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ANALYSIS OF OBSERVABILITY OF A DIFFERENTIAL EQUATION SYSTEM DESCRIBING A SYNCHRONOUS ELECTROMAGNETIC DRIVE

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Abstract: The contribution of this paper is to propose an observability analysis for a nonlinear differential equations model describing a synchronous electrical drive. In the modern control systems the trend is to reduce the state variables to be measured and to estimate through an observer all variables which are needed. In this sense, an analysis of the observability of the model is always necessary.

Throughout the paper a procedure for testing the observability is shown. Observability analysis allows to discover if the system is observable and in particular to discover those points or set in which the system is not observable. These points must be avoided from the observer and alternative procedure for the estimation of the state variable should be analyzed. The analysis of the presented drive model shows general results which can be extended to the family of such kinds of drives.

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1. Introduction and Motivation

One of the most important issues in control systems is represented by the sen-

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sorless control. With the term “sensorless control” the possibility to control a system with a minimal number of measured variables is to be intended. A recent application is shown in [1] in which the author presented a sensorless control of a drive just voltage and current inputs. Through a Kalman Filter all the rest of the variables needed for the control are estimated. This possibility is offered using an observer which consists of a model of the considered system together with a feedback between measured data and their estimation. Roughly speaking, this difference states a *innovation* which is able to correct the error in an arbitrarily short time. The most important issue in designing an observer is that the considered model of the system must be observable. It happens that the considered model is not global observable and we have *unobservable sets* as well as isolated points. These points should be avoided in the observer design; thus, a thorough analysis of the observability is important. Sensorless operations tend to perform poorly in a low-speed environment, since nonlinear observer-based algorithms work only if the rotor speed is high enough. In the low-speed region, an open loop control strategy must be considered. One of the first attempts to develop an open loop observer for a permanent motor drive is described in [2]. In more recent work [3], the authors proposed a nonlinear state observer for the sensorless control of a permanent magnet AC machine, which is based to a great extent on the work described in [4] and [5]. The approach presented in [4] and [5] consists of an observable linear system and a Lipschitz nonlinear part. The observer is basically a Luenenberger observer, in which the gain is calculated through a Lyapunov approach. In [3], the authors used a change of variables to obtain a nonlinear system consisting of an observable linear part and a Lipschitz nonlinear part. In the work presented here, our system does not satisfy the condition in [5]; thus, a Luenenberger observer is not feasible. The paper is organized as follows. Section 2 is devoted to the explanation of the mathematical model of the drive. Section 3 considers the observability analysis. The conclusions close the paper.

2. Mathematical Model of the Drive

The considered electromagnetic drive can be represented mathematically as follows:

$$\frac{\partial i_C(t)}{\partial t} = -\frac{R_C}{L_C} i_C(t) + \frac{u_{in}(t) - u_q(t)}{L_C} \tag{1}$$

$$\frac{\partial \varphi(t)}{\partial t} = \omega(t) \tag{2}$$

$$\frac{\partial \omega(t)}{\partial t} = \frac{M(\varphi(t), i_C(t)) + M_{fric} + M_k + M_d}{J}. \tag{3}$$

The following expression

$$M(\varphi(t), i_C(t)) = M_0(\varphi(t)) + M_i(\varphi(t), i_C(t)), \tag{4}$$

describes the torque generated by the drive, where

$$M_0(\varphi(t)) = k_2 \sin(\pi + \frac{2\pi\varphi(t)}{\varphi_{Pol}}) \tag{5}$$

$$M_i(\varphi, i_C(t)) = i_C(t) k_1 \sin(\frac{\pi\varphi(t)}{\varphi_{Pol}} + \frac{\pi}{2}). \tag{6}$$

In equation (1) the following expression

$$u_q(t) = k_1 \sin(\frac{\pi\varphi(t)}{\varphi_{Pol}} + \frac{\pi}{2})\omega(t) \tag{7}$$

represents the induced electromagnetic voltage; k_1 and k_2 are physical constants; and $\varphi_{Pol} = \frac{2\pi}{n}$ is the angular position of the stator poles of the motor, where there are $n \in \mathbb{N}$ poles. J is the inertia of the moving part of the drive system, and M_{fric} is the friction torque modeled by

$$M_{fric} = k_{d1} sign(\omega(t)) + k_{d2}\omega(t), \tag{8}$$

with the damping coefficients k_{d1} and, k_{d2}. The spring torque is represented because of the narrow variation around zero of variable $\varphi(t)$ by

$$M_k = k_f r sin(\varphi(t)) \simeq k_f r \varphi(t), \tag{9}$$

where r is the length of the lever, which can be mounted on the axes of the motor, and k_f is the elastic contact of the spring. Finally, M_d indicates the disturbance torque (gas pressure) acting on the armature. Equation (1) represents the electromagnetic system of the drive. Equations (3) and (2) describe the mechanical behaviour of the drive, and equation (3) also includes the magnetic system. Component $u_q(t)$ of equation (1), the electromotive back force, states the connections between the electric and the mechanical dynamics, and it has an electromagnetic nature. The state is represented by the coil current $i_C(t)$, the drive angular position $\varphi(t)$, and its angular velocity $\omega(t)$. A detailed explanation of this adopted model is available [6].

3. Analysis of the Nonlinear Observability

Here, an analysis of the observability is performed using well-known rank criteria [7, 8]. The rank criteria provide sufficient and necessary conditions for the observability of a nonlinear system. Moreover, for applications, it is useful to detect those sets in which the *observability level* of the state variables decreases; thus, a measurement of the observability is sometimes needed. The *unobservable sets* should be avoided in the observer design; thus, a thorough analysis of the observability is important. Sensorless operations tend to perform poorly in a low-speed environment because nonlinear observer-based algorithms work only if the rotor speed is fast enough. In the low-speed region, an open loop control strategy must be considered. One of the first attempts to develop an open loop observer for a permanent motor drive has been described [2]. In more recent work [3], the authors proposed a nonlinear state observer for the sensorless control of a permanent magnet AC machine, which is based primarily on other works [5] and [4]. The proposed approach [5] and [4] consists of an observable linear system and a Lipschitz nonlinear part. The observer is basically a Luenberger observer in which the gain is calculated through a Lyapunov approach. In [3], a change of variables has been used to obtain a nonlinear system consisting of an observable linear part and a Lipschitz nonlinear part.

3.1. Mathematical Derivation

It is useful to summarize briefly some well established facts about nonlinear systems such as Lie-derivatives and observability.

Definition 1. Given the following nonlinear system:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\mathbf{u}(t) \tag{10}$$

$$\mathbf{y}(t) = \mathbf{h}(\mathbf{x}), \tag{11}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$, $\mathbf{u}(t) \in \mathbb{R}^m$, and $\mathbf{y}(t) \in \mathbb{R}^p$, with $n, m, p \in \mathbb{N}$, a system in the form of (10), and (11) is said to be locally observable at a point $\mathbf{x}_0$, if all states $x(t)$ can be instantaneously distinguished by a judicious choice of input $\mathbf{u}(t)$ in a neighbourhood $\mathbf{U}$ of $\mathbf{x}_0$ [7, 8]. □

The following definition states the structure of the well known Lie-derivatives.

Definition 2. For a vector $\mathbf{x}(t) \in \mathbb{R}^n$, a real-valued function $\mathbf{h}(\mathbf{x}(t))$, which is the derivative of $\mathbf{h}(\mathbf{x}(t))$ along $\mathbf{f}$ [9], is denoted by

$$L_{\mathbf{f}}\mathbf{h}(\mathbf{x}(t)) = \sum_{i=1}^{n} \frac{d\mathbf{h}(\mathbf{x}(t))}{dx_i(t)} f_i(\mathbf{x}(t)) = \frac{d\mathbf{h}(\mathbf{x}(t))}{d\mathbf{x}(t)}\mathbf{f}(\mathbf{x}(t)), \tag{12}$$

where $n \in \mathbb{N}$ and the function $L_{\mathbf{f}}\mathbf{h}(\mathbf{x}(t))$ represents the derivative of $\mathbf{h}$ first along a vector field $\mathbf{f}(\mathbf{x}(t))$. Function $L^i_{\mathbf{f}}\mathbf{h}(\mathbf{x}(t))$ satisfies the recursion relation

$$dL^i_{\mathbf{f}}\mathbf{h}(\mathbf{x}(t)) = \frac{dL^{i-1}_{\mathbf{f}}\mathbf{h}(\mathbf{x}(t))}{d\mathbf{x}(t)}\mathbf{f}(\mathbf{x}(t)) \tag{13}$$

with $L^0_{\mathbf{f}}\mathbf{h}(\mathbf{x}(t)) = \mathbf{h}(\mathbf{x}(t))$. □

Definition 3. Observation space $\mathcal{O}$ of system (10) and (11) is the linear space of functions $\mathcal{M} \to \mathcal{R}$ over the field $\mathcal{R}$ spanned by all functions of the form

$$\mathcal{O} = \big\{L_{v_k}...L_{v_1}(h_i), \quad k \geq 0, \quad 1 \leq i \leq p, \quad v_k, ..., v_1 \in \{\mathbf{f}, \mathbf{g}_1, ...\mathbf{g}_m\}\big\}. \tag{14}$$

□

It is important to emphasize that the observation space consists of all linear combinations of the functions defined in (16) with real constant coefficients. Associated with the observation space $\mathcal{O}$ is its differential $d\mathcal{O}$, the codistribution defined as follows.

Definition 4. Let

$$d\mathcal{O} = span\{d\lambda : \lambda \in \mathcal{O}\} \tag{15}$$

be a codistribution, then the observability codistribution is the smallest codistribution Ω_O containing covectors $dh_1, ..., dh_p$ which is invariant with respect to the vector fields $\{\mathbf{f}, \mathbf{g}_1, ...\mathbf{g}_m\}$. □

In [7] it can be shown that if $d\mathcal{O}$ is nonsingular, then $d\mathcal{O} = \Omega_O$. Test criteria can be derived according to the local observability definitions [7, 8]. In particular, if $\mathbf{u}(t) = 0$, the system is called *zero input observable*, which is also important for this application because if a system is zero input observable, then it is also locally observable [10]. The system is locally observable at x_0 if the observability codistribution, Ω_O has rank n at x_0. This is called the observability rank condition. If x_0 is a regular point of $\Omega_O(x_0)$, the observability rank condition is necessary as well as sufficient. If the system has zero input, then the observability codistribution reduces to

$$\Omega_L = span\big\{L^k_f(dh_i), \quad 1 \leq i \leq p, \quad 0 \leq k \leq n-1\}\big\}. \tag{16}$$

When dim $\Omega_O(x_0) = n$ but dim $\Omega_L(x_0) < n$, the implication is that some states are distinguishable only under the action of control inputs. When this occurs, most control inputs do distinguish the states. There are a few singular inputs, notably $u = 0$, that do not. Thus, when dim $\Omega_L(x0) = n$ we will use the terminology observable for zero input at x_0.

Rank Condition 1. The system described in (10) and (11) is autonomous if $\mathbf{u}(t) = 0$. The following rank condition [7, 8] is used to determine the local observability for the nonlinear system stated in (10). The system is locally observable if and only if

$$\mathsf{dim}\big(d\mathcal{O}(\mathbf{x}_0)\big) = \left.\frac{\partial l(\mathbf{x}(t))}{\partial \mathbf{x}(t)}\right|_{\mathbf{x}_0} = n, \tag{17}$$

$$\text{where } l(\mathbf{x}(t)) = \begin{bmatrix} L^0_{\mathbf{f}}(h(\mathbf{x}(t))) \\ L_{\mathbf{f}}(h(\mathbf{x}(t))) \\ L^2_{\mathbf{f}}(h(\mathbf{x}(t))) \end{bmatrix}. \tag{18}$$

□

For the case considered here, if $M_d = 0$ in equations (1), (2), and (3), the following is true:

$$\mathbf{f}(\mathbf{x}(t)) = \begin{bmatrix} f_1(\mathbf{x}(t)) \\ f_2(\mathbf{x}(t)) \\ f_3(\mathbf{x}(t)) \end{bmatrix} = \begin{bmatrix} -\frac{R_C}{L_C} i_C(t) - \frac{u_q(t)}{L_C} \\ \omega(t) \\ \frac{M(\varphi(t), i_C(t)) + k_{d1} sign(\omega(t))}{J} + \frac{k_{d2}\omega(t) + k_f r \varphi(t)}{J} \end{bmatrix}.$$

Applying the above criterion with the following assumptions,

i) $\mathbf{x}(t) = \begin{bmatrix} i_C(t) \\ \varphi(t) \\ \omega(t) \end{bmatrix}$,

ii) $\mathbf{h}(\mathbf{x}(t)) = i(t)$, and

iii) L_C and R_C are constant functions with respect to state variable $\mathbf{x}(t)$.

According to these hypotheses, the following calculations are derived:

$$dL^0_{\mathbf{f}}\mathbf{h}(\mathbf{x}(t)) = \frac{\partial \mathbf{h}(\mathbf{x}(t))}{\partial \mathbf{x}(t)} = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}, \tag{19}$$

$$L_{\mathbf{f}}\mathbf{h}(\mathbf{x}(t)) = -\frac{R_{\mathrm{C}}}{L_{\mathrm{C}}} i_{\mathrm{C}} - \frac{u_q(t)}{L_{\mathrm{C}}}, \tag{20}$$

$$dL_{\mathbf{f}}\mathbf{h}(\mathbf{x}(t)) \quad = \quad \frac{1}{L_{\mathrm{C}}}\left[-R_C \quad -\tfrac{\partial u_q(t)}{\partial \varphi(t)} \quad -\tfrac{\partial u_q(t)}{\partial \omega(t)}\right], \tag{21}$$

$$\text{where } \frac{\partial u_q(t)}{\partial \varphi(t)} = \frac{k_1 \pi}{\varphi_{Pol}} \cos(\frac{\pi \varphi(t)}{\varphi_{Pol}} + \frac{\pi}{2})\omega(t) \tag{22}$$

$$\text{and } \frac{\partial u_q(t)}{\partial \omega(t)} = k_1 \sin(\frac{\pi \varphi(t)}{\varphi_{Pol}} + \frac{\pi}{2}). \tag{23}$$

According to Definition 2, then

$$L_{\mathbf{f}}^2\mathbf{h}(\mathbf{x}) = \frac{\partial L_{\mathbf{f}}\mathbf{h}(\mathbf{x}(t))}{\partial \mathbf{x}}\mathbf{f}(\mathbf{x}(t)), \tag{24}$$

$$\begin{aligned} &\frac{\partial L_{\mathbf{f}}\mathbf{h}(\mathbf{x}(t))}{\partial \mathbf{x}(t)}\mathbf{f}(\mathbf{x}(t)) \\ &\quad = -\frac{1}{L_C}\Big(R_C\big(-\frac{R_{\mathrm{C}}}{L_{\mathrm{C}}}i_{\mathrm{C}} - \frac{u_q(t)}{L_{\mathrm{C}}}\big) + \frac{\partial u_q(t)}{\partial \varphi(t)}\omega(t) + \frac{\partial u_q(t)}{\partial \omega(t)}\frac{\partial \omega(t)}{\partial t}\Big). \end{aligned} \tag{25}$$

For the sake of notation,

$$dL_{\mathbf{f}}^2\mathbf{h}(\mathbf{x}(t)) = \begin{bmatrix} M_1(t) & M_2(t) & M_3(t) \end{bmatrix}, \tag{26}$$

where $M_1(t)$, $M_2(t)$ and $M_3(t)$ are functions of $i_C(t)$, $\varphi(t)$ and $\omega(t)$. In particular, it is useful to note the terms

$$\begin{aligned} M_2 = \frac{R_{Coli}}{L_C^2}\frac{\partial u_q(t)}{\partial \varphi(t)} + \frac{1}{L_C}\Big(&\frac{\partial^2 u_q(t)}{\partial \varphi(t)^2}\omega(t) + \\ &\frac{\partial u_q(t)}{\partial \varphi(t)}\frac{\partial \omega}{\partial \varphi} + \frac{\partial\big(\frac{\partial u_q(t)}{\partial \omega(t)}\big)}{\partial \varphi}\frac{\partial \omega(t)}{\partial t} + \frac{\partial u_q(t)}{\partial \omega(t)}\frac{\partial\big(\frac{\partial \omega(t)}{\partial t}\big)}{\partial \varphi}\Big) \end{aligned} \tag{27}$$

$$\text{and } M_3(t) = \frac{R_C}{L_C^2}\frac{\partial u_q(t)}{\partial \omega(t)} + \frac{2}{L_C^2}\frac{k_1\pi}{\varphi_{Pol}}\cos(\frac{\pi\varphi(t)}{\varphi_{Pol}} + \frac{\pi}{2})\omega(t) + \frac{\partial u_q(t)}{\partial \omega(t)}\frac{\partial\big(\frac{\partial \omega(t)}{\partial t}\big)}{\partial \omega}. \tag{28}$$

Matrix $d\mathcal{O}(\mathbf{x}_0)$ becomes

$$d\mathcal{O}(x_0) = \begin{bmatrix} 1 & 0 & 0 \\ -R_C & -\frac{\partial u_q(t)}{\partial \varphi(t)} & -\frac{\partial u_q(t)}{\partial \omega(t)} \\ M_1(t) & M_2(t) & M_3(t) \end{bmatrix}. \tag{29}$$

If set $\mathbf{x}_0 = \{\omega(t) = 0, \frac{\partial \omega(t)}{\partial t} = 0\}$, (drive at the initial and final position) is considered, then matrix (29) is not full rank. In fact, given that $\frac{\partial u_q(t)}{\partial \varphi(t)} =$

$\frac{k_1\pi}{\varphi_{Pol}}\cos(\frac{\pi\varphi(t)}{\varphi_{Pol}}+\frac{\pi}{2})\omega(t)$, then $\frac{\partial u_q(t)}{\partial\varphi(t)}|_{\mathbf{x}_0}=0$, and considering Eq. (27) calculated in $\mathbf{x}_0$, it follows that $M_2(t)|_{\mathbf{x}_0}=0$. Thus, it is shown that the three rows of matrix (29) are linearly dependent, and matrix (29) is not full rank.

Rank Condition 2. The system described in (10) and (11) is not autonomous if $\mathbf{u}(t)\neq 0$. The following rank condition [7, 8] is used to determine the local observability for the nonlinear system stated in (10). The system is locally observable if and only if

$$\mathsf{dim}\big(d\mathcal{O}(\mathbf{x}_0)\big)=\frac{\partial l(\mathbf{x}(t))}{\partial\mathbf{x}(t)}\bigg|_{\mathbf{x}_0}=n, \tag{30}$$

$$\text{where } l(\mathbf{x}(t))=\begin{bmatrix} L_f^0(\mathbf{h}(\mathbf{x}(t))) \\ L_f(\mathbf{h}(\mathbf{x}(t))) \\ L_f^2(\mathbf{h}(\mathbf{x}(t))) \\ L_gL_f(\mathbf{h}(\mathbf{x}(t))) \end{bmatrix}. \tag{31}$$

□

According to (10) and the model in (1),

$$\mathbf{g}(\mathbf{x})=\begin{bmatrix}\frac{1}{L_C}\\ 0\\ 0\end{bmatrix}. \tag{32}$$

According to (21), it is straightforward to notice that

$$L_{\mathbf{g}}L_{\mathbf{f}}(\mathbf{h}(\mathbf{x}(t)))=-\frac{R_C}{L_C^2}. \tag{33}$$

In fact, as shown above,

$$L_{\mathbf{f}}\mathbf{h}(\mathbf{x}(t))=-\frac{R_{\mathrm{C}}}{L_{\mathrm{C}}}i_{\mathrm{C}}-\frac{u_q(t)}{L_{\mathrm{C}}}, \tag{34}$$

$$\text{and } dL_{\mathbf{f}}\mathbf{h}(\mathbf{x}(t))=\frac{1}{L_{\mathrm{C}}}\begin{bmatrix}-R_C & -\frac{\partial u_q(t)}{\partial\varphi(t)} & -\frac{\partial u_q(t)}{\partial\omega(t)}\end{bmatrix}, \tag{35}$$

then, according to Definition 2, it follows:

$$L_{\mathbf{g}}L_{\mathbf{f}}(\mathbf{h}(\mathbf{x}(t)))=dL_{\mathbf{f}}\mathbf{h}(\mathbf{x}(t))\mathbf{g}(\mathbf{x}). \tag{36}$$

To conclude, the vectorial product between (35) and (32) is equal to (33). The differential calculation is as follows:

$$dL_{\mathbf{g}}L_{\mathbf{f}}(\mathbf{h}(\mathbf{x}(t))) = [0 \quad 0 \quad 0]. \tag{37}$$

Thus, even for a *judicious choice* of input $\mathbf{u}(t)$, no contribution to the observability set is given compared with the autonomous case provided above.

4. Conclusions

The paper proposes an observability analysis for a nonlinear differential equations model describing a synchronous electrical drive. Throughout the paper a procedure for testing the observability is shown. The analysis of the presented drive model shows general results which can be extended to the family of such kinds of drives.

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