International Journal of Pure and Applied Mathematics
Volume 80 No. 4 2012, 597-608
ISSN: 1311-8080 (printed version)
url: http://www.ijpam.eu

ijpam.eu

SEMIGROUPS OF INJECTIVE PARTIAL LINEAR TRANSFORMATIONS WITH RESTRICTED RANGE: GREEN'S RELATIONS AND PARTIAL ORDERS

Kritsada Sangkhanan[1], Jintana Sanwong[2 §]

Department of Mathematics
Faculty of Science
Chiang Mai University
Chiangmai 50200, THAILAND

Abstract: Let V be any vector space and $I(V)$ the set of all partial injective linear transformations defined on V, that is, all injective linear transformations $\alpha : A \to B$ where A, B are subspaces of V. Then $I(V)$ is a semigroup under composition. Let W be a subspace of V. Define $I(V, W) = \{\alpha \in I(V) : V\alpha \subseteq W\}$. So $I(V, W)$ is a subsemigroup of $I(V)$. In this paper, we present the largest regular subsemigroup of $I(V, W)$ and determine its Green's relations. Furthermore, we study the natural partial order $\leq$ on $I(V, W)$ in terms of domains and images, compare $\leq$ with the subset order and find elements of $I(V, W)$ which are compatible.

AMS Subject Classification: 20M20
Key Words: regular elements, Green's relations, injective partial linear transformation semigroups, natural order, compatibility

1. Introduction

The partial transformation semigroup on the set X, denoted $P(X)$, is the set of all functions from a subset of X into X, with the operation of composition. In addition, the semigroups $T(X)$ and $I(X)$ are defined by:

Received: August 27, 2012

§Correspondence author

$$\begin{aligned} T(X) &= \{\alpha \in P(X) : \text{dom } \alpha = X\}, \\ I(X) &= \{\alpha \in P(X) : \alpha \text{ is injective}\}. \end{aligned}$$

$T(X)$ and $I(X)$ are called the full transformation semigroup and the symmetric inverse semigroup, respectively. It is well-known that $P(X)$, $T(X)$ are regular and $I(X)$ is an inverse semigroup.

Now, we introduce the partial transformation semigroup with restricted range. Let Y be a subset of X. We consider the semigroup $PT(X,Y)$, $T(X,Y)$ and $I(X,Y)$ defined by $PT(X,Y) = \{\alpha \in P(X) : X\alpha \subseteq Y\}$, $T(X,Y) = T(X) \cap PT(X,Y)$ and $I(X,Y) = I(X) \cap PT(X,Y)$. Clearly, $PT(X,X) = P(X)$, $T(X,X) = T(X)$, $I(X,X) = I(X)$ and $PT(X,\emptyset) = \{\emptyset\} = I(X,\emptyset)$.

In 2008, Sanwong and Sommanee [7] obtained the largest regular subsemigroup of $T(X,Y)$ and a class of its maximal inverse subsemigroups. Further, they characterized Green's relations on $T(X,Y)$. In [2], Fernandes and Sanwong proved that $PF = \{\alpha \in PT(X,Y) : X\alpha = Y\alpha\}$ and $I(Y)$ are the largest regular subsemigroups of $PT(X,Y)$ and $I(X,Y)$, respectively. Moreover, they determined Green's relations on $PT(X,Y)$ and $I(X,Y)$.

Analogously to $P(X)$, we can define a partial linear transformation on some vector spaces. Let V be any vector space, $P(V)$ the set of all linear transformations $\alpha : S \to T$ where S and T are subspaces of V, that is, every element $\alpha \in P(V)$, the domain and range of α are subspaces of V. Then we have $P(V)$ under composition is a semigroup and it is called the partial linear transformation semigroup of V. The full linear transformation semigroup, $T(V)$, and the injective linear transformation semigroup, $I(V)$ are defined as follows.

$$\begin{aligned} T(V) &= \{\alpha \in P(V) : \text{dom } \alpha = V\}, \\ I(V) &= \{\alpha \in P(V) : \alpha \text{ is injective}\}. \end{aligned}$$

Similarly, the linear transformation semigroups with restricted range can be defined as follows. For any vector space V and a subspace W of V, let $PT(V,W) = \{\alpha \in P(V) : V\alpha \subseteq W\}$, $T(V,W) = T(V) \cap PT(V,W)$ and $I(V,W) = I(V) \cap PT(V,W)$. Obviously, $PT(V,V) = P(V)$, $T(V,V) = T(V)$ and $I(V,V) = I(V)$. Hence we may regard $PT(V,W)$, $T(V,W)$ and $I(V,W)$ as generalizations of $P(V)$, $T(V)$ and $I(V)$, respectively.

Now, we deal with a natural partial order or Mitsch order [4] on any semigroup S defined by for $a, b \in S$,

$$a \leq b \text{ if and only if } a = xb = by,\ xa = a \text{ for some } x, y \in S^1.$$

Recently, Sangkhanan and Sanwong [5] characterized the natural partial order $\leq$ and the subset order $\subseteq$ on $PT(X,Y)$, described the meet and join of $\leq$

and $\subseteq$. They also compared $\leq$, $\subseteq$ with other partial orders and found elements of $PT(X,Y)$ which are compatible with $\leq$.

In 2005, Sullivan [8] studied the natural partial order $\leq$ and the subset order $\subseteq$ on $P(V)$. The author determined the meet and join of these two partial orders and also found all elements of $P(V)$ which are compatible with respect to $\leq$. In [6], the authors presented the largest regular subsemigroup and determine Green's relations on $PT(V,W)$. Furthermore, they studied the natural partial order $\leq$ on $PT(V,W)$ in terms of domains and images, compared $\leq$ with the subset order, characterized the meet and join of these two orders, and found elements of $PT(V,W)$ which are compatible.

In this paper, we describe the largest regular subsemigroup of $I(V,W)$ and characterized its Green's relations. Furthermore, we study the natural partial order $\leq$ on $I(V,W)$ in terms of domains and images, compare $\leq$ with the subset order and characterize elements of $I(V,W)$ which are compatible.

2. Regularity and Green's Relations on $I(V,W)$

We begin this section with the following simple result on $I(V,W)$ which will be used through out the paper. Here $V\alpha = \{v\alpha : v \in V \cap \text{dom } \alpha\}$.

Lemma 1. *If S and T are subspaces of V with $S \subseteq T$, then $S\alpha \subseteq T\alpha$ for all $\alpha \in I(V,W)$.*

For convenience, we adopt the convention used in [1] namely, if $\alpha \in P(X)$ then we write
$$\alpha = \begin{pmatrix} X_i \\ a_i \end{pmatrix}.$$
and take as understood that the subscript i belongs to some (unmentioned) index set I, the abbreviation $\{a_i\}$ denotes $\{a_i : i \in I\}$, and that $X\alpha = \{a_i\}$ and $a_i\alpha^{-1} = X_i$.

Similarly, we can use this notation for elements in $I(V)$. To define a linear transformation $\alpha \in I(V)$, we first choose a basis $\{e_i\}$ for a subspace of V and a subset $\{a_i\}$ of V, and then let $e_i\alpha = a_i$ for each $i \in I$ and extend this map linearly to V . To shorten this process, we simply say, given $\{e_i\}$ and $\{a_i\}$ within the context, then for each $\alpha \in I(V)$, we can write
$$\alpha = \begin{pmatrix} e_i \\ a_i \end{pmatrix}.$$
From this notation, it is easy to verify that $\{a_i\}$ is also linearly independent since $\ker \alpha = \langle 0 \rangle$.

A subspace U of V which is generated by a linearly independent subset $\{e_i\}$ of V is denoted by $\langle e_i \rangle$ and when we write $U = \langle e_i \rangle$, we mean that the set $\{e_i\}$ is a basis of U, and we have $\dim U = |I|$. For each $\alpha \in I(V)$, the kernel and the range of α are denoted by $\ker \alpha$ and $V\alpha$, respectively. If we write $U\alpha = \langle u_i\alpha \rangle$, it means that $u_i \in U \cap \text{dom } \alpha$ for all i. In addition, we can show that $\{u_i\}$ is linearly independent.

It is well-known that the injective linear transformation semigroup, $I(V)$, is regular but the following example shows that $I(V, W)$ is not regular when $V \neq W$.

Example. Let $W = \langle v_1 \rangle$ and $V = \langle v_1, v_2 \rangle$ and define $\alpha = \binom{v_2}{v_1}$. Since $\text{dom } \alpha = \langle v_2 \rangle$ and $v_2 \notin W$, we obtain for each $\beta \in I(V, W)$ $\text{dom } \beta\alpha = (\text{im } \beta \cap \text{dom } \alpha)\beta^{-1} = \langle 0 \rangle$ which implies that $\beta\alpha = \binom{0}{0}$. Thus $\alpha\beta\alpha = \binom{0}{0} \neq \alpha$. Therefore α is not regular.

Theorem 2. *Let $\alpha \in I(V, W)$. Then α is regular if and only if $\alpha \in I(W)$. Consequently, $I(W)$ is the largest regular subsemigroup of $I(V, W)$.*

Proof. Let α be regular. Then there is $\beta \in I(V, W)$ such that $\alpha = \alpha\beta\alpha$. Suppose that $\text{dom } \alpha = \langle v_i \rangle$ and $\alpha = \binom{v_i}{w_i}$. For each i, we obtain $v_i\alpha = v_i\alpha\beta\alpha = w_i\beta\alpha$ which implies that $v_i = w_i\beta \in W$ since α is injective. Hence $\text{dom } \alpha \subseteq W$. Conversely, let $\alpha \in I(W)$. Then we can write $\alpha = \binom{u_i}{w_i}$ where $\langle u_i \rangle$ and $\langle w_i \rangle$ are subspaces of W. Define $\beta = \binom{w_i}{u_i}$, we obtain $\alpha = \alpha\beta\alpha$, as required. $\square$

Lemma 3. *Let $\alpha, \beta \in I(V, W)$. Then $V\alpha \subseteq W\beta$ if and only if $\alpha = \gamma\beta$ for some $\gamma \in I(V, W)$.*

Proof. Let $V\alpha \subseteq W\beta$. We can write

$$\alpha = \begin{pmatrix} v_i \\ w_i \end{pmatrix}, \beta = \begin{pmatrix} w_i' & v_j \\ w_i & w_j \end{pmatrix}$$

where $\langle w_i' \rangle \subseteq W$. Let $\gamma = \binom{v_i}{w_i'}$. It is a routine matter to show that $\ker \gamma = \langle 0 \rangle$. Hence, we have $\gamma \in I(V, W)$ and $\alpha = \gamma\beta$. The converse is clear. $\square$

By the above lemma, we obtain the following result immediately.

Lemma 4. *Let $\alpha \in I(V, W)$. If $\beta \in I(W)$, then $V\alpha \subseteq V\beta$ if and only if $\alpha = \gamma\beta$ for some $\gamma \in I(V, W)$.*

Now, we characterize Green's relations on $I(V, W)$ as follows.

Theorem 5. *Let $\alpha, \beta \in I(V, W)$. Then $\alpha\mathcal{L}\beta$ if and only if ($\alpha, \beta \in I(W)$ and $V\alpha = V\beta$) or ($\alpha, \beta \in I(V, W) \setminus I(W)$ and $\alpha = \beta$).*

Proof. Assume that $\alpha\mathcal{L}\beta$. Then $\alpha = \lambda\beta$ and $\beta = \mu\alpha$ for some $\lambda, \mu \in I(V,W)^1$. If $\alpha \in I(W)$ and ($\lambda = 1$ or $\mu = 1$), then $\beta = \alpha \in I(W)$ and so $V\alpha = V\beta$. On the other hand, suppose that $\alpha \in I(W)$ and $\lambda, \mu \in I(V,W)$. Let $v \in \text{dom } \beta$. Then $v\beta = v\mu\alpha = (v\mu\lambda)\beta$ and so $v = v\mu\lambda$ since β is injective. Thus $\text{dom } \beta \subseteq W$, that is, $\beta \in I(W)$. From $\alpha = \lambda\beta$ and $\beta = \mu\alpha$, we obtain $V\alpha \subseteq V\beta \subseteq V\alpha$ by Lemma 4. Now, suppose that $\alpha \in I(V,W) \setminus I(W)$. If $\lambda, \mu \in I(V,W)$, then $v\alpha = v\lambda\beta = v\lambda\mu\alpha$ and thus $v = v\lambda\mu \in W$, for all $v \in \text{dom } \alpha$, since α is injective. Hence $\text{dom } \alpha \subseteq W$, that is $\alpha \in I(W)$, which is a contradiction. Therefore $\lambda = 1$ or $\mu = 1$ and so $\beta = \alpha \in I(V,W) \setminus I(W)$.

The converse is clear by Lemma 4. □

Theorem 6. *Let $\alpha, \beta \in I(V,W)$. Then* $\text{dom } \alpha \subseteq \text{dom } \beta$ *if and only if $\alpha = \beta\gamma$ for some $\gamma \in I(V,W)$. Consequently, $\alpha\mathcal{R}\beta$ in $I(V,W)$ if and only if* $\text{dom } \alpha = \text{dom } \beta$.

Proof. If $\alpha = \beta\gamma$ for some $\gamma \in I(V,W)$, then clearly $\text{dom } \alpha \subseteq \text{dom } \beta$. Conversely, suppose that $\text{dom } \alpha \subseteq \text{dom } \beta$. Then we can write

$$\alpha = \begin{pmatrix} v_i \\ w_i \end{pmatrix}, \beta = \begin{pmatrix} v_i & v_j \\ w'_i & w_j \end{pmatrix}$$

Now, being $\gamma = \binom{w'_i}{w_i}$, we have $\gamma \in I(V,W)$ and $\alpha = \beta\gamma$, as required. □

Corollary 7. *Let $\alpha, \beta \in I(V,W)$. If $\alpha\mathcal{R}\beta$, Then $\alpha, \beta \in I(W)$ or $\alpha, \beta \in I(V,W) \setminus I(W)$.*

Proof. Suppose that $\alpha\mathcal{R}\beta$. If $\alpha \in I(W)$, then $\text{dom } \beta = \text{dom } \alpha \subseteq W$ which implies that $\beta \in I(W)$. On the other hand, if $\alpha \in I(V,W) \setminus I(W)$, then $\text{dom } \beta = \text{dom } \alpha \nsubseteq W$ and so $\beta \notin I(W)$. □

As a direct consequence of Theorem 5, Theorem 6 and Corollary 7, we have the following corollary.

Corollary 8. *Let $\alpha, \beta \in I(V,W)$. Then $\alpha\mathcal{H}\beta$ if and only if ($\alpha, \beta \in I(W)$, $V\alpha = V\beta$ and* $\text{dom } \alpha = \text{dom } \beta$*) or ($\alpha, \beta \in I(V,W) \setminus I(W)$ and $\alpha = \beta$).*

Theorem 9. *Let $\alpha, \beta \in I(V,W)$. Then $\alpha\mathcal{D}\beta$ if and only if ($\alpha, \beta \in I(W)$ and* $\dim(\text{dom } \alpha) = \dim(\text{dom } \beta)$*) or ($\alpha, \beta \in I(V,W) \setminus I(W)$ and* $\text{dom } \alpha = \text{dom } \beta$*).*

Proof. Assume that $\alpha\mathcal{L}\gamma\mathcal{R}\beta$ for some $\gamma \in I(V,W)$. Since $\alpha\mathcal{L}\gamma$, if $\alpha \in I(W)$, then $\gamma \in I(W)$ and $V\alpha = V\gamma$ by Theorem 5. Furthermore, from $\gamma\mathcal{R}\beta$, it follows that $\beta \in I(W)$ and $\text{dom } \gamma = \text{dom } \beta$ by Corollary 7 and Theorem 6. Hence $\dim(\text{dom } \alpha) = \dim(V\alpha) = \dim(V\gamma) = \dim(\text{dom } \gamma) = \dim(\text{dom } \beta)$. On the other hand, if $\alpha \in I(V,W) \setminus I(W)$, then $\gamma \in I(V,W) \setminus I(W)$ and $\alpha = \gamma$ by Theorem 5. It follows that $\beta \in I(V,W) \setminus I(W)$ and dom $\gamma =$ dom β, whence dom $\alpha =$ dom $\gamma =$ dom β.

Conversely, assume that the conditions hold. If $\alpha, \beta \in I(W)$ and dim(dom α) = dim(dom β), then we can write

$$\alpha = \begin{pmatrix} u_i \\ w_i \end{pmatrix} \text{ and } \beta = \begin{pmatrix} v_i \\ w_i' \end{pmatrix}$$

where $\langle u_i \rangle, \langle v_i \rangle, \langle w_i \rangle, \langle w_i' \rangle \subseteq W$. Hence, being $\gamma = \binom{v_i}{w_i} \in I(W)$, we have $V\alpha = V\gamma$ and dom $\gamma =$ dom β which implies that $\alpha \mathcal{L} \gamma \mathcal{R} \beta$. On the other hand, if $\alpha, \beta \in I(V,W) \setminus I(W)$ and dom $\alpha =$ dom β, then $\alpha \mathcal{R} \beta$ and so $\alpha \mathcal{D} \beta$, as required. □

In order to characterize the $\mathcal{J} - relation$ on $I(V,W)$, the following lemma is needed.

Lemma 10. *Let $\alpha, \beta \in I(V,W)$. If $\alpha = \lambda\beta\mu$ for some $\lambda \in I(V,W)$ and $\mu \in I(V,W)^1$, then* $\dim(V\alpha) \leq \dim(W\beta)$.

Proof. Since $V\alpha = (V\lambda)\beta\mu \subseteq W\beta\mu$, we have $\dim(V\alpha) \leq \dim(W\beta\mu)$. Let $W\beta\mu = \langle w_i\mu \rangle$ where $\{w_i\} \subseteq W\beta$. Then $\langle w_i \rangle \subseteq W\beta$ which implies that

$$\dim(W\beta\mu) = \dim\langle w_i\mu \rangle = \dim\langle w_i \rangle \leq \dim(W\beta).$$

Therefore, $\dim(V\alpha) \leq \dim(W\beta)$. □

Theorem 11. *Let $\alpha, \beta \in I(V,W)$. Then $\alpha \mathcal{J} \beta$ if and only if* dom $\alpha =$ dom β *or* $\dim(V\alpha) = \dim(W\alpha) = \dim(W\beta) = \dim(V\beta)$.

Proof. Assume that $\alpha \mathcal{J} \beta$. Then $\alpha = \lambda\beta\mu$ and $\beta = \lambda'\alpha\mu'$ for some $\lambda, \lambda', \mu, \mu' \in I(V,W)^1$. If $\lambda = 1 = \lambda'$, then $\alpha = \beta\mu$ and $\beta = \alpha\mu'$ and so $\alpha \mathcal{R} \beta$. Thus dom $\alpha =$ dom β. If either λ or λ' belongs to $I(V,W)$, then $\alpha = \sigma\beta\delta$ and $\beta = \sigma'\alpha\delta'$ for some $\sigma, \sigma' \in I(V,W)$ and $\delta, \delta' \in I(V,W)^1$. Thus, by Lemma 10, it follows that

$$\dim(W\beta) \geq \dim(V\alpha) \geq \dim(W\alpha) \geq \dim(V\beta) \geq \dim(W\beta).$$

Whence $\dim(V\alpha) = \dim(W\alpha) = \dim(W\beta) = \dim(V\beta)$.

Conversely, assume that the conditions hold. If dom $\alpha =$ dom β, then $\alpha \mathcal{R} \beta$, and so $\alpha \mathcal{J} \beta$. If $\dim(V\alpha) = \dim(W\alpha) = \dim(W\beta) = \dim(V\beta)$, then by using the equality $\dim(V\alpha) = \dim(W\beta)$, we can write

$$\alpha = \begin{pmatrix} u_i \\ w_i \end{pmatrix} \text{ and } \beta = \begin{pmatrix} v_i & v_j \\ w_i' & w_j \end{pmatrix}$$

where $\langle v_i \rangle \subseteq W$ and $v_j \notin W$ for all j. Now, define $\lambda = \binom{u_i}{v_i}$ and $\mu = \binom{w_i'}{w_i}$. Thus $\lambda, \mu \in I(V,W)$ and $\alpha = \lambda\beta\mu$. Similarly, by using the equality $\dim(V\beta) = \dim(W\alpha)$, we can find $\lambda', \mu' \in I(V,W)$ such that $\beta = \lambda'\alpha\mu'$. Therefore, $\alpha\mathcal{J}\beta$, as required. □

Corollary 12. *If $\alpha, \beta \in I(W)$, then $\alpha\mathcal{J}\beta$ on $I(V,W)$ if and only if $\alpha\mathcal{D}\beta$ on $I(V,W)$.*

Proof. In general, we have $\mathcal{D} \subseteq \mathcal{J}$. Let $\alpha, \beta \in I(W)$ and $\alpha\mathcal{J}\beta$ on $I(V,W)$. Then $\text{dom } \alpha = \text{dom } \beta$ or $\dim(V\alpha) = \dim(W\alpha) = \dim(W\beta) = \dim(V\beta)$. If $\text{dom } \alpha = \text{dom } \beta$, then

$$\dim(V\alpha) = \dim(\text{dom } \alpha / \ker \alpha) = \dim(\text{dom } \beta / \ker \beta) = \dim(V\beta).$$

Thus, both cases imply $\dim(\text{dom } \alpha) = \dim(V\alpha) = \dim(V\beta) = \dim(\text{dom } \beta)$ and $\alpha\mathcal{D}\beta$ on $PT(V,W)$ by Theorem 9. □

Theorem 13. *$\mathcal{D} = \mathcal{J}$ on $I(V,W)$ if and only if $\dim W$ is finite or $V = W$.*

Proof. It is clear that if $V = W$, then $I(V,W) = I(V)$ which follows that $\mathcal{D} = \mathcal{J}$ by Corollary 12. Suppose that $\dim W$ is finite. Let $\alpha, \beta \in I(V,W)$ with $\alpha\mathcal{J}\beta$. If $\text{dom } \alpha = \text{dom } \beta$, then $\alpha\mathcal{R}\beta$ and hence $\alpha\mathcal{D}\beta$. Now, assume that $\dim(V\alpha) = \dim(W\alpha) = \dim(W\beta) = \dim(V\beta)$. Since $\dim W$ is finite, we have $\dim(V\alpha), \dim(V\beta)$ are finite, and it follows that $V\alpha = W\alpha$ and $V\beta = W\beta$. Let $v \in \text{dom } \alpha$. Then $v\alpha = w\alpha$ for some $w \in W$ from which it follows that $v = w \in W$ since α is injective, so $\alpha \in I(W)$. Similarly, we obtain $\beta \in I(W)$. In addition, we have $\dim(\text{dom } \alpha) = \dim(V\alpha) = \dim(V\beta) = \dim(\text{dom } \beta)$. Therefore, $\alpha\mathcal{D}\beta$ and the other containment is clear.

Conversely, suppose that $\dim W$ is infinite and $W \subsetneq V$. Let $W = \langle v_i \rangle$ and $V = \langle v_i \rangle \oplus \langle v_j \rangle$. Then there is an infinite countable subset $\{u_n\}$ of $\{v_i\}$ where $n \in \mathbb{N}$. Let $v \in \{v_j\}$ and define α, β by

$$\alpha = \begin{pmatrix} v & u_n \\ u_1 & u_{2n} \end{pmatrix}, \beta = \begin{pmatrix} v & u_{2n} \\ u_1 & u_{4n} \end{pmatrix}.$$

Then $\alpha, \beta \in I(V,W) \setminus I(W)$ and $\dim(V\alpha) = \dim(W\alpha) = \aleph_0 = \dim(W\beta) = \dim(V\beta)$, so $\alpha\mathcal{J}\beta$. Since $\text{dom } \alpha \neq \text{dom } \beta$, we have α and β are not $\mathcal{D}$-related on $I(V,W)$. □

3. Partial Orders

Recall that the natural partial order on any semigroup S is defined by

$$a \leq b \text{ if and only if } a = xb = by, \ xa = a \text{ for some } x, y \in S^1,$$

or equivalently

$$a \leq b \text{ if and only if } a = wb = bz, az = a \text{ for some } w, z \in S^1. \tag{1}$$

Since $I(V, W)$ is not regular in general, we use (1) to define the partial order on the semigroup $I(V, W)$, that is for each $\alpha, \beta \in I(V, W)$

$$\alpha \leq \beta \text{ if and only if } \alpha = \gamma\beta = \beta\mu,\ \alpha = \alpha\mu \text{ for some } \gamma, \mu \in I(V, W)^1.$$

We note that if $W \subsetneq V$, then $I(V, W)$ has no identity elements. So, in this case, $I(V, W)^1 \neq I(V, W)$.

Now, we aim to characterize this partial order on $I(V, W)$ as follows.

Theorem 14. *Let $\alpha, \beta \in I(V, W)$. Then $\alpha \leq \beta$ if and only if $\alpha = \beta$ or the following statements hold.*
(1) $V\alpha \subseteq W\beta$.
(2) $\text{dom } \alpha \subseteq \text{dom } \beta$.
(3) *For each $v \in \text{dom } \beta$, if $v\beta \in V\alpha$, then $v \in \text{dom } \alpha$ and $v\alpha = v\beta$.*

Proof. Suppose that $\alpha \leq \beta$. Then there exist $\gamma, \mu \in I(V, W)^1$ such that $\alpha = \gamma\beta = \beta\mu$ and $\alpha = \alpha\mu$. If $\gamma = 1$ or $\mu = 1$, then $\alpha = \beta$. If $\gamma, \mu \in I(V, W)$, then (1) and (2) hold by Lemma 3 and Theorem 6. If $v \in \text{dom } \beta$ and $v\beta \in V\alpha$, then $v\beta = w\alpha$ for some $w \in V$, thus

$$v\beta = w\alpha = w\alpha\mu = v\beta\mu = v\alpha.$$

Therefore, $v \in \text{dom } \alpha$ and $v\alpha = v\beta$. Conversely, assume that the conditions (1)-(3) hold. Again by Lemma 3 and Theorem 6, there exist $\gamma, \mu \in I(V, W)$ such that $\alpha = \gamma\beta = \beta\mu$. Now, we prove that $V\alpha \subseteq \text{dom } \mu$, by letting $w \in V\alpha$. Then there is $v \in \text{dom } \alpha$ such that $v\alpha = w$. Since $\alpha = \gamma\beta$, we have $w = v\alpha = v\gamma\beta$. By (3), $v\gamma \in \text{dom } \alpha$ and $v\gamma\alpha = v\gamma\beta$. Thus $v\gamma\beta = v\gamma\alpha = v\gamma\beta\mu = w\mu$ which implies that $w \in \text{dom } \mu$. So $V\alpha \subseteq \text{dom } \mu$. Hence

$$\text{dom } \alpha\mu = (\text{im } \alpha \cap \text{dom } \mu)\alpha^{-1} = (\text{im } \alpha)\alpha^{-1} = \text{dom } \alpha.$$

For each $v \in \text{dom } \alpha$, $v\alpha = v\gamma\beta$. Again by (3), $v\gamma \in \text{dom } \alpha$ and $v\gamma\alpha = v\gamma\beta$. Thus

$$v\alpha = v\gamma\beta = v\gamma\alpha = v\gamma\beta\mu = v\alpha\mu.$$

Therefore, $\alpha = \alpha\mu$. □

If we regard $\alpha, \beta \in I(X)$ as subsets of $X \times X$, it is easy to see that

$$\alpha \subseteq \beta \text{ if and only if } \text{dom } \alpha \subseteq \text{dom } \beta \text{ and } x\alpha = x\beta \text{ for all } x \in \text{dom } \alpha.$$

And we also have $\subseteq$ is a partial order on $I(X)$. Since $I(V, W)$ is a subsemigroup of $I(X)$, it is clear that $\subseteq$ is also a partial order on $I(V, W)$.

In [3], we have the natural partial order and the subset order are the same on $I(X)$ but the following example shows that it is false on $I(V, W)$ when $W \subsetneq V$.

Example. Let $V = \langle u, v, w\rangle$ and $W = \langle v, w\rangle$. Define

$$\alpha = \begin{pmatrix} u \\ v \end{pmatrix}, \ \beta = \begin{pmatrix} u & v \\ v & w \end{pmatrix}.$$

Then $\alpha \subseteq \beta$ but $V\alpha = \langle v\rangle \nsubseteq \langle w\rangle = W\beta$. Therefore $\alpha \nleq \beta$.

Theorem 15. *For $\alpha, \beta \in I(V, W)$, $\alpha \leq \beta$ implies $\alpha \subseteq \beta$.*

Proof. Let $\alpha, \beta \in I(V, W)$ be such that $\alpha \leq \beta$. By Theorem 14, we have $\operatorname{dom} \alpha \subseteq \operatorname{dom} \beta$, and for each $v \in \operatorname{dom} \alpha$, $v\alpha \in V\alpha \subseteq W\beta$ which implies that $v\alpha = w\beta$ for some $w \in W$, hence $w\beta \in V\alpha$. Again by Theorem 14, we have $w \in \operatorname{dom} \alpha$ and $w\alpha = w\beta$. So $v\alpha = w\alpha$ and $v = w$ since α is injective. Thus $v\alpha = v\beta$ and therefore $\alpha \subseteq \beta$. □

By the above lemma, we can see that $\leq \subseteq \subseteq$ on $I(V, W)$. So it is clear that the meet and join of these two partial orders are $\leq$ and $\subseteq$, respectively.

To determine when these two relations on $I(V, W)$ are equal, we note that the zero linear transformation is a zero map having domain as $\langle 0\rangle$ and denoted by 0. It is easy to verify that $0 \leq \alpha$, for all $\alpha \in I(V, W)$.

Theorem 16. *On $I(V, W)$, $\subseteq = \leq$ if and only if $V = W$ or $\dim W = 1$.*

Proof. If $V = W$, then $I(V, W) = I(V)$. Let $\alpha, \beta \in I(V)$ be such that $\alpha \subseteq \beta$. Then $V\alpha \subseteq V\beta$ and $\operatorname{dom} \alpha \subseteq \operatorname{dom} \beta$. Let $v \in \operatorname{dom} \beta$ be such that $v\beta \in V\alpha$. We obtain $v\beta = u\alpha$ for some $u \in V$ and then $v\beta = u\alpha = u\beta$ since $\alpha \subseteq \beta$. Thus $u = v$ which implies that $v \in \operatorname{dom} \alpha$ and $v\alpha = v\beta$. Therefore, $\alpha \leq \beta$ by Theorem 14. If $\dim W = 1$, let $W = \langle w\rangle$, thus

$$I(V, W) = \left\{ \begin{pmatrix} v \\ w \end{pmatrix} : v \in V \right\} \cup \{0\}.$$

So we see that for each $\alpha, \beta \in I(V, W)$ with $\alpha \subseteq \beta$, then $\alpha \leq \beta$. It is concluded that $\subseteq = \leq$.

Now, suppose that $\subseteq = \leq$. If $W \subsetneq V$ and $\dim W > 1$, then we can write $W = \langle w_i\rangle$ and $V = \langle w_i\rangle \oplus \langle v_j\rangle$. Choose $v \in \{v_j\}$ and $w, w' \in \{w_i\}$ such that $w \neq w'$. Define

$$\alpha = \begin{pmatrix} v \\ w \end{pmatrix}, \ \beta = \begin{pmatrix} v & w \\ w & w' \end{pmatrix}.$$

Then it is clear that $\alpha \subseteq \beta$. Since $\subseteq = \leq$, we get $\alpha \leq \beta$ from which follows that $\langle w \rangle = V\alpha \subseteq W\beta = \langle w' \rangle$ and this leads to a contradiction. Therefore, $V = W$ or $\dim W = 1$. □

Now, we deal with the compatibility of elements in $I(V, W)$. Let $\preceq$ be a partial order on a semigroup S. An element $c \in S$ is said to be left [right] compatible if $ca \preceq cb$ [$ac \preceq bc$] for each $a, b \in S$ such that $a \preceq b$.

It is easy to see that every element in $I(V, W)$ is left and right compatible with respect to $\subseteq$.

To characterize all elements in $I(V, W)$ those compatible with respect to $\leq$, we first prove the following lemma.

Lemma 17. *Let* $\dim W = 1$ *and* $\alpha, \beta \in I(V, W)$. *If* $\alpha \leq \beta$, *then* $\alpha = \beta$ *or* $\alpha = 0$.

Proof. Suppose that $\alpha \leq \beta$ and $\alpha \neq 0$. So $1 \leq \dim(V\alpha) \leq \dim W = 1$, and that $V\alpha = W$. For each $v \in \operatorname{dom} \beta$, $v\beta \in V\beta \subseteq W = V\alpha$ and hence $v \in \operatorname{dom} \alpha$ and $v\alpha = v\beta$ by Theorem 14(3). Thus $\operatorname{dom} \beta \subseteq \operatorname{dom} \alpha$. Since $\alpha \leq \beta$, we have $\operatorname{dom} \alpha \subseteq \operatorname{dom} \beta$. Therefore, $\operatorname{dom} \alpha = \operatorname{dom} \beta$ which implies that $\alpha = \beta$. □

By Lemma 17, we can show that if $\dim W = 1$, then γ is left and right compatible with respect to $\leq$.

It is clear that for each $\gamma \in I(V, W)$, $\operatorname{dom} \gamma \subseteq W$ if and only if $W\gamma = V\gamma$. We use this fact to prove the following theorem.

Theorem 18. *Let* $\dim W > 1$ *and* $0 \neq \gamma \in I(V, W)$. *Then*

(1) γ *is left compatible with respect to* $\leq$ *if and only if* $\operatorname{dom} \gamma \subseteq W$ *or* $\dim(V\gamma) = 1$.

(2) γ *is always right compatible with respect to* $\leq$.

Proof. (1) Suppose that $\operatorname{dom} \gamma \nsubseteq W$ and $\dim(V\gamma) > 1$. Then $W\gamma \subsetneq V\gamma$. We can write $V\gamma = W\gamma \oplus \langle w_i \rangle$. Choose $w_{i_1} \in \{w_i\}$ and $w \in V\gamma$ such that $\{w_{i_1}, w\}$ is linearly independent. Define $\alpha, \beta \in I(V, W)$ by

$$\alpha = \begin{pmatrix} w_{i_1} \\ w_{i_1} \end{pmatrix}, \beta = \begin{pmatrix} w_{i_1} & w \\ w_{i_1} & w \end{pmatrix}.$$

We can see that $\alpha \leq \beta$. Since $V\gamma\alpha = \langle w_{i_1} \rangle \neq \langle w_{i_1}, w \rangle = V\gamma\beta$, we get $\gamma\alpha \neq \gamma\beta$. Since $w_{i_1} \in V\gamma\alpha$ but $w_{i_1} \notin W\gamma\beta$, we have $V\gamma\alpha \nsubseteq W\gamma\beta$ which follows that $\gamma\alpha \nleq \gamma\beta$. Therefore, γ is not left compatible.

Conversely, let $\alpha, \beta \in I(V, W)$ be such that $\alpha \leq \beta$. Then $\alpha \subseteq \beta$ by Theorem 15. If $\operatorname{dom} \gamma \subseteq W$, then $W\gamma = V\gamma$. So we have $V\gamma\alpha \subseteq V\gamma\beta = W\gamma\beta$. By the left compatibility of γ with respect to $\subseteq$, we obtain $\gamma\alpha \subseteq \gamma\beta$ which implies that dom $\gamma\alpha \subseteq$ dom $\gamma\beta$. Let $v \in$ dom $\gamma\beta$ and $v\gamma\beta \in V\gamma\alpha$. Then $(v\gamma)\beta \in V\alpha$ from which follows that $v\gamma\alpha = v\gamma\beta$ since $\alpha \leq \beta$. Therefore, $\gamma\alpha \leq \gamma\beta$. Now, if $\dim(V\gamma) = 1$, then we can write $\gamma = \binom{v}{w}$. If $w \notin$ dom α, then $\gamma\alpha = 0$ from which follows that $\gamma\alpha \leq \gamma\beta$. If $w \in$ dom α, then $w \in$ dom β since dom $\alpha \subseteq$ dom β and hence dom $\gamma\alpha = \{v\} =$ dom $\gamma\beta$. Furthermore, $v\gamma\alpha = w\alpha = w\beta = v\gamma\beta$ since $\alpha \subseteq \beta$. Therefore, $\gamma\alpha = \gamma\beta$.

(2) Let $\alpha, \beta \in I(V,W)$ be such that $\alpha \leq \beta$. Then $\alpha \subseteq \beta$ from which it follows that $\alpha\gamma \subseteq \beta\gamma$. So dom $\alpha\gamma \subseteq$ dom $\beta\gamma$. Since $\alpha \leq \beta$, we have $V\alpha\gamma \subseteq W\beta\gamma$. Let $v\beta\gamma \in V\alpha\gamma$. Then $v\beta\gamma = w\alpha\gamma$ for some $w \in V$. Since γ is injective, we get $v\beta = w\alpha \in V\alpha$ which implies that $v\beta = v\alpha$. Hence $v\beta\gamma = v\alpha\gamma$, and therefore $\alpha\gamma \leq \beta\gamma$. □

Acknowledgments

The first author thanks the Development and Promotion of Science and Technology talents project, Thailand, for its financial support. He also thanks the Graduate School, Chiang Mai University, Chiangmai, Thailand, for its financial support that he received during the preparation of this paper.

References

[1] A. H. Clifford and G. B. Preston, *The Algebraic Theory of Semigroups*, Vol. 2, Mathematical Surveys, No. 7, American Mathematical Society, Providence, RI (1967).

[2] V. H. Fernandes and J. Sanwong, On the ranks of semigroups of transformations on a finite set with restricted range, *Algebra Colloq.*, in press.

[3] J. M. Howie, *Fundamentals of Semigroup Theory*, Oxford University Press, New York (1995).

[4] H. Mitsch, A natural partial order for semigroups, *Proc. Amer. Soc.* **97**, No.3 (1986), 384-388.

[5] K. Sangkhanan and J. Sanwong, Partial orders on semigroups of partial transformations with restricted range, *Bull. Aust. Math. Soc.* **86**, No.1 (2012), 100-118.

[6] K. Sangkhanan and J. Sanwong, Green's relations and partial orders on semigroups of partial linear transformations with restricted range, submitted.

[7] J. Sanwong and W. Sommanee, Regularity and Green's relations on a semigroup of transformation with restricted range, *Int. J. Math. Math. Sci.* **2008** (2008), Art. ID 794013, 11pp.

[8] R. P. Sullivan, Partial orders on linear transformation semigroups, *Proceeding of the Royal Society of Edinburgh.* **135A** (2005), 413-437.