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ON CAYLEY ISOMORPHISMS OF LEFT AND RIGHT GROUPS

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Abstract: In this paper, we investigate the characterization of CI-graphs on Cayley digraphs of left groups. We also determine which Cayley digraphs of right groups with given connection sets are CI-graphs.

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1. Introduction

Let S be a semigroup and A a subset of S. The Cayley digraph $\mathrm{Cay}(S, A)$ of S relative to a connection set A is defined as the graph with the vertex set S and the arc set $E(\mathrm{Cay}(S, A))$ consisting of those ordered pairs (x, y) such that $xa = y$ for some $a \in A$. Clearly, if A is an empty set, then $\mathrm{Cay}(S, A)$ is an empty graph.

Arthur Cayley (1821-1895) introduced Cayley graphs of groups in 1878. One of the first investigations on Cayley graphs of algebraic structures can be found in Maschke's Theorem from 1896 about groups of genus zero, that is, groups which possess a generating system such that the Cayley graph is planar.

Cayley graphs of groups have been extensively studied and many interesting results have been obtained, see for examples [1], [8], [9], [10], [11], and [17]. The Cayley graphs of semigroups have been considered by many authors. Many new interesting results on Cayley graphs of semigroups have recently appeared in various journals, see for examples [3], [4], [5], [6], [7], [8], [13], [14], [15], and

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[16]. In the investigation of the Cayley graphs of semigroups, it is first of all interesting to find the analogous of natural conditions which have been used in the group case.

A Cayley digraph $\mathrm{Cay}(S, A)$ is called a *CI-graph* of a semigroup S, CI stands for *Cayley Isomorphism*, if whenever B is a subset of S for which $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$, there exists an automorphism σ of S such that $\sigma(A) = B$. A semigroup S is called a *CI-semigroup* if all of its Cayley digraphs are CI-graphs.

Necessary and sufficient conditions have been found for Cayley graphs of groups to be CI-graphs and for groups to be CI-groups, see for examples [9], [10], [11], and [12]. Such a problem is called Cayley isomorphism. Here we shall investigate this problem on left and right groups which both of them are the cartesian product between a group and a semigroup. Graphs considered in this paper are directed graphs. The terminology and notation which related to our paper will be defined in the next section.

2. Basic Definitions and Results

Let (V_1, E_1) and (V_2, E_2) be digraphs. A mapping $\varphi : V_1 \to V_2$ is called a *digraph homomorphism* if $u, v \in E_1$ implies $((\varphi(u)), (\varphi(v))) \in E_2$, i.e. φ preserves arcs. We write $\varphi : (V_1, E_1) \to (V_2, E_2)$. A digraph homomorphism $\varphi : (V, E) \to (V, E)$ is called a *digraph endomorphism*. If $\varphi : (V_1, E_1) \to (V_2, E_2)$ is a bijective digraph homomorphism and φ^{-1} is also a digraph homomorphism, then φ is called a *digraph isomorphism*. A digraph isomorphism $\varphi : (V, E) \to (V, E)$ is called a *digraph automorphism*.

A digraph (V, E) is called a *semigroup* (*group*) *digraph* or *digraph of a semigroup* (*group*) if there exists a semigroup (group) S and a connection set $A \subseteq S$ such that (V, E) is isomorphic to the Cayley graph $\mathrm{Cay}(S, A)$.

A semigroup S is called a *left* (*right*) *zero semigroup* if, for any $x, y \in S$, $xy = x$ ($xy = y$).

A semigroup S is called a *left* (*right*) *group* if $S = G \times L_n$ ($S = G \times R_n$) where G is a group and L_n (R_n) is an n-element left (right) zero semigroup. Then the operation on a left group S is defined by $(g, l)(g', l') = (gg', l)$ for $g, g' \in G$ and $l, l' \in L_n$. Similarly, the operation on a right group S is defined by $(g, r)(g', r') = (gg', r')$ for $g, g' \in G$ and $r, r' \in R_n$.

Now we recall some lemmas and theorems which are needed in the sequel.

Theorem 2.1. [11] *A cyclic group G is called a 2-DCI-group, that is, all Cayley digraphs of G of valency at most 2 are CI-graphs.*

The following lemmas give the structure of the Cayley digraphs of left groups and right groups, respectively. From now on, p_i denotes the projection map on the i^{th} coordinate of an ordered pair.

Let $(V_1, E_1), (V_2, E_2), ..., (V_n, E_n)$ be graphs and $V_i \cap V_j = \emptyset$ for all $i \neq j$. The *disjoint union* of $(V_1, E_1), (V_2, E_2), ..., (V_n, E_n)$ is defined as $\dot{\bigcup}_{i=1}^{n}(V_i, E_i) := (V_1 \cup V_2 \cup ... \cup V_n, E_1 \cup E_2 \cup ... \cup E_n)$.

Lemma 2.2. [16] *Let $S = G \times L_n$ be a left group and $A \subseteq S$. Then the following conditions hold:*

1. *for each $i \in \{1, 2, ..., n\}$,*
   $\mathrm{Cay}(G \times \{l_i\}, p_1(A) \times \{l_i\}) \cong \mathrm{Cay}(G, p_1(A))$
2. $\mathrm{Cay}(S, A) = \dot{\bigcup}_{i=1}^{n}\mathrm{Cay}(G \times \{l_i\}, p_1(A) \times \{l_i\})$.

Lemma 2.3. [16] *Let $S = G \times R_n$ be a right group and $A \subseteq S$. If $A \subseteq G \times \{r_i\}$ where $i \in \{1, 2, ..., n\}$, then* $\mathrm{Cay}(G \times \{r_i\}, A) \cong \mathrm{Cay}(G, p_1(A))$.

The next lemma shows the condition when any two Cayley digraphs of a given right group with a one-element connection set are isomorphic.

Lemma 2.4. [12] *Let $S = G \times R_n$ be a right group, and $(g, r), (g', r') \in S$ where $g, g' \in G$ and $r, r' \in R_n$. Then* $\mathrm{Cay}(S, \{(g, r)\})$
$\cong \mathrm{Cay}(S, \{(g', r')\})$ *if and only if* $|g| = |g'|$.

3. Main Results

This section is divided into two parts. We first characterize CI-graphs of left groups. We will end the section by introducing about CI-graphs of right groups which the connection set is a subset of $G \times \{r_i\}$ where $\{r_i\}$ is a singleton subset of the n-element right zero semigroup R_n.

3.1. CI-Graphs of Left Groups

We start with the lemma that will be used in Theorem 3.2. The condition for two Cayley digraphs of an arbitrary left group which can be isomorphic will be given.

Lemma 3.1. *Let $S = G \times L_n$ be a left group and $A, B \subseteq S$. Then* $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$ *if and only if* $\mathrm{Cay}(G, p_1(A)) \cong \mathrm{Cay}(G, p_1(B))$.

Proof. ($\Longrightarrow$) Let $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$ and $i \in \{1, 2, ..., n\}$. By Lemma 2.2, we have $\dot{\bigcup}_{i=1}^{n}\mathrm{Cay}(G \times \{l_i\}, p_1(A) \times \{l_i\}) \cong \dot{\bigcup}_{i=1}^{n}\mathrm{Cay}(G \times \{l_i\}, p_1(B) \times \{l_i\})$

and $\text{Cay}(G, p_1(A)) \cong \text{Cay}(G \times \{l_i\}, p_1(A) \times \{l_i\}) \cong \text{Cay}(G \times \{l_i\}, p_1(B) \times \{l_i\}) \cong \text{Cay}(G, p_1(B))$ as required.

($\Longleftarrow$) Let $\text{Cay}(G, p_1(A)) \cong \text{Cay}(G, p_1(B))$. Then $\text{Cay}(G \times \{l_i\}, p_1(A) \times \{l_i\}) \cong \text{Cay}(G \times \{l_i\}, p_1(B) \times \{l_i\})$ for all $i \in \{1, 2, ..., n\}$ by Lemma 2.2 (1). Therefore $\dot{\bigcup}_{i=1}^{n}\text{Cay}(G \times \{l_i\}, p_1(A) \times \{l_i\}) \cong \dot{\bigcup}_{i=1}^{n}\text{Cay}(G \times \{l_i\}, p_1(B) \times \{l_i\})$. Thus we get $\text{Cay}(S, A) \cong \text{Cay}(S, B)$ by Lemma 2.2 (2). □

The next result characterizes the CI-graphs of left groups.

Theorem 3.2. *Let $S = G \times L_n$ be a left group and $A \subseteq S$. Then $\text{Cay}(S, A)$ is a CI-graph if and only if $n = 1$ and $\text{Cay}(G, p_1(A))$ is a CI-graph.*

Proof. ($\Longrightarrow$) Let $\emptyset \neq A \subseteq G \times L_n$ and let $\text{Cay}(S, A)$ be a CI-graph and $n \neq 1$. We start the proof by choosing an element $(g, l_i) \in A$ to consider. Since $n \neq 1$, so $n \geq 2$. Then there exists $k \in \{1, 2, ..., n\}$ such that $k \neq i$ and $l_k \in L_n$. We will consider the following two cases:

Case 1: if there exists $(g, l_k) \in A$, consider $B = A \setminus \{(g, l_k)\}$. We will see that $p_1(A) = p_1(B)$ and $\text{Cay}(G, p_1(A)) \cong \text{Cay}(G, p_1(B))$. Thus we have $\text{Cay}(S, A) \cong \text{Cay}(S, B)$ by Lemma 3.1, but $|A| \neq |B|$. So it is easy to see that there is no any functions $f \in \text{Aut}(S)$ such that $f(A) = B$ which satisfy the definition of CI-graph.

Case 2: if $(g, l_k) \notin A$, consider $B = A \cup \{(g, l_k)\}$. Similarly to the case 1, $\text{Cay}(S, A) \cong \text{Cay}(S, B)$, but we can't find any functions $f \in \text{Aut}(S)$ such that $f(A) = B$ since $|A| \neq |B|$. It contradicts the assumption by these two cases. Therefore $n = 1$.

Next, we will show that $\text{Cay}(G, p_1(A))$ is a CI-graph. Suppose that

$$\text{Cay}(G, p_1(A)) \cong \text{Cay}(G, X).$$

Take $B = X \times \{l_1\}$, then $p_1(B) = X$. By Lemma 3.1, we get $\text{Cay}(S, A) \cong \text{Cay}(S, B)$. Since $\text{Cay}(S, A)$ is a CI-graph, there exists $\alpha \in \text{Aut}(S)$ such that $\alpha(A) = B$. Define $f : G \to G$ by $g \mapsto p_1(\alpha(g, l_1))$. Since $\alpha \in \text{Aut}(G \times L_1)$, we have f is bijective. Therefore f is a group homomorphism since $f(g_1)f(g_2) = p_1(\alpha(g_1, l_1))p_1(\alpha(g_2, l_1)) = p_1(\alpha(g_1, l_1)\alpha(g_2, l_1)) = p_1(\alpha(g_1 g_2, l_1)) = f(g_1 g_2)$ for $g_1, g_2 \in G$. Moreover, $f(p_1(A)) = p_1(\alpha(A)) = p_1(B) = X$. Hence $f \in \text{Aut}(G)$ and $f(p_1(A)) = p_1(B) = X$. Thus $\text{Cay}(G, p_1(A))$ is a CI-graph.

($\Longleftarrow$) Let $\text{Cay}(G, p_1(A))$ be a CI-graph. Let $n = 1$. Suppose that $\text{Cay}(G \times L_1, A) \cong \text{Cay}(G \times L_1, B)$. So, by Lemma 3.1, we have $\text{Cay}(G, p_1(A)) \cong \text{Cay}(G, p_1(B))$. Since $\text{Cay}(G, p_1(A))$ is a CI-graph, there exists $\alpha \in \text{Aut}(G)$ such that $\alpha(p_1(A)) = p_1(B)$. Then we define $\beta : G \times \{l_1\} \to G \times \{l_1\}$ by $\beta(g, l_1) =$

$(\alpha(g), l_1)$. Since $\alpha \in \mathrm{Aut}(G)$, it is easy to see that β is also bijective. Therefore β is a group homomorphism since $\beta(g_1, l_1)\beta(g_2, l_1) = (\alpha(g_1), l_1)(\alpha(g_2), l_1) = (\alpha(g_1)\alpha(g_2), l_1) = (\alpha(g_1g_2), l_1) = \beta(g_1g_2, l_1) = \beta((g_1, l_1)(g_2, l_1))$ for $(g_1, l_1), (g_2, l_1) \in G \times \{l_1\}$. In addition, $\beta(A) = \beta(p_1(A) \times \{l_1\}) = \alpha(p_1(A)) \times \{l_1\} = p_1(B) \times \{l_1\} = B$. Hence $\mathrm{Cay}(S, A)$ is a CI-graph. □

The next example shows that if $n \geq 2$, then $\mathrm{Cay}(S, A)$ is not a CI-graph.

Example 1. Let $S = \mathbb{Z}_5 \times L_2$. Consider $A = \{(\overline{1}, l_1), (\overline{1}, l_2)\}$ and $B = \{(\overline{1}, l_1)\}$.

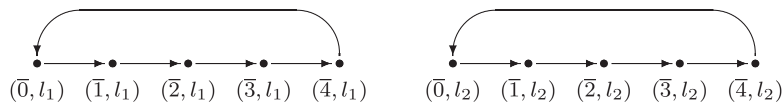

Figure 1: $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$

By the definition of a Cayley digraph, we have $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$, see Figure 1. Since $|A| \neq |B|$, then we can't find any automorphisms f in S such that $f(A) = B$.

3.2. CI-Graphs of Right Groups

The next lemma will be useful for the proof of Lemma 3.4. We mention about the degree of vertices of right groups. Let $\overrightarrow{d}(u)$ denote the *in-degree* of an arbitrary vertex u of a given right group S.

Lemma 3.3. *Let $S = G \times R_n$ be a right group and $A \subseteq S$. Let $i \in \{1, 2, ..., n\}$. Then $A \cap (G \times \{r_i\}) = \emptyset$ if and only if $\overrightarrow{d}(u) = 0$ for all $u \in (G \times \{r_i\})$.*

Proof. Let $i \in \{1, 2, ..., n\}$.

($\Longrightarrow$) Assume that $A \cap (G \times \{r_i\}) = \emptyset$. Suppose that there exists $u \in (G \times \{r_i\})$ such that $\overrightarrow{d}(u) \neq 0$. Hence there exists an element $a \in A$ such that $xa = u$ for some $x \in S$. Since S is a right group, we have $a \in (G \times \{r_i\})$. Then $a \in A \cap (G \times \{r_i\})$, contrary to $A \cap (G \times \{r_i\}) = \emptyset$. Therefore $\overrightarrow{d}(u) = 0$ for all $u \in (G \times \{r_i\})$.

($\Longleftarrow$) Let $u, v \in (G \times \{r_i\})$ and $\overrightarrow{d}(u) = 0$, $\overrightarrow{d}(v) = 0$. Suppose that $A \cap (G \times \{r_i\}) \neq \emptyset$. So there exists an element $a \in A \cap (G \times \{r_i\})$ such that (u, v) is an arc in $\mathrm{Cay}(S, A)$, and then $\overrightarrow{d}(v) \neq 0$, a contradiction. Hence $A \cap (G \times \{r_i\}) = \emptyset$. □

The following lemma gives the conditions when any two Cayley digraphs of an arbitrary right group which each of its connection set is a subset of the cartesian product of a group G and a singleton subset of the n-element right zero semigroup R_n. Throughout the proof, N_0^H denotes the number of vertices u in a graph H such that $\overrightarrow{d}(u) = 0$.

Lemma 3.4. *Let $S = G \times R_n$ be a right group. Let $A \subseteq G \times \{r_i\}$ where $i \in \{1, 2, ..., n\}$. Then $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$ if and only if the following conditions hold:*

1. *$B \subseteq G \times \{r_j\}$ for some $j \in \{1, 2, ..., n\}$,*
2. *there exists a graph isomorphism*

$$f : \mathrm{Cay}(G \times \{r_i\}, A) \to \mathrm{Cay}(G \times \{r_j\}, B)$$

such that $((g, r_k), (g', r_i)) \in E(\mathrm{Cay}(S, A))$ if and only if

$$(f(g, r_k), f(g', r_i)) \in E(\mathrm{Cay}(S, B)) \text{ for any } k \in \{1, 2, ..., n\}.$$

Proof. ($\Longrightarrow$) Let $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$.

1. Suppose that $B \nsubseteq G \times \{r_j\}$ for all $j \in \{1, 2, ..., n\}$. Then $|\{j | B \cap (G \times \{r_j\}) = \emptyset\}| \neq |\{j | A \cap (G \times \{r_j\}) = \emptyset\}|$. By Lemma 3.3, $N_0^{\mathrm{Cay}(S,B)} = |\{j | B \cap (G \times \{r_j\}) = \emptyset\}||G|$ and $N_0^{\mathrm{Cay}(S,A)} = |\{j | A \cap (G \times \{r_j\}) = \emptyset\}||G|$. Therefore $N_0^{\mathrm{Cay}(S,A)} \neq N_0^{\mathrm{Cay}(S,B)}$, which contradicts $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. Then $B \subseteq G \times \{r_j\}$ for some $j \in \{1, 2, ..., n\}$.

2. Since $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$, there exists a graph isomorphism $s : \mathrm{Cay}(S, A) \to \mathrm{Cay}(S, B)$. Next, we can define $t : \mathrm{Cay}(G \times \{r_i\}, A) \to \mathrm{Cay}(G \times \{r_j\}, B)$ as the restriction of s to $G \times \{r_i\}$, i.e. $t = s_{|G \times \{r_i\}}$ by Lemma 3.3. It is obvious that t is also a graph isomorphism by the definition of s. Therefore $\mathrm{Cay}(G \times \{r_i\}, A) \cong \mathrm{Cay}(G \times \{r_j\}, B)$. The statement $((g, r_k), (g', r_i)) \in E(\mathrm{Cay}(S, A))$ if and only if $(t(g, r_k), t(g', r_i)) \in E(\mathrm{Cay}(S, B))$ for any $k \in \{1, 2, ..., n\}$ is also true by the assumption.

($\Longleftarrow$) We define $\varphi : \mathrm{Cay}(S, A) \to \mathrm{Cay}(S, B)$ by

$$\varphi(g, r) = \begin{cases} (p_1 f(g, r_i), r_j), & \text{if} \quad r = r_i \\ (p_1 f(g, r_i), r_i), & \text{if} \quad r = r_j \\ (p_1 f(g, r_i), r), & \text{otherwise.} \end{cases}$$

By the assumption, it is obviously concluded that φ is a graph isomorphism from $\mathrm{Cay}(S, A)$ to $\mathrm{Cay}(S, B)$. Therefore $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. □

Now we introduce the theorem about being CI-graphs of any right groups with a one-element connection set. Theorem 2.1 will be helpful in the proof.

Theorem 3.5. *Let $S = G \times R_n$ be a right group where G is a cyclic group and R_n is an n-element right zero semigroup. Let $(a, r_i) \in S$ where $i \in \{1, 2, ..., n\}$. Then $\mathrm{Cay}(S, \{(a, r_i)\})$ is a CI-graph.*

Proof. Suppose that $\mathrm{Cay}(S, \{(a, r_i)\}) \cong \mathrm{Cay}(S, \{(b, r_j)\})$ where $(b, r_j) \in S$ for some $j \in \{1, 2, ..., n\}$. By Theorem 2.1, we know that $\mathrm{Cay}(G, \{a\})$ is a CI-graph. So for all $b \in G$ such that $\mathrm{Cay}(G, \{b\}) \cong \mathrm{Cay}(G, \{a\})$, there exists $\alpha \in \mathrm{Aut}(G)$ such that $\alpha(a) = b$. Then we define $t : S \to S$ by

$$t(g, r) = \begin{cases} (\alpha(g), r_j), & \text{if} \quad r = r_i \\ (\alpha(g), r_i), & \text{if} \quad r = r_j \\ (\alpha(g), r), & \text{otherwise.} \end{cases}$$

It is obvious that t is bijective. Let $(g, r), (g', r') \in S$. Since S is a right group, there are only 3 cases to be considered depend on r'.

Case 1: $r' = r_i$. Then $t((g, r)(g', r_i)) = t(gg', r_i) = (\alpha(gg'), r_j)$ and

$$t(g, r)t(g', r_i) = (p_1(t(g, r))\alpha(g'), r_j) = (\alpha(g)\alpha(g'), r_j) = (\alpha(gg'), r_j).$$

Case 2: $r' = r_j$. Then $t((g, r)(g', r_j)) = t(gg', r_j) = (\alpha(gg'), r_i)$ and

$$t(g, r)t(g', r_j) = (p_1(t(g, r))\alpha(g'), r_i) = (\alpha(g)\alpha(g'), r_i) = (\alpha(gg'), r_i).$$

Case 3: $r' \neq r_i \neq r_j$. Then $t((g, r)(g', r')) = t(gg', r')$
$= (\alpha(gg'), r')$ and

$$t(g, r)t(g', r') = (p_1(t(g, r))\alpha(g'), r') = (\alpha(g)\alpha(g'), r') = (\alpha(gg'), r').$$

Thus we have t is a semigroup homomorphism. Since $t \in \mathrm{Aut}(S)$ and $t(a, r_i) = (\alpha(a), r_j) = (b, r_j)$, $\mathrm{Cay}(S, \{(a, r_i)\})$ is a CI-graph. □

The following lemma is similar to Lemma 3.1. We give the condition for two Cayley digraphs of a right group can be isomorphic. The connection set which will be considered is a subset of the cartesian product of a group G and a one-element subset of the right zero semigroup R_n.

Lemma 3.6. *Let $S = G \times R_n$ be a right group, $A \subseteq G \times \{r_i\}$ where $i \in \{1, 2, ..., n\}$, and $B \subseteq S$. Then $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$ if and only if $\mathrm{Cay}(G, p_1(A)) \cong \mathrm{Cay}(G, p_1(B))$.*

Proof. Let $i \in \{1, 2, ..., n\}$ and $A \subseteq G \times \{r_i\}$.

($\Longrightarrow$) Let $\text{Cay}(S, A) \cong \text{Cay}(S, B)$. By Lemma 3.4, there exists $j \in \{1, 2, ..., n\}$ such that $B \subseteq G \times \{r_j\}$ and $\text{Cay}(G \times \{r_i\}, A)$ $\cong \text{Cay}(G \times \{r_j\}, B)$. Therefore, by Lemma 2.3, we have $\text{Cay}(G, p_1(A))$ $\cong \text{Cay}(G, p_1(B))$.

($\Longleftarrow$) Let $\text{Cay}(G, p_1(A)) \cong \text{Cay}(G, p_1(B))$. Then there exists $\varphi : \text{Cay}(G, p_1(A)) \to \text{Cay}(G, p_1(B))$ which is a digraph isomorphism. We define $f : \text{Cay}(S, A) \to \text{Cay}(S, B)$ by

$$f(g, r) = \begin{cases} (\varphi(g), r_j), & \text{if} \quad r = r_i, \\ (\varphi(g), r_i), & \text{if} \quad r = r_j, \\ (\varphi(g), r), & \text{otherwise.} \end{cases}$$

It is obvious that f is bijective. Let $(g, r_a), (g', r_b) \in \text{Cay}(S, A)$ and $((g, r_a), (g', r_b)) \in E(\text{Cay}(S, A))$. There exists $(a, r_i) \in A$ such that $(g', r_b) = (g, r_a)(a, r_i)$. Then $g' = ga$ and $r_b = r_i$. Hence $(g, g') \in E(\text{Cay}(G, p_1(A)))$ and $f(g', r_b) = f(g', r_i) = (\varphi(g'), r_j)$. Thus we have $(\varphi(g), \varphi(g')) \in E(\text{Cay}(G, p_1(B)))$ by the assumption. Then there exists $b \in p_1(B)$ such that $\varphi(g') = \varphi(g)b$. Since

$$\begin{aligned} f(g', r_b) = (\varphi(g'), r_j) = (\varphi(g)b, r_j) = (\varphi(g), r_a)(b, r_j) \\ = f(g, r_a)(b, r_j), (f(g, r_a), f(g', r_b)) \in E(\text{Cay}(S, B)), \end{aligned}$$

where $(b, r_j) \in B$. Thus we have f preserves arcs, and then f^{-1} preserves arcs can prove in the same way. Therefore $\text{Cay}(S, A) \cong \text{Cay}(S, B)$. □

Here we come to our main theorem of the right group. The preceding lemma will be used in the proof.

Theorem 3.7. *Let $S = G \times R_n$ be a right group and $A \subseteq G \times \{r_i\}$ where $i \in \{1, 2, ..., n\}$. Then $\text{Cay}(S, A)$ is a CI-graph if and only if $\text{Cay}(G, p_1(A))$ is a CI-graph.*

Proof. Let $i \in \{1, 2, ..., n\}$.

($\Longrightarrow$) Let $\text{Cay}(S, A)$ be a CI-graph. Suppose that $\text{Cay}(G, p_1(A))$ $\cong \text{Cay}(G, B)$. Take $X = B \times \{r_j\}$ for some $j \in \{1, 2, ..., n\}$. By Lemma 3.6, we get $\text{Cay}(S, A) \cong \text{Cay}(S, X)$. So there exists $f \in \text{Aut}(S)$ such that $f(A) = X$. Define $\varphi : G \to G$ by $g \mapsto p_1(f(g, r_i))$. Clearly, φ is bijective. Then φ is also a group homomorphism since $\varphi(g_1)\varphi(g_2) = p_1(f(g_1, r_i))p_1(f(g_2, r_i)) = p_1(f(g_1, r_i)f(g_2, r_i)) = p_1 f(g_1 g_2, r_i) = \varphi(g_1 g_2)$. Let $t \in \varphi(p_1(A))$, i.e. $t = p_1(f(x, r_i))$ for some $(x, r_i) \in A$. Then $t \in p_1(f(A)) = p_1(X) = B$. Conversely, let $t \in B = p_1(X)$, i.e. $t = p_1(t, r_j)$. Since $f(A) = X$, there exists $(h, r_i) \in$

A such that $f(h, r_i) = (t, r_j)$ and thus $t = p_1(f(h, r_i)) \in \varphi(p_1(A))$. Hence $\varphi(p_1(A)) = B$. Therefore $\mathrm{Cay}(G, p_1(A))$ is a CI-graph.

($\Longleftarrow$) Let $\mathrm{Cay}(G, p_1(A))$ be a CI-graph. Suppose that $\mathrm{Cay}(S, A) \cong \mathrm{Cay}(S, B)$. By Lemma 3.6, we have $\mathrm{Cay}(G, p_1(A)) \cong \mathrm{Cay}(G, p_1(B))$ where $B \subseteq G \times \{r_j\}$ for some $j \in \{1, 2, ..., n\}$. Then there exists $f \in \mathrm{Aut}(G)$ such that $f(p_1(A)) = p_1(B)$. Define $\varphi : S \to S$ by

$$\varphi(g, r) = \begin{cases} (f(g), r_j), & \text{if} \quad r = r_i \\ (f(g), r_i), & \text{if} \quad r = r_j \\ (f(g), r), & \text{otherwise.} \end{cases}$$

It is easy to check that φ is bijective. About to prove that φ is a semigroup homomorphism is similar to Theorem 3.5. Next, we will prove that $\varphi(A) = B$. Let $t \in \varphi(A) = \varphi(p_1(A) \times \{r_i\})$. Then $t = \varphi(x, r_i)$ for some $x \in p_1(A)$. So $t = (f(x), r_j) \in B$. Therefore $\varphi(A) \subseteq B$. Conversely, let $t \in B$. Suppose that $t = (g, r_j)$ for some $g \in G$. Since $f(p_1(A)) = p_1(B)$, there exists $h \in p_1(A)$, i.e. $(h, r_i) \in A$ such that $f(h) = g$. Hence $t = (f(h), r_j) = \varphi(h, r_i) \in \varphi(A)$. Therefore $B \subseteq \varphi(A)$. So we can conclude that $\mathrm{Cay}(S, A)$ is a CI-graph. □

We now show another example which can be concluded by Theorem 3.7.

Example 2. Let $G = \mathbb{Z}_9$ and $S = \mathbb{Z}_9 \times R_n$. Consider $A = \{\overline{1}, \overline{4}, \overline{6}, \overline{7}\}$ and $B = \{\overline{1}, \overline{3}, \overline{4}, \overline{7}\}$.

Define $\beta : \mathrm{Cay}(G, A) \to \mathrm{Cay}(G, B)$ by $0 \mapsto 6, 1 \mapsto 1, 2 \mapsto 2, 3 \mapsto 3, 4 \mapsto 7, 5 \mapsto 8, 6 \mapsto 0, 7 \mapsto 4$ and $8 \mapsto 5$. We have $\mathrm{Cay}(G, A) \cong \mathrm{Cay}(G, B)$, but there is no Cayley isomorphisms mapping A to B, that is, $\mathrm{Cay}(G, A)$ is not a CI-graph. Therefore, by Theorem 3.7, we can conclude that $\mathrm{Cay}(S, A \times \{r_i\})$ is not a CI-graph for all $i \in \{1, 2, ..., n\}$.

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References

[1] N. Biggs, *Algebraic Graph Theory*, Cambridge University Press, Cambridge (1993).

[2] G. Chartrand, L. Lesniak, *Graphs and digraphs*, Chapman and Hall, London (1996).

[3] Y. Hao, Y. Luo, On the Cayley graphs of left (right) groups, *Southeast Asian Bull. Math.*, **34** (2010), 685-691.

[4] A. V. Kelarev, On Undirected Cayley Graphs, *Australas. J. Combin.*, **25** (2002), 73-78.

[5] A. V. Kelarev, C. E. Praeger, On transitive Cayley graphs of groups and semigroups, *European J. Combin.*, **24** (2003), 59-72.

[6] A. V. Kelarev, S. J. Quinn, A combinatorial property and Cayley graphs of semigroups, *Semigroup Forum*, **66** (2003), 89-96.

[7] A. V. Kelarev, Labelled Cayley graphs and minimal automata, *Australas. J. Combin.*, **30** (2004), 95-101.

[8] U. Knauer, *Algebraic graph theory*, W. de Gruyter, Berlin (2011).

[9] C. H. Li, Isomorphisms of Connected Cayley digraphs, *Graphs Combin.*, **14** (1998), 37-44.

[10] C. H. Li, S. Zhou, On isomorphisms of minimal Cayley graphs and digraphs, *Graphs Combin.*, **17** (2001), 307-314.

[11] C.H. Li, On isomorphisms of finite Cayley graphs - a survey, *Discrete Math.*, **256** (2002), 301-334.

[12] J. Meksawang, S. Panma, U. Knauer, Characterization of finite simple semigroup digraphs, *Alg. Dis. Mthm.*, **12** (2011), 53-68.

[13] S. Panma, U. Knauer, Sr. Arworn, On transitive Cayley graphs of right (left) groups and of Clifford semigroups, *Thai J. Math.*, **2** (2004), 183-195.

[14] S. Panma, U. Knauer, Sr. Arworn, On transitive Cayley graphs of strong semilattice of right (left) groups, *Discrete Math.*, **309** (2009), 5393-5403.

[15] S. Panma, N. Na Chiangmai, U. Knauer, Sr. Arworn, Characterizations of Clifford semigroup digraphs, *Discrete Math.*, **306** (2006), 1247-1252.

[16] S. Panma, Characterization of Cayley graphs of rectangular groups, *Thai J. Math.*, **8** (2010), 535-543.

[17] A. T. White, *Graphs, Groups and Surfaces*, Elsevier, Amsterdam (2001).