] \to R$, defined by $g(t) = f(tx + (1-t)y)$, is similar s-convex in the first sense on $[0, 1]$.

The above lemma is proved by the similar way as in [8].

Theorem 2.1. *Let $f, g : I \subset [0, b^*] \to R$ be nonnegative mappings, where $b^* > 0$ and $a, b \in I$ with $a < b$. If $f \in K^1_{s_1^*}([a, b])$ and $g \in K^1_{s_2^*}([a, b])$ for all $x, y \in [a, b]$ and some fixed $s_1, s_2 \in (0, 1]$, then the following inequality holds:*

$$\begin{aligned}
&\frac{1}{b-a}\int_a^b\int_a^b A(fg; x, y)dydx \\
&\leq \Big\{1 - R_1(s_1, s_2)\Big\}(b-a)A(fg; a, b) \\
&\quad + R_1(s_1, s_2)(b-a)B(f; a, b; s_1)B(g; a, b; s_2) \qquad (1)
\end{aligned}$$

where

$$R_1(s_1, s_2) = \frac{1}{s_1+1} + \frac{1}{s_2+1} - \frac{2}{s_1+s_2+1}.$$

Proof. Since $f \in K^1_{s_1^*}([a, b])$ and $g \in K^1_{s_2^*}([a, b])$ for all $x, y \in [a, b]$ and some fixed $s_1, s_2 \in (0, 1]$, we have

$$\begin{aligned}
f(tx + (1-t)y) &\leq t^{s_1}f(x) + (1 - t^{s_1})f(y) \\
g(tx + (1-t)y) &\leq t^{s_2}g(x) + (1 - t^{s_2})g(y)
\end{aligned}$$

for all $t \in [0, 1]$.

Since f and g are nonnegative, we have

$$\begin{aligned}
&f(tx + (1-t)y)g(tx + (1-t)y) \\
&\leq t^{s_1+s_2}f(x)g(x) + t^{s_1}(1 - t^{s_2})f(x)g(y) \\
&\quad + t^{s_2}(1 - t^{s_1})f(y)g(x) + (1 - t^{s_1})(1 - t^{s_2})f(y)g(y). \qquad (2)
\end{aligned}$$

Integrating both sides of the above inequalities (2) over t on $[0, 1]$ we obtain

$$\begin{aligned}
&\int_0^1 f(tx + (1-t)y)g(tx + (1-t)y)dt \\
&\leq \Big\{\frac{1}{s_1+s_2+1}\Big\}f(x)g(x) + \Big\{\frac{s_2}{(s_1+1)(s_1+s_2+1)}\Big\}f(x)g(y)
\end{aligned}$$

$$\begin{aligned}
&+ \Big\{\frac{s_1}{(s_2+1)(s_1+s_2+1)}\Big\}f(y)g(x) \\
&+ \Big\{\frac{s_1+s_2+2}{s_1+s_2+1} - \frac{s_1+s_2+2}{(s_1+1)(s_2+1)}\Big\}f(y)g(y) \qquad (3)
\end{aligned}$$

Integrating both sides of the above inequalities (3) over (x, y) on $[a, b] \times [a, b]$ we obtain

$$\begin{aligned}&\frac{1}{b-a}\int_a^b\int_a^b A(fg; x, y)dydx\\&\leq \Big\{1 - R_1(s_1, s_2)\Big\}(b-a)A(fg; a, b)\\&\quad + R_1(s_1, s_2)(b-a)^2 B(f; a, b; s_1)B(g; a, b; s_2),\end{aligned}$$

which completes the proof.

Remark 1. If we choose $s_1 = s_2 = 1$ in Theorem 2.1, then the inequality (1) reducded to the following inequality:

$$\begin{aligned}&\frac{1}{b-a}\int_a^b\int_a^b A(fg; x, y)dydx\\&\leq \frac{2}{3}\int_a^b f(x)g(x)dx + \frac{1}{12}(b-a)\\&\times\Big[\big\{f(a)g(a) + f(b)g(b)\big\} + \big\{f(a)g(b) + f(b)g(a)\big\}\Big].\end{aligned}$$

Theorem 2.2. *Let $f, g : I \subset [0, b^*] \to R$ be nonnegative mappings, where $b^* > 0$ and $a, b \in I$ with $a < b$. If $f \in K^1_{s_1^*}([a, b])$ and $g \in K^1_{s_2^*}([a, b])$ for all $x \in [a, b]$ and some fixed $s_1, s_2 \in (0, 1]$, then the following inequality holds:*

$$\begin{aligned}&\frac{1}{b-a}\int_a^b A(fg; x, A(a, b))dx\\&\leq \Big\{\frac{1}{s_1+s_2+1}\Big\}A(fg; a, b)\\&\quad + \Big\{\frac{s_1+s_2}{s_1+s_2+1}\Big\}B(f; a, b; s_1)B(g; a, b; s_2).\end{aligned} \tag{4}$$

Proof. Since $f \in K^1_{s_1^*}([a, b])$ and $g \in K^1_{s_2^*}([a, b])$ for some fixed $s_1, s_2 \in (0, 1]$, we have

$$\begin{aligned}f(tx + (1-t)A(a, b)) &\leq t^{s_1} f(x) + (1 - t^{s_1})f(A(a, b))\\g(tx + (1-t)A(a, b)) &\leq t^{s_2} g(x) + (1 - t^{s_2})g(A(a, b))\end{aligned}$$

for all $x \in [a, b]$ and all $t \in [0, 1]$ and also since f and g are nonnegative, we have

$$f(tx + (1-t)A(a, b))g(tx + (1-t)A(a, b))$$

$$\leq t^{s_1+s_2} f(x)g(x) + t^{s_1}(1-t^{s_2}) f(x)g(A(a,b)) + t^{s_2}(1-t^{s_1}) f(A(a,b))g(x) + (1-t^{s_1})(1-t^{s_2}) f(A(a,b))g(A(a,b)). \tag{5}$$

Integrating both sides of the above inequalities (5) over t on $[0,1]$ we obtain

$$A(fg; x, A(a,b)) \leq \Big\{\frac{1}{s_1+s_2+1}\Big\} f(x)g(x) + \Big\{\frac{s_2}{(s_1+1)(s_1+s_2+1)}\Big\} f(x)g(A(a,b)) + \Big\{\frac{s_1}{(s_2+1)(s_1+s_2+1)}\Big\} f(A(a,b))g(x) + \Big\{\frac{s_1+s_2+2}{s_1+s_2+1} - \frac{s_1+s_2+2}{(s_1+1)(s_2+1)}\Big\} f(A(a,b))g(A(a,b)). \tag{6}$$

As in the proof of Lemma 1 and Theorem 2.1 given above, also the product fg is integrable on $[a,b]$.

Now by integrating both sides of the above inequalities (6) on $[a,b]$ and by using the right half of the Hadamard's inequality and the similar s-convexity of f and g in the first sense, we observe that

$$\int_a^b A(fg; x, A(a,b))dx \leq \Big\{\frac{1}{s_1+s_2+1}\Big\}(b-a)A(fg;a,b) + \Big\{\frac{s_2}{s_1+s_2+1}\Big\}(b-a) f(A(a,b))g(A(a,b)) + \Big\{\frac{s_1}{(s_2+1)(s_1+s_2+1)}\Big\} f(A(a,b))(b-a)B(g;a,b;s_2) + \Big\{\frac{s_1+s_2+2}{s_1+s_2+1} - \frac{s_1+s_2+2}{(s_1+1)(s_2+1)}\Big\} f(A(a,b))g(A(a,b))(b-a). \tag{7}$$

By Theorem 1.1 and the inequality (7), we have

$$\int_a^b A(fg; x, A(a,b))dx \leq \Big\{\frac{1}{s_1+s_2+1}\Big\}(b-a)A(fg;a,b) + (b-a)\Big\{\frac{s_1+s_2}{s_1+s_2+1}\Big\} B(f;a,b;s_1)B(g;a,b;s_2),$$

which completes the proof.

Theorem 2.3. *Let $f, g : I \subset [0, b^*] \to R$ be nonnegative mappings, where $b^* > 0$ and $a, b \in I$ with $a < b$. If $f \in K^1_{s_1^*}([a, b])$ and $g \in K^1_{s_2^*}([a, b])$ for all $y \in [a, b]$ and some fixed $s_1, s_2 \in (0, 1]$, then the following inequality holds:*

$$\begin{aligned}&\int_a^b A(fg; A(a,b), y)dy\\&\le \Big\{1 - R_2(s_1, s_2)\Big\}(b-a)A(fg; a, b)\\&\quad + R_2(s_1, s_2)(b-a)\Big\{\frac{g(a) + s_2 g(b)}{s_2 + 1}\Big\}\Big\{\frac{f(a) + s_1 f(b)}{s_1 + 1}\Big\},\end{aligned}$$

where

$$R_2(s_1, s_2) = \frac{s_1 + s_2 + 2}{(s_1 + 1)(s_2 + 1)} - \frac{1}{s_1 + s_2 + 1}.$$

Proof. Since $f \in K^1_{s_1^*}([a, b])$ and $g \in K^1_{s_2^*}([a, b])$ for all $y \in [a, b]$ and some fixed $s_1, s_2 \in (0, 1]$, and also f and g are nonnegative, we have

$$\begin{aligned}&f(t(A(a,b)) + (1-t)y)g(t(A(a,b)) + (1-t)y)\\&\le t^{s_1+s_2} f(A(a,b))g(A(a,b)) + t^{s_1}(1 - t^{s_2})f(A(a,b))g(y)\\&\quad + t^{s_2}(1 - t^{s_1})f(y)g(A(a,b)) + (1 - t^{s_1})(1 - t^{s_2})f(y)g(y). \qquad (8)\end{aligned}$$

Integrating both sides of the above inequalities (8) over t on $[0, 1]$, we obtain

$$\begin{aligned}&A(fg; A(a,b), y)\\&\le \Big\{\frac{1}{s_1 + s_2 + 1}\Big\}f(A(a,b))g(A(a,b))\\&\quad + \Big\{\frac{s_2}{(s_1 + 1)(s_1 + s_2 + 1)}\Big\}f(A(a,b))g(y)\\&\quad + \Big\{\frac{s_1}{(s_2 + 1)(s_1 + s_2 + 1)}\Big\}f(y)g(A(a,b))\\&\quad + \Big\{\frac{s_1 + s_2 + 2}{s_1 + s_2 + 1} - \frac{s_1 + s_2 + 2}{(s_1 + 1)(s_2 + 1)}\Big\}f(y)g(y). \qquad (9)\end{aligned}$$

As in the proof of Lemma 1 and Theorem 2.1 given above, the product fg is integrable on $[a, b]$.

Now by integrating both sides of the above inequalities (9) on $[a, b]$ and by using the right half of the Hadamard's inequality and the similar s-convexity of f and g in the first sense, we observe that

$$\int_a^b A(fg; A(a,b), y)dy$$

$$
\begin{aligned}
&= \int_a^b \int_0^1 f(t(A(a,b)) + (1-t)y)g(t(A(a,b)) + (1-t)y)dtdy \\
&\leq \Big\{\frac{s_1+s_2+2}{s_1+s_2+1} - \frac{s_1+s_2+2}{(s_1+1)(s_2+1)}\Big\}(b-a)A(fg;a,b) \\
&\quad + \Big\{\frac{1}{s_1+s_2+1}\Big\}(b-a)f(A(a,b))g(A(a,b)) \\
&\quad + \Big\{\frac{s_2}{(s_1+1)(s_1+s_2+1)}\Big\}(b-a)f(A(a,b))A(g;a,b) \\
&\quad + \Big\{\frac{s_1}{(s_2+1)(s_1+s_2+1)}\Big\}(b-a)A(f;a,b)g(A(a,b)) \\
&\leq \Big\{1 - R_2(s_1,s_2)\Big\}(b-a)A(fg;a,b) \\
&\quad + R_2(s_1,s_2)(b-a)B(f;a,b;s_1)B(g;a,b;s_2), \qquad (10)
\end{aligned}
$$

which completes the proof.

Theorem 2.4. *Let $f, g : I \subset [0, b^*] \to R$ be nonnegative mappings, where $b^* > 0$ and $a, b \in I$ with $a < b$. If $f \in K^1_{s_1^*}([a,b])$ and $g \in K^1_{s_2^*}([a,b])$ for all $x \in [a,b]$ and some fixed $s_1, s_2 \in (0,1]$, then the following inequality holds:*

$$
\begin{aligned}
&f(A(a,b))A(g;a,b) + g(A(a,b))A(f;a,b)\Big\} \\
&\leq \frac{(2^{s_1}-1)(2^{s_2}-1)+1}{2^{s_1+s_2}}A(fg;a,b)
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{1}{2^{s_1+s_2}}\Big[\Big\{\frac{s_1(2^{s_1}-1)}{(s_2+1)(s_1+s_2+1)} + \frac{s_2(2^{s_2}-1)}{(s_1+1)(s_1+s_2+1)}\Big\}f(a)g(a) \\
&+ \Big\{\frac{2^{s_2}-1}{s_1+s_2+1} + (2^{s_1}-1)\Big(\frac{s_1+s_2+2}{s_1+s_2+1} - \frac{s_1+s_2+2}{(s_1+1)(s_2+1)}\Big)\Big\}f(a)g(b) \\
&+ \Big\{\frac{2^{s_1}-1}{s_1+s_2+1} + (2^{s_2}-1)\Big(\frac{s_1+s_2+2}{s_1+s_2+1} - \frac{s_1+s_2+2}{(s_1+1)(s_2+1)}\Big)\Big\}f(b)g(a) \\
&+ \Big\{\frac{s_2(2^{s_1}-1)}{(s_1+1)(s_1+s_2+1)} + \frac{s_1(2^{s_2}-1)}{(s_2+1)(s_1+s_2+1)}\Big\}f(b)g(b)\Big] \\
&+ fg(A(a,b)). \qquad (11)
\end{aligned}
$$

Proof. Since $f \in K^1_{s_1^*}([a,b])$ and $g \in K^1_{s_2^*}([a,b])$ for all $t \in [0,1]$, we observe that

$$
f(A(a,b)) \leq \frac{1}{2^{s_1}}\Big\{f\big(ta+(1-t)b\big) + (2^{s_1}-1)f\big((1-t)a+tb\big)\Big\},
$$

$$g(A(a,b)) \le \frac{1}{2^{s_2}}\Big\{g\big(ta+(1-t)b\big)+(2^{s_2}-1)g\big((1-t)a+tb\big)\Big\}.$$

Then by noting that if $l \le m$ and $r \le w$ then the inequality $lw + mr \le lr + mw$ holds for $l, m, r, w \in R$, we have

$$\begin{aligned}
&\frac{1}{2^{s_2}}f(A(a,b))\Big\{g\big(ta+(1-t)b\big)+(2^{s_2}-1)g\big((1-t)a+tb)\big)\Big\}\\
&\quad+\frac{1}{2^{s_1}}g(A(a,b))\Big\{f\big(ta+(1-t)b\big)+(2^{s_1}-1)f\big((1-t)a+tb\big)\Big\}\\
&\le \frac{1}{2^{s_1+s_2}}\Big\{f\big(ta+(1-t)b\big)g\big(ta+(1-t)b\big)\\
&\qquad+(2^{s_1}-1)(2^{s_2}-1)f\big((1-t)a+tb\big)g\big((1-t)a+tb\big)\Big\}\\
&\quad+\frac{2^{s_1}-1}{2^{s_1+s_2}}\Big\{t^{s_1+s_2}f(b)g(a)+t^{s_1}(1-t^{s_2})f(b)g(b)\\
&\qquad+(1-t^{s_1})t^{s_2}f(a)g(a)+(1-t^{s_1})(1-t^{s_2})f(a)g(b)\Big\}\\
&\quad+\frac{2^{s_2}-1}{2^{s_1+s_2}}\Big\{t^{s_1+s_2}f(a)g(b)+t^{s_1}(1-t^{s_2})f(a)g(a)\\
&\qquad+(1-t^{s_1})t^{s_2}f(b)g(b)+(1-t^{s_1})(1-t^{s_2})f(b)g(a)\Big\}\\
&\qquad\quad+f(A(a,b))g(A(a,b)).
\end{aligned}\tag{12}$$

By substituting $ta + (1-t)b = x$ or $(1-t)a + tb = x$ and integrating both sides of the above inequalities (12) over $[0, 1]$, we obtain the inequality (11).

Theorem 2.5. *Let $f, g : I \subset [0, b^*] \to R$ be nonnegative mappings, where $b^* > 0$ and $a, b \in I$ with $a < b$. If $f \in K^1_{s_1^*}([a, b])$ and $g \in K^1_{s_2^*}([a, b])$ for all $x, y \in [a, b]$ and some fixed $s_1, s_2 \in (0, 1]$, then the following inequality holds:*

$$\begin{aligned}
&\int_a^b\int_a^b \frac{1}{(y-x)}\Big[g(x)\int_x^y\Big(\frac{y-z}{y-x}\Big)^{s_2}f(z)dz\\
&\qquad\qquad+g(y)\int_x^y\Big\{1-(\frac{y-z}{y-x})^{s_2}\Big\}f(z)dz\\
&+f(x)\int_x^y\Big(\frac{y-z}{y-x}\Big)^{s_1}g(z)dz\\
&\qquad\qquad+f(y)\int_x^y\Big\{1-(\frac{y-z}{y-x})^{s_1}\Big\}g(z)dz\Big]dydx\\
&\le (b-a)^2B(f;a,b;s_1)B(g;a,b;s_2)\\
&\quad\times\Big\{\frac{s_2}{(s_1+1)(s_1+s_2+1)}+\frac{s_1}{(s_2+1)(s_1+s_2+1)}\Big\}
\end{aligned}$$

$$+\left\{\frac{s_1+s_2+3}{s_1+s_2+1}-\frac{s_1+s_2+2}{(s_1+1)(s_2+1)}\right\}(b-a)^2A(fg;a,b)$$
$$+\int_a^b\int_a^b A(fg;x,y)dydx.$$

Proof. By the facts that $f \in K^1_{s_1^*}([a,b])$ and $g \in K^1_{s_2^*}([a,b])$ for all $x, y \in [a,b]$ and all $t \in [0,1]$ and by using the fact that if $e \le f$ and $p \le r$ then $er + fp \le ep + fr$ for $e, f, p, r \in R$, we get

$$\begin{aligned}
&f(tx+(1-t)y)\left\{t^{s_2}g(x)+(1-t^{s_2})g(y)\right\}\\
&\quad+g(tx+(1-t)y)\left\{t^{s_1}f(x)+(1-t^{s_1})f(y)\right\}\\
&\le\left\{t^{s_2}g(x)+(1-t^{s_2})g(y)\right\}\left\{t^{s_1}f(x)+(1-t^{s_1})f(y)\right\}\\
&\quad+f(tx+(1-t)y)g(tx+(1-t)y)
\end{aligned}$$

and we obtain

$$\begin{aligned}
&g(x)t^{s_2}f(tx+(1-t)y)+g(y)(1-t^{s_2})f(tx+(1-t)y)\\
&\quad+f(x)t^{s_1}g(tx+(1-t)y)+f(y)(1-t^{s_1})g(tx+(1-t)y)\\
&\le f(x)g(x)t^{s_1+s_2}+f(x)g(y)t^{s_1}(1-t^{s_2})\\
&\quad+f(y)g(x)t^{s_2}(1-t^{s_1})+f(y)g(y)(1-t^{s_1})(1-t^{s_2})\\
&\quad+f(tx+(1-t)y)g(tx+(1-t)y). \qquad (13)
\end{aligned}$$

By Lemma 1, $f(tx+(1-t)y)$ and $g(tx+(1-t)y)$ are similar s-convex in the first sense on $[0,1]$, which implies that they are integrable on $[0,1]$ and consequently $f(tx+(1-t)y)g(tx+(1-t)y)$ is also integrable on $[0,1]$. Similarly, since f and g are integrable on $[a,b]$, the product fg is integrable on $[a,b]$. Integrating both sides of the above inequality (13) over $[0,1]$, we get

$$\begin{aligned}
&\int_0^1\left\{g(x)t^{s_2}+g(y)(1-t^{s_2})\right\}f(tx+(1-t)y)dt\\
&\quad+\int_0^1\left\{f(x)t^{s_1}+f(y)(1-t^{s_1})\right\}g(tx+(1-t)y)dt\\
&\le f(x)g(x)\int_0^1 t^{s_1+s_2}dt+f(x)g(y)\int_0^1 t^{s_1}(1-t^{s_2})dt\\
&\quad+f(y)g(x)\int_0^1 t^{s_2}(1-t^{s_1})dt+f(y)g(y)\int_0^1(1-t^{s_1})(1-t^{s_2})dt\\
&\quad+\int_0^1 f(tx+(1-t)y)g(tx+(1-t)y)dt. \qquad (14)
\end{aligned}$$

By (14), we have

$$
\begin{aligned}
&\frac{1}{(y-x)^{s_2+1}}\Big[g(x)\int_x^y (y-z)^{s_2} f(z)dz \\
&\qquad\qquad + g(y)\int_x^y \big\{(y-x)^{s_2}-(y-z)^{s_2}\big\}f(z)dz\Big] \\
&+\frac{1}{(y-x)^{s_1+1}}\Big[f(x)\int_x^y (y-z)^{s_1} g(z)dz \\
&\qquad\qquad + f(y)\int_x^y \big\{(y-x)^{s_1}-(y-z)^{s_1}\big\}g(z)dz\Big] \\
&\le f(x)g(x)\Big\{\frac{1}{s_1+s_2+1}\Big\} \\
&\quad + f(y)g(y)\Big\{\frac{s_1+s_2+2}{s_1+s_2+1}-\frac{s_1+s_2+2}{(s_1+1)(s_2+1)}\Big\} \\
&\quad + f(x)g(y)\Big\{\frac{s_2}{(s_1+1)(s_1+s_2+1)}\Big\} \\
&\quad + f(y)g(x)\Big\{\frac{s_1}{(s_2+1)(s_1+s_2+1)}\Big\} \\
&\quad + A(fg;x,y). \qquad (15)
\end{aligned}
$$

By integrating both sides of the above inequalities (15) on $[a,b]\times[a,b]$ and using Theorem 1.1 we obtain

$$
\begin{aligned}
&\int_a^b\int_a^b \frac{1}{(y-x)}\Big[g(x)\int_x^y \Big(\frac{y-z}{y-x}\Big)^{s_2} f(z)dz \\
&\qquad\qquad + g(y)\int_x^y \Big\{1-(\frac{y-z}{y-x})^{s_2}\Big\}f(z)dz \\
&\quad + f(x)\int_x^y \Big(\frac{y-z}{y-x}\Big)^{s_1} g(z)dz \\
&\qquad\qquad + f(y)\int_x^y \Big\{1-(\frac{y-z}{y-x})^{s_1}\Big\}g(z)dz\Big]dydx \\
&\le (b-a)^2 B(f;a,b;s_1)B(g;a,b;s_2) \\
&\quad \times\Big\{\frac{s_2}{(s_1+1)(s_1+s_2+1)}+\frac{s_1}{(s_2+1)(s_1+s_2+1)}\Big\} \\
&\quad + \int_a^b f(x)g(x)dx\Big\{\frac{s_1+s_2+2}{s_1+s_2+1}-\frac{s_1+s_2+2}{(s_1+1)(s_2+1)}\Big\}(b-a) \\
&\quad + \int_a^b\int_a^b A(fg;x,y)dydx,
\end{aligned}
$$

which completes the proof.

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