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ON TERNARY DIOPHANTINE EQUATION $(py + g(z))(qy + h(z)) = ax^2 + bx + c$

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Abstract: Let a, b, c, p, q be given positive integers with $a = pq$ and let $g(z)$ and $h(z)$ be positive integer valued arithmetical functions such that there is no integer between $g(z)/p$ and $h(z)/q$ for any positive integer z. Consider the ternary diophantine equation $(py + g(z))(qy + h(z)) = ax^2 + bx + c$ in variables x, y, z. In this paper, we find a real number γ such that all positive integral solutions (x, y, z) of the equation satisfy $x \leq \gamma$.

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1. Introduction

Let $f(x, y, z)$ be a polynomial in variables x, y and z with integer coefficients such that the equation $f(x, y, z) = 0$ has only finitely many solutions (x, y, z) in integers.

Problem 1. Give an upper bound for $|x|$ when (x, y, z) is an integer solution of $f(x, y, z) = 0$, in terms of the degree of f and of the coefficients of f.

The same problem has been studied by many authors for the diophantine equation $f(x, y) = 0$ in variables x and y. For example, see [1], [2], [3] and [5]. Recently, in [4], we have given an elementary algorithm to produce an upper bound for $|x|$ when (x, y) is an integral solution of the equation $y^p = x^{kp} + a_1x^{kp-1} + ... + a_{kp}$ in which right is not a perfect pth power.

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In this paper, we deal with Problem 1 and our aim is to prove Theorem 1. Through out this paper, $\mathbb{N}$ is the set of all positive integers and

$$\wp(p,q,b,c) = \max\left\{c, \frac{(b-q)(b-p-1)}{pq} - c\right\}$$

for any positive integers p, q, b and c.

Theorem 1. *Let p, q, $a = pq$, b and c be positive integers and $f(x) = ax^2 + bx + c$ be a polynomial in variable x. Let $f(x) \neq (px+\alpha)(qx+\beta)$ for any two integers α and β. If $g(z)$ and $h(z)$ are positive integer valued arithmetical functions such that for each $z \in \mathbb{N}$, there is no integer between $g(z)/p$ and $h(z)/q$, then all positive integer solutions (x,y,z) of the equation*

$$(py + g(z))(qy + h(z)) = f(x)$$

satisfy

$$x \leq \wp\,(p,q,b,c)\,.$$

2. Preliminaries

In this section, we prove some lemmas which are used to prove Theorem 1. Through out this section p, q, $a = pq$, b and c are in $\mathbb{N}$ and $f(x) = ax^2 + bx + c$ is a polynomial in variable x.

Lemma 1. *There is a positive integer $m > \wp\,(p,q,b,c)$ such that $f(m) = (pm + b_1)\,(qm + b_2)$ for some $b_1, b_2 \in \mathbb{N}$ if and only if $f(x) = (px + b_1)\,(qx + b_2)$.*

Proof. Let there be an integer $m > \wp\,(p,q,b,c)$ such that$f(m) = (pm + b_1)(qm + b_2)$ for some $b_1, b_2 \in \mathbb{N}$. Therefore

$$bm + c = (pb_2 + qb_1)m + b_1 b_2. \tag{1}$$

Suppose $f(x) \neq (px + b_1)\,(qx + b_2)$, then we have $b \neq pb_2 + qb_1$ or $c \neq b_1 b_2$. If $b \neq pb_2 + qb_1$, then $c - b_1 b_2 = m\,(pb_2 + qb_1 - b) \neq 0$, by (1). Similarly if $c \neq b_1 b_2$, then $b \neq pb_2 + qb_1$. Therefore we have

$$b \neq pb_2 + qb_1 \quad \text{and} \quad c \neq b_1 b_2.$$

Since $c \neq b_1 b_2$, $c < b_1 b_2$ or $c > b_1 b_2$. Guess that $c < b_1 b_2$. Then $c = b_1 b_2 - t_1$ for some $t_1 \in \mathbb{N}$. Put c in (1). Then we obtain $t_1 = m(b - pb_2 - qb_1)$. So $t_1 = mt$

for some $t \in \mathbb{N}$. Therefore $b = pb_2 + qb_1 + t$ and $c = b_1 b_2 - mt$. After removing b_1 from the last two equations, we have $mtq = b_2(b - t - pb_2) - qc$. Therefore

$$m \leq \frac{b_2(b-1-p)}{q} - c, \tag{2}$$

Since $b > pb_2 + qb_1$, $b_2 < (b - qb_1)/p$. That is,

$$b_2 \leq (b-q)/p. \tag{3}$$

Remove b_2 from (2) and (3). Then we have

$$m \leq \frac{(b-q)(b-p-1)}{pq} - c.$$

This is a contradiction to given. So our guess is wrong.

Next we consider the case $c > b_1 b_2$. Then $c = b_1 b_2 + t_1$ for some $t_1 \in \mathbb{N}$. Put c in (1). Then we have $t_1 = mt$ for some $t \in \mathbb{N}$. Therefore $c = b_1 b_2 + mt$. So $m \leq mt = c - b_1 b_2 < c$. This is a contradiction to given.

From the above two cases, we have $c = b_1 b_2$ and $b = pb_2 + qb_1$. This proves that $f(x) = (px + b_1)(qx + b_2)$. Converse part of this lemma is obvious. □

Lemma 2. *Let b_1 and b_2 be positive integers such that there is no integer between b_1/p and b_2/q. If $f(x) \neq (px+i)(qx+j)$ for any integers i and j, then all solutions $(x, y) \in \mathbb{N} \times \mathbb{N}$ of the equation*

$$(py + b_1)(qy + b_2) = f(x)$$

satisfy

$$x \leq \wp(p, q, b, c).$$

Proof. Let $(x, y) = (\alpha, \beta) \in \mathbb{N} \times \mathbb{N}$ be a solution of the given equation. Then

$$(p\beta + b_1)(q\beta + b_2) = f(\alpha). \tag{4}$$

If $\alpha \leq \beta$. Then $\beta = \alpha + l$ for some nonnegative integer l. So from (4), we obtain

$$(p\alpha + (pl + b_1))(q\alpha + (ql + b_2)) = f(\alpha).$$

Since $(pl + b_1), (ql + b_2) \in \mathbb{N}$ and $f(x) \neq (px+i)(qx+j)$ for any integers i and j, by Lemma 1, $\alpha \leq \wp(p, q, b, c)$. Assume that $\alpha > \beta$. Then $\beta = \alpha - l$ for some $l \in \mathbb{N}$. So from (4), we get

$$(p\alpha + (b_1 - pl))(q\alpha + (b_2 - ql)) = f(\alpha).$$

If $b_1 - lp$ and $b_2 - lq$ both are negative, then

$$p\alpha + (b_1 - pl) < p\alpha \text{ and } q\alpha + (b_2 - ql) < q\alpha.$$

So $f(\alpha) = (p\alpha + (b_1 - pl))(q\alpha + (b_2 - ql)) < a\alpha^2$. This is imposible. Suppose that one of $b_1 - lp$ and $b_2 - lq$ is positive and another one is negative. Without loss of generality, we can assume that $b_1 - lp < 0$ and $b_2 - lq > 0$. This implies that $b_1/p < l$ and $b_2/q > l$. That is, there is an integer l such that

$$b_1/p < l < b_2/q.$$

This is a contradiction to given. Therefore $b_1 - lp$ and $b_2 - lq$ both are positive. So by Lemma 1, $\alpha \leq \wp(p, q, b, c)$. This proves the lemma. □

3. Proof of Theorem 1

Fix a positive integer z. Since $f(x) \neq (px + i)(qx + j)$ for any integers i and j, by Lemma 2, any positive integral solutions (x, y) of the equation

$$(py + g(z))\,(qy + h(z)) = f(x)$$

satisfies

$$x \leq \wp\,(p, q, b, c)\,.$$

Since $\wp\,(p, q, b, c)$ does not depend z, the bound is constant for any positive integer z. This implies that any positive integral solutions (x, y, z) of the given equation satisfies $x \leq \wp\,(p, q, b, c)\,.$

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