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HERMITE-HADAMARD-LIKE AND SIMPSON-LIKE TYPE INTEGRAL INEQUALITIES VIA φ-CONVEXITY

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Abstract: In this article, we obtain some Hermite-Hadamard-like and Simpson-like type integral inequalities for functions whose derivatives in absolute value at certain powers are φ-convex.

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1. Introduction

The following double inequality is known in the literature as Hermite-Hadamard integral inequality for convex functions: For a convex function $f : I \subset R \to R$ defined on the interval I of real numbers and $a, b \in I$ with $a < b$, the following double inequality holds:

$$f(\frac{a+b}{2}) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

This inequality plays an important role in convex analysis and it has a huge literature dealing with its applications, various generalizations and refinements [3, 5, 7, 8, 13, 14, 15, 16, 17].

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Suppose that $f : [a, b] \to R$ is a four times continuously differentiable function on (a, b) and $\| f^{(4)} \| = \sup | f^{(4)} | < \infty$. The following inequality

$$\Big|\frac{1}{6}\{f(a) + 4f(\frac{a+b}{2}) + f(b)\} - \frac{1}{b-a}\int_a^b f(x)dx\Big| \\ \leq \frac{1}{2880} \| f^{(4)} \|_{\infty} (b-a)^4$$

is well known in the literature as Simpson's inequality.

For recent result of the Simpson's inequality, see [1, 2, 4, 6, 15, 16, 18].

Let $f, \varphi : K \to R$, where K is a nonempty closed set in R^n, be continuous functions. We recall the following results, which are due to Noor[9, 10] as follows:

Definition 1. Let $u \in K$. Then the set K is said to be φ-convex at u with respect to φ if

$$u + te^{i\varphi}(v - u) \in K, \quad \forall u, v \in K, \quad t \in [0, 1].$$

Definition 2. (a) The function f on φ-convex set K is said to be φ-convex with respect to φ if

$$f(u + te^{i\varphi}(v - u)) \leq (1 - t)f(u) + tf(v), \quad \forall u, v \in K, \quad t \in [0, 1].$$

(b) The function f is said to be φ-concave with respect to φ if $-f$ is φ-convex with respect to φ.

Note that every convex function is a φ-convex function, but the converse is not true.

In [11, 12], Özdemir, Avci and Akdemir established the following theorems:

Theorem 1.1. *Let $f : K \to (0, \infty)$ be a differentiable function on K^0, $a, b \in K$ with $a < a + e^{i\varphi}(b - a)$. If $| f' |$ is a φ-convex function on K^0, then the following inequality holds:*

$$\Big|\frac{1}{6}\{f(a) + 4f(\frac{2a + e^{i\varphi}(b-a)}{2}) + f(a + e^{i\varphi}(b-a))\} \\ - \frac{1}{e^{i\varphi}(b-a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx\Big| \\ \leq e^{i\varphi}(b-a)\big(\frac{5}{72}\big)\big\{\big|f'(a)\big| + \big|f'(b)\big|\big\}.$$

Theorem 1.2. *Let $f : K \to (0, \infty)$ be a differentiable function on K^0, $a, b \in K$ with $a < a + e^{i\varphi}(b - a)$. If $| f' |^q$ is a φ-convex function on K^0 for some fixed $q > 1$ with $p = \frac{q}{q-1}$, then the following inequality holds:*

$$\begin{aligned}&\Big|\frac{1}{6}\{f(a) + 4f(\frac{2a + e^{i\varphi}(b - a)}{2}) + f(a + e^{i\varphi}(b - a))\}\\ &\qquad - \frac{1}{e^{i\varphi}(b - a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx\Big|\\ &\leq e^{i\varphi}(b - a)\Big(\frac{1 + 2^{p+1}}{6^{p+1}(p + 1)}\Big)^{\frac{1}{p}}\\ &\quad\times\Big[\{\frac{3}{8}|f'(a)|^q + \frac{1}{8}|f'(b)|^q\}^{\frac{1}{q}} + \{\frac{1}{8}|f'(a)|^q + \frac{3}{8}|f'(b)|^q\}^{\frac{1}{q}}\Big].\end{aligned}$$

Theorem 1.3. *Let $f : K \to (0, \infty)$ be a differentiable function on K^0, $a, b \in K$ with $a < a + e^{i\varphi}(b - a)$. If $| f' |^q$ is a φ-convex function on K^0 for some fixed $q \geq 1$ then the following inequality holds:*

$$\begin{aligned}&\Big|\frac{1}{6}\{f(a) + 4f(\frac{2a + e^{i\varphi}(b - a)}{2}) + f(a + e^{i\varphi}(b - a))\}\\ &\qquad - \frac{1}{e^{i\varphi}(b - a)}\int_a^{a+e^{i\varphi}(b-a)} f(x)dx\Big|\\ &\leq e^{i\varphi}(b - a)\Big(\frac{5}{72}\Big)^{1-\frac{1}{q}}\Big[\{\frac{61|f'(a)|^q + 29|f'(b)|^q}{1296}\}^{\frac{1}{q}}\\ &\qquad + \{\frac{29|f'(a)|^q + 61|f'(b)|^q}{1296}\}^{\frac{1}{q}}\Big].\end{aligned}$$

In this article we establish some generalizations of Hermite-Hadamard-like and Simpson-like type inequalities for functions whose derivatives in absolute values are φ-convex, which is a generalization of Theorem 1.1.

2. Main Results

Throughout this section, let $K = [a, a + e^{i\varphi}(b - a)]$ and $0 \leq \varphi \leq \frac{\pi}{2}$.

To generalize Theorem 1.1, we need the following lemma.

Lemma 1. *Let $K \subset R$ be a φ-convex subset and $f : K \to (0, \infty)$ be a differentiable function on the interior K^0 of K, and $a, b \in K$ with $a < a + e^{i\varphi}(b - a)$. If f' is an integrable function on $[a, a + e^{i\varphi}(b - a)]$, then for $r \geq 2$ and $h \in (0, 1)$ with $\frac{1}{r} \leq h \leq \frac{r-1}{r}$ the following identity holds:*

$$\big|S(f, \varphi, r, h)\big|$$

$$=^{put} \left| \frac{1}{r} \{ f(a) + (r-2) f(a + he^{i\varphi}(b-a)) + f(a + e^{i\varphi}(b-a)) \} - \frac{1}{e^{i\varphi}(b-a)} \int_a^{a+e^{i\varphi}(b-a)} f(x)dx \right|$$

$$= e^{i\varphi}(b-a) \int_0^1 p(h,r,t) f'(a + te^{i\varphi}(b-a))dt, \tag{1}$$

where

$$p(h,r,t) = \begin{cases} t - \frac{1}{r} & t \in [0,h] \\ t - \frac{r-1}{r} & t \in (h,1] \end{cases}$$

for each $t \in [0,1]$.

Proof. Since K is a φ-convex subset, for $a, b \in K$ and $t \in [0,1]$ we have $a + e^{i\varphi}(b-a) \in K$. Integrating by parts implies that

$$\int_0^h (t - \frac{1}{r}) f'(a + te^{i\varphi}(b-a))dt + \int_h^1 (t - \frac{r-1}{r}) f'(a + te^{i\varphi}(b-a))dt$$

$$= \Big[(t - \frac{1}{r}) \frac{f'(a + te^{i\varphi}(b-a))}{e^{i\varphi}(b-a)} \Big]_0^h - \int_0^h \frac{f'(a + te^{i\varphi}(b-a))}{e^{i\varphi}(b-a)} dt$$

$$+ \Big[(t - \frac{r-1}{r}) \frac{f'(a + te^{i\varphi}(b-a))}{e^{i\varphi}(b-a)} \Big]_h^1 - \int_h^1 \frac{f'(a + te^{i\varphi}(b-a))}{e^{i\varphi}(b-a)} dt$$

$$= \frac{1}{re^{i\varphi}(b-a)} \Big[f(a) + (r-2) f(a + he^{i\varphi}(b-a)) + f(a + e^{i\varphi}(b-a)) \Big]$$

$$- \frac{1}{[e^{i\varphi}(b-a)]^2} \int_a^{a+e^{i\varphi}(b-a)} f(x)dx,$$

If we change the variable $x = a + te^{i\varphi}(b-a)$ and multiply the resulting equality with $e^{i\varphi}(b-a)$ we get the desired result.

Theorem 2.1. *Let $K \subset R$ be a φ-convex subset and $f : K \to (0, \infty)$ be a differentiable function on the interior K^0 of K, and $a, b \in K$ with $a < a + e^{i\varphi}(b-a)$. If f' is an integrable function on $[a, a + e^{i\varphi}(b-a)]$ and $| f' |$ is a φ-convex function on K^0, then for $r \geq 2$ and $h \in (0,1)$ with $\frac{1}{r} \leq h \leq \frac{r-1}{r}$ the following inequality holds:*

$$\big| S(f, \varphi, r, h) \big| \leq e^{i\varphi}(b-a) \Big\{ \mu_{11} \mid f'(a) \mid + \mu_{12} \mid f'(b) \mid \Big\}, \tag{2}$$

where

$$\mu_{11} = \frac{6 - 3r + (2 - 6h + 9h^2 - 4h^3)r^2}{6r^2},$$

$$\mu_{12} = \frac{6 - 3r + (1 - 3h^2 + 4h^3)r^2}{6r^2}.$$

Proof. From Lemma 2 and using the φ-convexith of $| f' |$, we have

$$\begin{aligned}
&\left|S(f, \varphi, r, h)\right| \\
&\leq e^{i\varphi}(b - a)\Big\{ \int_0^h | t - \frac{1}{r} || f'(a + te^{i\varphi}(b - a)) | dt \\
&\quad + \int_h^1 | t - \frac{r-1}{r} || f'(a + te^{i\varphi}(b - a)) | dt \Big\} \\
&\leq e^{i\varphi}(b - a)\Big[\int_0^{\frac{1}{r}} (\frac{1}{r} - t)\big\{(1 - t) | f'(a) | + t | f'(b) | \big\}dt \\
&\quad + \int_{\frac{1}{r}}^h (t - \frac{1}{r})\big\{(1 - t) | f'(a) | + t | f'(b) | \big\}dt \\
&\quad + \int_h^{\frac{r-1}{r}} (\frac{r-1}{r} - t)\big\{(1 - t) | f'(a) | + t | f'(b) | \big\}dt \\
&\quad + \int_{\frac{r-1}{r}}^1 (t - \frac{r-1}{r})\big\{(1 - t) | f'(a) | + t | f'(b) | \big\}dt\Big],
\end{aligned}$$

which completes the proof by the simple calculations.

Corollary 1. *In Theorem 2.1*
(a) If we give $h = \frac{1}{2}$ and $r = 6$ then we get

$$\left|S(f, \varphi, 6, \frac{1}{2})\right| \leq e^{i\varphi}(b - a)\frac{5}{72}\big(| f'(a) | + | f'(b) | \big),$$

which implies that Theorem 2.1 is a generalization of Theorem 1.1.
(b)If we give $h = \frac{1}{2}$ and $r = 2$ then we get

$$\left|S(f, \varphi, 6, \frac{1}{2})\right| \leq e^{i\varphi}(b - a)\frac{1}{8}\big(| f'(a) | + | f'(b) | \big).$$

Theorem 2.2. *Let $K \subset R$ be a φ-convex subset and $f : K \to (0, \infty)$ be a differentiable function on the interior K^0 of K, and $a, b \in K$ with $a < a + e^{i\varphi}(b - a)$. If f' is an integrable function on $[a, a + e^{i\varphi}(b - a)]$ and $| f' |^q$ is a φ-convex function on K^0 for some fixed $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then for $r \geq 2$ and $h \in (0, 1)$ with $\frac{1}{r} \leq h \leq \frac{r-1}{r}$ the following inequality holds:*

$$\left|S(f, \varphi, r, h)\right|$$

$$\leq e^{i\varphi}(b-a)\Big[\Big\{\frac{1+(rh-1)^{p+1}}{r^{p+1}(p+1)}\Big\}^{\frac{1}{p}}$$
$$\times\Big\{\big(\frac{2h-h^2}{2}\big)\mid f'(a)\mid^q+\big(\frac{h^2}{2}\big)\mid f'(b)\mid^q\Big\}^{\frac{1}{q}}$$
$$+\Big\{\frac{1+(r-rh+1)^{p+1}}{r^{p+1}(p+1)}\Big\}^{\frac{1}{p}}$$
$$\times\Big\{\big(\frac{(1-h)^2}{2}\big)\mid f'(a)\mid^q+\big(\frac{1-h^2}{2}\big)\mid f'(b)\mid^q\Big\}^{\frac{1}{q}}\Big].$$

Proof. From Lemma 2 and using the φ-convexith of $\mid f' \mid$, we have

$$\big|S(f,\varphi,r,h)\big|$$
$$\leq e^{i\varphi}(b-a)\Big\{\int_0^h \mid t-\frac{1}{r}\mid\mid f'(a+te^{i\varphi}(b-a))\mid dt$$
$$+\int_h^1 \mid t-\frac{r-1}{r}\mid\mid f'(a+te^{i\varphi}(b-a))\mid dt\Big\}$$
$$\leq e^{i\varphi}(b-a)\Big[\Big(\int_0^h \mid t-\frac{1}{r}\mid^p dt\Big)^{\frac{1}{p}}\Big(\int_0^h \mid f'(a+te^{i\varphi}(b-a))\mid^q dt\Big)^{\frac{1}{q}}$$
$$+\Big(\int_0^h \mid t-\frac{r-1}{r}\mid^p dt\Big)^{\frac{1}{p}}\Big(\int_0^h \mid f'(a+te^{i\varphi}(b-a))\mid^q dt\Big)^{\frac{1}{q}}\Big]. \qquad (3)$$

Note that

$$(i)\int_0^h \mid t-\frac{1}{r}\mid^p dt=\frac{1+(rh-1)^{p+1}}{r^{p+1}(p+1)},$$
$$(ii)\int_0^h \mid t-\frac{r-1}{r}\mid^p dt=\frac{1+(r-rh+1)^{p+1}}{r^{p+1}(p+1)},$$
$$(iii)\int_0^h \mid f'(a+te^{i\varphi}(b-a))\mid^q dt$$
$$\leq\big(\frac{2h-h^2}{2}\big)\mid f'(a)\mid^q+\big(\frac{h^2}{2}\big)\mid f'(b)\mid^q,$$

$$(iv)\int_h^1 \mid f'(a+te^{i\varphi}(b-a))\mid^q dt$$
$$\leq\big(\frac{(1-h)^2}{2}\big)\mid f'(a)\mid^q+\big(\frac{1-h^2}{2}\big)\mid f'(b)\mid^q . \qquad (4)$$

By (3) and (4) we get the desired result.

Corollary 2. *In Theorem 2.2*
(a) If we give $h = \frac{1}{2}$ *and* $r = 6$ *then we get*

$$\begin{aligned} &\left|S(f,\varphi,6,\frac{1}{2})\right| \\ &\le e^{i\varphi}(b-a)\Big(\frac{2^{p+1}+1}{6^{p+1}(p+1)}\Big)^{\frac{1}{p}}\Big[\{\frac{3}{8}\mid f'(a)\mid^q+\frac{1}{8}\mid f'(b)\mid^q\}^{\frac{1}{q}} \\ &\qquad +\{\frac{1}{8}\mid f'(a)\mid^q+\frac{3}{8}\mid f'(b)\mid^q\}^{\frac{1}{q}}\Big], \end{aligned}$$

which implies that Theorem 2.2 is a generalization of Theorem 1.2.
(b)If we give $h = \frac{1}{2}$ *and* $r = 2$ *then we get*

$$\begin{aligned} &\left|S(f,\varphi,2,\frac{1}{2})\right| \\ &\le e^{i\varphi}(b-a)\Big(\frac{1}{2^{p+1}(p+1)}\Big)^{\frac{1}{p}}\Big[\{\frac{3}{8}\mid f'(a)\mid^q+\frac{1}{8}\mid f'(b)\mid^q\}^{\frac{1}{q}} \\ &\qquad +\{\frac{1}{8}\mid f'(a)\mid^q+\frac{3}{8}\mid f'(b)\mid^q\}^{\frac{1}{q}}\Big]. \end{aligned}$$

Theorem 2.3. *Under the assumptions of THeorem 2.2, we have the following inequalities holds:*

$$\begin{aligned} &\left|S(f,\varphi,r,h)\right| \\ &\le e^{i\varphi}(b-a)\Big\{\frac{2+(rh-1)^{p+1}+(r-rh+1)^{p+1}}{r^{p+1}(p+1)}\Big\}^{\frac{1}{p}} \\ &\qquad \times\Big\{\frac{\mid f'(a)\mid^q+\mid f'(b)\mid^q}{2}\Big\}^{\frac{1}{q}}. \end{aligned}$$

Proof. From Lemma 2 and using the φ-convexity of $\mid f' \mid$, we have

$$\begin{aligned} \left|S(f,\varphi,r,h)\right| &\le e^{i\varphi}(b-a)\Big(\int_0^1 \mid p(h,r,t)\mid^p dt\Big)^{\frac{1}{p}} \\ &\quad \times\Big(\int_0^1 \mid f'(a+te^{i\varphi}(b-a))\mid^q dt\Big)^{\frac{1}{q}} \end{aligned}$$

$$\begin{aligned} &\le e^{i\varphi}(b-a)\Big(\int_0^h \mid t-\frac{1}{r}\mid^p dt+\int_h^1 \mid t-\frac{r-1}{r}\mid^p dt\Big)^{\frac{1}{p}} \\ &\quad \times\Big(\int_0^1 (1-t)dt\mid f'(a)\mid^q+\int_0^1 tdt\mid f'(b)\mid^q\Big)^{\frac{1}{q}}. \end{aligned} \tag{5}$$

By (4)(i) and (ii), we get the desired result.

Corollary 3. *In Theorem 2.3*
(a) If we give $h = \frac{1}{2}$ *and* $r = 6$*, then we get*

$$\begin{aligned}&\big|S(f,\varphi,6,\frac{1}{2})\big|\\&\le e^{i\varphi}(b-a)\Big\{\frac{2^{p+2}+2}{6^{p+1}(p+1)}\Big\}^{\frac{1}{p}}\Big\{\frac{|\, f'(a)\,|^q+|\, f'(b)\,|^q}{2}\Big\}^{\frac{1}{q}}.\end{aligned}$$

(b)If we give $h = \frac{1}{2}$ *and* $r = 2$*, then we get*

$$\begin{aligned}&\big|S(f,\varphi,2,\frac{1}{2})\big|\\&\le e^{i\varphi}(b-a)\Big\{\frac{1}{2^{p}(p+1)}\Big\}^{\frac{1}{p}}\Big\{\frac{|\, f'(a)\,|^q+|\, f'(b)\,|^q}{2}\Big\}^{\frac{1}{q}}.\end{aligned}$$

Theorem 2.4. *Let* $K \subset R$ *be a* φ*-convex subset and* $f : K \to (0,\infty)$ *be a differentiable function on the interior* K^0 *of* K*, and* $a, b \in K$ *with* $a < a + e^{i\varphi}(b-a)$*. If* f' *is an integrable function on* $[a, a+e^{i\varphi}(b-a)]$ *and* $|\, f'\,|^q$ *is a* φ*-convex function on* K^0 *for some fixed* $q \ge 1$ *with* $\frac{1}{p}+\frac{1}{q}=1$*, then for* $r \ge 2$ *and* $h \in (0,1)$ *with* $\frac{1}{r} \le h \le \frac{r-1}{r}$ *the following inequality holds:*

$$\begin{aligned}&\big|S(f,\varphi,r,h)\big|\\&\le e^{i\varphi}(b-a)\Big[\Big(\frac{1+(rh-1)^{p+1}}{r^{p+1}(p+1)}\Big)^{\frac{1}{p}}\Big(\mu_{21}\,|\, f'(a)\,|^q+\mu_{22}\,|\, f'(b)\,|^q\Big)^{\frac{1}{q}}\\&\quad+\Big(\frac{1+(r-rh+1)^{p+1}}{r^{p+1}(p+1)}\Big)^{\frac{1}{p}}\Big(\mu_{23}\,|\, f'(a)\,|^q+\mu_{24}\,|\, f'(b)\,|^q\Big)^{\frac{1}{q}}\Big],\end{aligned}$$

where

$$\begin{aligned}\mu_{21}&=\frac{(3-2h)h^2r^3-3(2-h)hr^2+6r-3}{6r^3},\\\mu_{22}&=\frac{2h^3r^3-3h^2r^2+3}{6r^3},\\\mu_{23}&=\frac{2(1-h)^3r^3-3(1-h)^2r^2+2}{6r^3},\end{aligned}$$

$$\mu_{21}=\frac{(1-h)^2(1+2h)r^3-3(1-h^2)r^2+6r-2}{6r^3}.$$

Proof. From Lemma 2 and using the φ-convexity of $|\, f'\,|$, we have

$$\big|S(f,\varphi,r,h)\big|$$

$$\leq e^{i\varphi}(b-a)\Big\{\int_0^h \mid t-\frac{1}{r} \mid\mid f'(a+te^{i\varphi}(b-a)) \mid dt$$
$$+\int_h^1 \mid t-\frac{r-1}{r} \mid\mid f'(a+te^{i\varphi}(b-a)) \mid dt\Big\}$$

$$\leq e^{i\varphi}(b-a)\Big[\Big(\int_0^h \mid t-\frac{1}{r} \mid dt\Big)^{\frac{1}{p}}$$
$$\times\Big(\int_0^h \mid t-\frac{1}{r} \mid\mid f'(a+te^{i\varphi}(b-a)) \mid^q dt\Big)^{\frac{1}{q}}$$
$$+\Big(\int_h^1 \mid t-\frac{r-1}{r} \mid dt\Big)^{\frac{1}{p}}$$
$$\times\Big(\int_h^1 \mid t-\frac{r-1}{r} \mid\mid f'(a+te^{i\varphi}(b-a)) \mid^q dt\Big)^{\frac{1}{q}}\Big]. \tag{6}$$

Note that

$$(i)\int_0^h \mid t-\frac{1}{r} \mid\mid f'(a+te^{i\varphi}(b-a)) \mid^q dt$$
$$\leq \int_0^{\frac{1}{r}} \big(\frac{1}{r}-t\big)\big\{(1-t)\mid f'(a)\mid^q +t\mid f'(b)\mid^q\big\}dt$$
$$+\int_{\frac{1}{r}}^h \big(t-\frac{1}{r}\big)\big\{(1-t)\mid f'(a)\mid^q +t\mid f'(b)\mid^q\big\}dt$$
$$=\mu_{21}\mid f'(a)\mid^q +\mu_{22}\mid f'(b)\mid^q, \tag{7}$$

$$(ii)\int_h^1 \mid t-\frac{r-1}{r} \mid\mid f'(a+te^{i\varphi}(b-a)) \mid^q dt$$
$$\leq \int_h^{\frac{r-1}{r}} \big(\frac{r-1}{r}-t\big)\big\{(1-t)\mid f'(a)\mid^q +t\mid f'(b)\mid^q\big\}dt$$
$$+\int_{\frac{r-1}{r}}^1 \big(t-\frac{r-1}{r}\big)\big\{(1-t)\mid f'(a)\mid^q +t\mid f'(b)\mid^q\big\}dt$$
$$=\mu_{23}\mid f'(a)\mid^q +\mu_{24}\mid f'(b)\mid^q. \tag{8}$$

By (6),(7) and (8) we get the desired result.

Corollary 4. *In Theorem 2.4*
(a) If we give $h=\frac{1}{2}$ and $r=6$ then we get

$$\big|S(f,\varphi,6,\frac{1}{2})\big| \leq e^{i\varphi}(b-a)\Big\{\frac{2^{p+1}+1}{6^{p+1}(p+1)}\Big\}^{\frac{1}{p}}\Big\{\frac{5}{72}\Big\}^{\frac{1}{q}}$$

$$\times\Big[\Big\{\frac{61}{90}\mid f'(a)\mid^q+\frac{29}{90}\mid f'(b)\mid^q\Big\}^{\frac{1}{q}}$$
$$+\Big\{\frac{29}{90}\mid f'(a)\mid^q+\frac{61}{90}\mid f'(b)\mid^q\Big\}^{\frac{1}{q}}\Big],$$

which implies that Theorem 2.4 is a generalization of Theorem 1.3.

(b)If we give $h=\frac{1}{2}$ and $r=2$ then we get

$$\big|S(f,\varphi,2,\frac{1}{2})\big|\le e^{i\varphi}(b-a)\Big\{\frac{1}{2^{p+1}(p+1)}\Big\}^{\frac{1}{p}}$$
$$\times\Big[\Big\{\frac{5}{48}\mid f'(a)\mid^q+\frac{1}{48}\mid f'(b)\mid^q\Big\}^{\frac{1}{q}}$$
$$+\Big\{\frac{1}{48}\mid f'(a)\mid^q+\frac{5}{48}\mid f'(b)\mid^q\Big\}^{\frac{1}{q}}\Big].$$

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