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ITERATION SCHEME FOR COMMON FIXED POINTS OF TWO ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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Abstract: In this paper, we consider a generalized iterative process and analyze a two-step iteration scheme for asymptotically nonexpansive mappings in uniformly convex Banach space. Our iteration scheme includes Ishikawa type and Mann type iterations as special cases. The results obtained in this paper represent an extension as well as refinement of previous known results. A new iterative scheme for approximating common fixed points of two asymptotically nonexpansive mapping is defined and we have proved weak and strong convergence theorems in a uniformly convex Banach space.

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1. Introduction

Let E be a uniformly convex Banach space, K be a nonempty closed convex subset of E. Throughout this paper, $\mathbb{N}$ denotes the set of all positive integers

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and $F(T) := x : Tx = x$. A mapping $T : K \to K$ is said to be asymptotically nonexpansive if for a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$, if $\| T^n x - T^n y \| \leq k_n \| x - y \|$, for all $x, y \in K$ and for all $n \in \mathbb{N}$. This class of asymptotically nonexpansive mappings was to introduced by Goebel and Kirk [8] in 1972.They proved that,if K is a nonempty bounded closed convex subset of a uniformly convex Banach space E, then every asymptotically nonexpansive self-mapping T of K has a fixed point. The fixed point iteration process for asymptotically nonexpansive mapping in Banach spaces including Mann and Ishikawa iterations processes have been studied extensively by many authors;see ([2]-[24]).

In 1991, Schu ([16], [17]) introduced a modified Mann iteration process to approximate fixed points of asymptotically nonexpansive self-map defined on nonempty closed convex and bounded subset of Hilbert space H.

In 2001, Xu and Ori [24] introduced the following implicit iteration scheme for common fixed points of a finite family of nonexpansive mappings $\{T_i\}_{i=1}^{\mathbb{N}}$ in Hilbert spaces:

$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, n \geq 1,$$

where $T_n = T_{nmodN}$, and they proved weak convergence theorem.

In 2008 Zhao et al. [25] introduced the following iteration scheme for common fixed points of nonexpansive mapping T in Banach space and proved weak and strong convergence theorems.:

$$x_n = \alpha_n x_{n-1} + \beta_n T x_{n-1} + \gamma_n T x_n, n \geq 1,$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0, 1]$, and $\alpha_n + \beta_n + \gamma_n = 1$.

The Picard and Mann [12] iteration schemes for a mapping $T : K \to K$ are defined by

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = T^n x_n \end{cases} \tag{1}$$

and

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n x_n, n \in \mathbb{N} \end{cases} \tag{2}$$

where $\{\alpha_n\}$ is in $(0, 1)$. It is well-known that Picard iteration scheme converges for contractions but not converges for nonexpansive mapping whereas Mann iteration scheme converges for nonexpansive. Several authors have been studied weak and strong convergence problems of iterative sequence (with errors) for asymptotically nonexpansive type mappings in a Hilbert space or a Banach space (see [2],[13], [14],[16]).

In 2007, Agrawal et al. [1] introduced the following iteration process:

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = (1-\alpha_n)T^n x_n + \alpha_n T^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, n \in \mathbb{N} \end{cases} \tag{3}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are in (0,1). They showed that this process converges at a rate same as that of Picard iteration and faster than Mann iteration for contractions.

The above process deals with one mapping only. The case of two mappings in iterative processes has also remained under study since Das and Debata [3] gave and studied a two mappings process. Also see, for example [10] and [19]. The problem of approximating common fixed points of finitely many mapping plays an important role in applied mathematics especially in the theory of evaluation equation and the minimization problems. See ([4], [5],[6], [20]) for example.

In 2001, Khan and Takahashi [10] approximated the fixed points of two asymptotically nonexpansive mappings $S, T : K \to K$ through the sequence $\{x_n\}$ given by

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, n \in \mathbb{N} \end{cases} \tag{4}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$.

Recently, Khan et al. [9] modified the iteration process (4) to the case of two mappings as follows:

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = (1-\alpha_n)T^n x_n + \alpha_n S^n y_n, \\ y_n = (1-\beta_n)x_n + \beta_n T^n x_n, n \in \mathbb{N} \end{cases} \tag{5}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$.

In this paper, we introduced a new implicit iteration scheme as below:

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = \alpha_n x_n + \beta_n T^n x_n + \gamma_n S^n y_n, \\ y_n = \alpha_n' x_n + \beta_n' S^n x_n + \gamma_n' T^n x_n, n \in \mathbb{N}, \end{cases} \tag{6}$$

where $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n^{'}\}, \{\beta_n^{'}\}$ and $\{\gamma_n^{'}\}$ are sequences in $[0,1]$ and $\alpha_n + \beta_n + \gamma_n = \alpha_n^{'} + \beta_n^{'} + \gamma_n^{'} = 1$, for fixed points of asymptotically nonexpansive mapping T in uniformly convex Banach space.

Observe that in (6) if we set $T = I$, then the scheme will reduce to:

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = (1-\gamma_n)x_n + \gamma_n S^n y_n, \\ y_n = (1-\beta_n^{'})x_n + \beta_n^{'} S^n x_n, n \in \mathbb{N}, \end{cases} \tag{7}$$

where $\{\gamma_n\}$ and $\{\beta_n^{'}\}$are sequences in $[0,1]$.

Now, we observe that in (7) if we set $S = T$, $\gamma_n = \alpha_n$ and $\beta_n^{'} = 0$ then the scheme will reduce to the Mann type iteration scheme given by:

$$\begin{cases} x_1 = x \in K, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n x_n, n \in \mathbb{N} \end{cases} \tag{8}$$

Above iteration scheme is introduced in 1991, by Schu [16] for modified Mann iteration process to approximate fixed points of asympticaly nonexpansive self-map, where $T : K \to K$ is an asymptotically nonexpansive mapping with a sequence $\{k_n\}$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $\{\alpha_n\}$ is a sequence in $(0,1)$ satisfying the condition $\delta \leq \alpha_n \leq 1-\delta$ for all $n \in \mathbb{N}$ and for some $\delta > 0$. Then the sequence $\{x_n\}$ converges weakly to a fixed point of T.

2. Preliminaries

Let us now gather some pre-requisites. Let $X = \{x \in E : \|x\| = 1\}$ and E^* be the dual of E. The space E has :
(i) Gâteaux differentiable norm if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}$$

exists for each $x, y \in E$;
(ii) Frèchet differentiable norm (see e.g. [18]) for each x in E, the above limit exists and is attained uniformly for y in E and in this case, it is also well-known that

$$\langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 \leq \frac{1}{2}\|x+h\|^2 \leq \langle h, J(x)\rangle + \frac{1}{2}\|x\|^2 + b(\|h\|)$$

for all $x, h \in E$, where J is the Frèchet derivative of the function $\frac{1}{2}\|.\|^2$ at $x \in E$, $\langle ., . \rangle$ is the dual pairing between E and E^*, and b is an increasing

function defined on $[0,\infty)$ such that $\lim_{t\to 0}\frac{b(t)}{t}=0$;
(iii) Opial's condition [15] if for any sequence $\{x_n\}$ in $E, x_n \rightharpoonup x$ implies that

$$\limsup_{n\to\infty}\|x_n - x\| < \limsup_{n\to\infty}\|x_n - y\|$$

for all $y \in E$ with $y \neq x$.

Following are the definitions and lemma used to prove the results in the next section.

Definition 1. Let K be a nonempty closed convex subset of a uniformly convex Banach space E. A mapping $T : K \to K$ is said to be asymptotically nonexpansive on K if there exists a sequence k_n, $k_n \geq 1$ with $\lim k_n = 1$, such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|,$$

for each $x, y \in K$ and each $n \geq 1$. If $k_n = 1$, then T is known as a nonexpansive mapping.

Definition 2. A mapping $T : K \to K$ is uniformly k-Lipschitzian if for some $k > 0, \| T^n x - T^n y \| \leq k \| x - y \| \; \forall x, y \in K$ and for all $n \in N$.

Definition 3. Let E be a uniformly convex Banach space, K be a nonempty closed convex subset of E, and $T : K \to K$ be an asymptotically nonexpansive mapping. Then $I - T$ is said to be demi-closed at 0, if $x_n \to x$ converges weakly and $x_n - Tx_n \to 0$ converges strongly, then it implies that $x \in K$ and $Tx = x$.

Definition 4. [7] Suppose two mappings $S, T : K \to K$, where K is a subset of a normed space E, said to be satisfy Condition (A') if there exists a nondecreasing function $F : [0,\infty) \to [0,\infty)$ with $F(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that either $\|x - Sx\| \geq f(d(x, F))$ or $\|x - Tx\| \geq f(d(x, F))$ for all $x \in K$ where $d(x, F) = inf\{\|x - p\| : p \in F = F(S) \cap F(T)\}$.

Lemma 1. (see [21], Lemma 1) *Let $\{a_n\}, \{b_n\}$ and $\{\delta_n\}$ be sequence of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq (1 + \delta_n)a_n + b_n.$$

If $\sum_{n=1}^{\infty} \delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.In particular,if $\{a_n\}$ has a subsequence converging to 0,then $\lim_{n\to\infty} a_n = 0$.

Lemma 2. (see [16]) *Suppose that E be a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of E such that $limsup_{n\to\infty}\|x_n\| \leq r$, $limsup_{n\to\infty}\|y_n\| \leq r$ and $lim_{n\to\infty}\|t_n x_n + (1 - t_n)y_n\| = r$ hold for some $r \geq 0$.Then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Lemma 3. *Let K be a normed linear space and K be a nonempty convex subset of X. Let $T : K \to K$ be an asymptotically nonexpansive mapping with nonempty fixed point set $F(T)$ and a sequence $k_n \geq 1$ of positive real numbers such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$, $k_n \to 1$ as $n \to \infty$. Let a sequence $\{x_n\}$ defined by (6), then $\lim_{n\to\infty} \|x_n - p\|$ exists for any $p \in F(T)$.*

Proof. Let $p \in F(T)$, using (6), we get

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n x_n + \beta_n T^n x_n + \gamma_n S^n y_n - p\| \\
&= \|\alpha_n(x_n - p) + \beta_n(T^n x_n - p) + \gamma_n(S^n y_n - p)\| \\
&\leq \alpha_n\|x_n - p\| + \beta_n k_n\|x_n - p\| + \gamma_n k_n\|y_n - p\| \\
&\leq \alpha_n\|x_n - p\| + \beta_n k_n\|x_n - p\| + \gamma_n k_n(\alpha_n^{'}\|x_n - p\| \\
&\qquad + \beta_n^{'}\|S^n x_n - p\| + \gamma_n^{'}\|T^n x_n - p\|) \\
&\leq \alpha_n\|x_n - p\| + \beta_n k_n\|x_n - p\| + \gamma_n\alpha_n^{'} k_n\|x_n - p\| \\
&\qquad + \gamma_n\beta_n^{'} k_n^2\|x_n - p\| + \gamma_n\gamma_n^{'} k_n^2\|x_n - p\| \\
&\leq (\alpha_n + \beta_n k_n + \gamma_n\alpha_n^{'} k_n + \gamma_n\beta_n^{'} k_n^2 + \gamma_n\gamma_n^{'} k_n^2)\|x_n - p\| \\
&\leq [1 + \beta_n(k_n - 1) + \gamma_n(k_n^2 - 1) - \gamma_n\alpha_n^{'} k_n(k_n - 1)]\|x_n - p\| \\
&\leq [1 + \{\beta_n + \gamma_n(k_n + 1) - \gamma_n\alpha_n^{'} k_n\}(k_n - 1)]\|x_n - p\| \\
&\leq [1 + \{\beta_n + \gamma_n + k_n\gamma_n(1 - \alpha_n^{'})\}(k_n - 1)]\|x_n - p\| \\
&\leq [1 + \{1 - \alpha_n + k_n\gamma_n(1 - \alpha_n^{'})\}(k_n - 1)]\|x_n - p\|
\end{aligned}$$

Since $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $k_n \to 1$ as $n \to \infty$, consequently, the condition of Lemma 3 follows from Lemma 1. This completes the proof. $\square$

3. Main Result

In this section, we have proved the approximate common fixed points of two asymptotically nonexpansive mappings for weak and strong convergence results, using iteration process. In the consequence, F denotes the set of common fixed point of the mappings T and S.

Theorem 5. *Let E be a uniformly convex Banach space and K be a nonempty closed convex subset of E. Let $T, S : K \to K$ be two asymptotically nonexpansive self mappings of K with $\sum_{n=1}^{\infty}(k_n-1) < \infty$ satisfying the iteration (6). Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\alpha_n^{'}\}, \{\beta_n^{'}\}$ and, $\{\gamma_n^{'}\}$ are real sequences in [0,1] such* *that* $\alpha_n + \beta_n + \gamma_n = 1 = \alpha_n^{'} + \beta_n^{'} + \gamma_n^{'}$. *If* $F \neq \phi$, $k_n \to 1$ *as* $n \to \infty$, *then* $\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|x_n - Sx_n\| = 0$.

Proof. Let $p \in F$. Since $\lim_{n\to\infty} \|x_n - p\|$ exists as proved in Lemma3. Suppose $\lim_{n\to\infty} \|x_n - p\| = c$, where $c \geq 0$ is a real number. Now suppose $c > 0$.
Thus

$$\begin{aligned}
\|y_n - p\| &= \|\alpha_n^{'} x_n + \beta_n^{'} S^n x_n + \gamma_n^{'} T^n x_n - p\| \\
&\leq \alpha_n^{'} \|x_n - p\| + \beta_n^{'} \|S^n x_n - p\| + \gamma_n^{'} \|T^n x_n - p\| \\
&\leq \alpha_n^{'} \|x_n - p\| + \beta_n^{'} k_n \|x_n - p\| + \gamma_n^{'} k_n \|x_n - p\| \\
&\leq (\alpha_n^{'} + \beta_n^{'} k_n + \gamma_n^{'} k_n) \|x_n - p\| \leq \{1 + \alpha_n^{'}(k_n - 1)\} \|x_n - p\| \\
&\leq \{1 + (1 - \alpha_n^{'})(k_n - 1)\} \|x_n - p\|.
\end{aligned}$$

Taking lim sup on both sides of the above inequality, we get

$$\limsup_{n\to\infty} \|y_n - p\| \leq c \tag{9}$$

Also from

$$\|T^n x_n - p\| \leq k_n \|x_n - p\|$$

for all $n = 1, 2, ...$, we have

$$\limsup_{n\to\infty} \|T^n x_n - p\| \leq c \tag{10}$$

Next, we consider

$$\|S^n y_n - p\| \leq k_n \|y_n - p\|$$

Taking lim sup on both sides of the above inequality and using (9), we have

$$\limsup_{n\to\infty} \|S^n y_n - p\| \leq c.$$

Furthermore,

$$\begin{aligned}
c &= \lim_{n\to\infty} \|x_{n+1} - p\| \\
&\leq \lim_{n\to\infty} (\alpha_n \|x_n - p\| + \beta_n \|T^n x_n - p\| + \gamma_n \|S^n y_n - p\|) \\
&= (1 - \gamma_n) \|T^n x_n - p\| + \gamma_n \|S^n y_n - p\|
\end{aligned}$$

by Lemma 2, we get

$$\lim_{n\to\infty} \|T^n x_n - S^n y_n\| = 0. \tag{11}$$

Now,

$$\begin{aligned}\|x_{n+1}-p\| &= \|\alpha_n x_n + \beta_n T^n x_n + \gamma_n S^n y_n - p\| \\ &= \|\alpha_n T^n x_n + \beta_n T^n x_n + \gamma_n S^n y_n - p\| \\ &\le (1-\gamma_n)T^n x_n + \gamma_n S^n y_n - p\| \\ &\le \|T^n x_n - p\| + \gamma_n \|T^n x_n - S^n y_n\|\end{aligned}$$

which yields that

$$c \le \liminf_{n\to\infty} \|T^n x_n - p\|.$$

So that (10) gives

$$\lim_{n\to\infty} \|T^n x_n - p\| = c.$$

In turn,

$$\begin{aligned}\|T^n x_n - p\| &\le \|T^n x_n - S^n y_n\| + \|S^n y_n - p\| \\ &\le \|T^n x_n - S^n y_n\| + k_n \|y_n - p\|\end{aligned}$$

which implies that

$$c \le \liminf_{n\to\infty} \|y_n - p\|. \tag{12}$$

By (9) and (12), we have

$$\lim_{n\to\infty} \|y_n - p\| = c. \tag{13}$$

Again, we get

$$c = \lim_{n\to\infty} \|y_n - p\| = \lim_{n\to\infty} (1-\gamma_n{}')\|x_n - p\| + \gamma_n{}'\|T^n x_n - p\|$$

gives by Lemma 2 that

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{14}$$

Notice that

$$\|y_n - x_n\| = \gamma_n{}'\|T^n x_n - x_n\|.$$

Hence by (14)

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{15}$$

Now

$$\begin{aligned}\|x_{n+1}-x_n\| &= \|\alpha_n x_n+\beta_n T^n x_n+\gamma_n S^n y_n - x_n\|\\ &\le (1-\alpha_n)\|T^n x_n - x_n\| + \gamma_n\|T^n x_n - S^n y_n\|.\end{aligned}$$

This gives

$$\lim_{n\to\infty}\|x_{n+1}-x_n\| = 0. \tag{16}$$

so that

$$\|x_{n+1}-y_n\| \le \|x_{n+1}-x_n\| + \|y_n - x_n\| \to 0\text{as } n\to\infty.$$

This gives

$$\lim_{n\to\infty}\|x_{n+1}-y_n\| = 0. \tag{17}$$

Moreover, from

$$\|x_{n+1}-S^n y_n\| \le \|x_{n+1}-x_n\| + \|x_n - T^n x_n\| + \|T^n x_n - S^n y_n\|$$

which is gives that

$$\lim_{n\to\infty}\|x_{n+1}-S^n y_n\| = 0. \tag{18}$$

Using (11), (14) and (15) we obtained

$$\begin{aligned}\|x_n - S^n x_n\| &\le \|x_n - T^n x_n\| + \|T^n x_n - S^n y_n\| + \|S^n y_n - S x_n\|\\ &\le \|x_n - T^n x_n\| + \|T^n x_n - S^n y_n\| + k\|y_n - x_n\|\end{aligned}$$

gives that

$$\lim_{n\to\infty}\|x_n - S^n x_n\| = 0.$$

and

$$\begin{aligned}\|x_{n+1}-Sx_{n+1}\| &\le \|x_{n+1}-S^{n+1}x_{n+1}\| + \|S^{n+1}x_{n+1}-Sx_{n+1}\|\\ &\le \|x_{n+1}-S^{n+1}x_{n+1}\| + k\|S^n x_{n+1}-x_{n+1}\|\\ &\le \|x_{n+1}-S^{n+1}x_{n+1}\| + k(\|S^n x_{n+1}-S^n y_n\| + \|S^n y_n - x_{n+1}\|)\\ &\le \|x_{n+1}-S^{n+1}x_{n+1}\| + k^2\|x_{n+1}-y_n\| + k\|S^n y_n - x_{n+1}\|\end{aligned}$$

this implies that

$$\lim_{n\to\infty}\|x_n - Sx_n\| = 0.$$

Now

$$\begin{aligned}\|x_{n+1} - Tx_{n+1}\| &\leq \|x_{n+1} - T^{n+1}x_{n+1}\| \\ &\quad + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| + \|T^{n+1}x_n - Tx_{n+1}\| \\ &\leq \|x_{n+1} - T^{n+1}x_{n+1}\| + k\|x_{n+1} - x_n\| + k\|T^n x_n - x_{n+1}\| \\ &= \|x_{n+1} - T^{n+1}x_{n+1}\| + k\|x_{n+1} - x_n\| + k\alpha_n\|T^n x_n - S^n y_n\|\end{aligned}$$

which is yields

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0.$$

This completes the proof. □

Theorem 6. *Let E be a uniformly convex Banach space satisfying Opial condition and K, T, S and $\{x_n\}$ be taken as Theorem 5. If $F(S) \cap F(T) \neq \phi, I - T$ and $I - S$ are demiclosded at zero, then $\{x_n\}$ converges weakly to a common fixed point of S and T.*

Proof. Let $p \in F(S) \cap F(T)$. Then as proved in Lemma 3 $\lim_{n\to\infty} \|x_n - p\|$ exist. Since E is uniformly convex. Thus there exists subsequences $\{x_{n_k}\}$ of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $z_1 \in K$. From Theorem 5, we have

$$\lim_{n\to\infty} \|x_{n_k} - Tx_{n_k}\| = 0, \text{ and } \lim_{n\to\infty} \|x_{n_k} - Sx_{n_k}\| = 0.$$

Since $I - T$ and $I - S$ are demiclosed at zero, therefore $Sz_1 = z_1$. Similarly $Tz_1 = z_1$. Again in the same way, we can prove that $z_2 \in F(S) \cap F(T)$. Next, we prove the uniqueness. From Lemma 3 the limits $\lim_{n\to\infty} \|x_n - z_2\|$ exists. For this suppose that $z_1 \neq z_2$, then by the Opial's condition

$$\begin{aligned}\lim_{n\to\infty} \|x_n - z_1\| &= \lim_{n_i\to\infty} \|x_{n_i} - z_1\| < \lim_{n_i\to\infty} \|x_{n_i} - z_2\| \\ &= \lim_{n\to\infty} \|x_n - z_2\| = \lim_{n_j\to\infty} \|x_{n_j} - z_2\| \\ &< \lim_{n_j\to\infty} \|x_{n_j} - z_1\| \lim_{n\to\infty} \|x_n - z_1\|.\end{aligned}$$

This is a contradiction so $z_1 = z_2$. Hence $\{x_n\}$ converges weakly to a common fixed point of T and S. □

Theorem 7. *Let E be a real uniformly convex Banach space and K, S, T, F, $\{x_n\}$ be as in Theorem 5. Then $\{x_n\}$ converges strongly to a point of F if and only if*

$$\liminf_{n\to\infty} d(x_n, F) = 0.$$

Proof. Necessity is evident, let $\lim inf_{n\to\infty} d(x_n, F) = 0$. From Lemma 3, $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F$, so that $\lim_{n\to\infty} d(x_n, F)$ exists.Since by hypothesis,$\lim inf_{n\to\infty} d(x_n, F) = 0$,so that,we get

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

But $\{x_n\}$ is Cauchy sequence and therefore converges to p. We know that $\lim_{n\to\infty} d(x_n, F) = 0$, we obtained $d(p, F) = 0$, therefore $p \in F$. Using theorem 7, we obtain a strong convergence theorem of the iteration scheme (6) under the Condition (A') as below: □

Theorem 8. *Let E be a uniformly convex Banach space and K, S, T, F, $\{x_n\}$ be as in Theorem 5.Let S, T satisfy the Condition (A') and $F \neq \phi$.Then $\{x_n\}$ converges strongly to a point of F.*

Proof. We proved in Theorem 5, i.e.

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0 = \lim_{n\to\infty} \|x_n - Tx_n\|$$

Then from the definition of Condition (A'), we obtain

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

or

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} \|x_n - Sx_n\| = 0.$$

In above cases, we get

$$\lim_{n\to\infty} f(d(x_n, F)) = 0.$$

But $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0, f(r) > 0$ for all $r \in (0, \infty)$, so that we get $\lim_{n\to\infty} d(x_n, F) = 0$. All the condition of Theorem 7 are satisfied, therefore by its conclusion $\{x_n\}$ converges to strongly to a fixed point of F. □

Corollary 9. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E. Suppose T be a asymptotically nonexpansive mapping of K. Let $\{x_n\}$ be defined by the iteration (7), where $\{\gamma_n\}$ and $\{\beta_n^{'}\}$ in $[0, 1]$ for all $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Suppose $S = T$ in the above theorem. □

Corollary 10. *Let K be a nonempty closed convex subset of a uniformly convex Banach space E. Suppose T be a asymptotically nonexpansive mapping of K. Let $\{x_n\}$ be defined by the iteration (8), where $\{\gamma_n\}$ and $\{\beta_n^{'}\}$ in $[0, 1]$ for all $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Suppose $T = I$ in the above theorem. □

Corollary 11. *. Suppose E be a Banach space satisfying Opial condition and let K and T be taken as in (5). Let $F(T) \neq \phi$. Now if the mapping $I - T$ is demiclosed at zero, then $\{x_n\}$ defined by (3) converges weakly to a fixed point of T.*

Corollary 12. *. Let E be a uniformly convex Banach space which has a Frechet differentiable norm and let K and T be taken as theorem 5. Let $F(T) \neq \phi$ taken $\{x_n\}$ defined by (8) converges weakly to a fixed point of T.*

Corollary 13. *. Let E be a uniformly convex Banach space satisfying Opial condition and let K and T be taken as in Theorem 5. Let $F(T) \neq \phi$. If the mapping $I - T$ is demiclosed at zero, then $\{x_n\}$ defined by (8) converges weakly to a fixed point of T.*

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